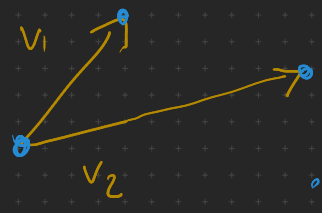
Trace formulas on Tori

Def: $\Lambda \subset \mathbb{R}^n$ is a (rank n) lattice if

- $\mathrm{Span}_{\mathbb{R}}(\Lambda) = \mathbb{R}^n$
- $\exists\ \phi : \mathbb{Z}^n \longrightarrow \Lambda$ ϕ group isomorphism of $(\mathbb{Z}^n, +)$

Example $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$



$$\phi : (a_1, a_2) \longmapsto \sum_{i=1}^{2} a_i v_i$$

$\mathbb{Z}^2$

Example Let $(v_i)_{i=1}^n$ be a basis of $\mathbb{R}^n$

$\phi : \mathbb{Z}^n \to \mathbb{R}^n$

$\mathrm{Image}\ \phi =: \Lambda$

$$\phi(a_1, a_2, \dots, a_n) := \sum_{i=1}^{n} a_i v_i$$

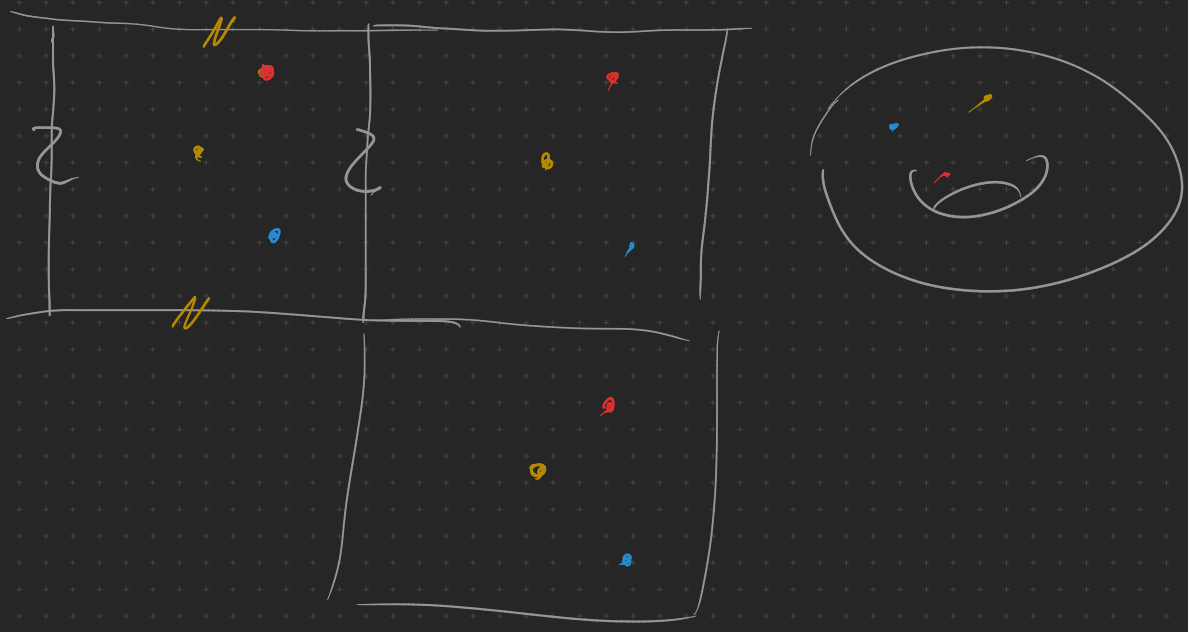
Def: Given a full rank lattice $\Lambda \subset \mathbb{R}^n$

$$T_\Lambda = \mathbb{R}^n / \sim_\Lambda$$

torus associated to Λ.

def'd of $x \sim_\Lambda y \overset{\text{def}}{\iff} x - y \in \Lambda$

Example: $\mathbb{Z}^2 = \Lambda \subset \mathbb{R}^2$



Def Let Λ be a lattice in $\mathbb{R}^n$, define

$$\mathbb{R}^n \supset \Lambda^* = \left\{ \xi \in \mathbb{R}^n \mid \langle \xi, x \rangle \in \mathbb{Z} \;\; \forall x \in \Lambda \right\}$$

Λ^*: dual lattice to Λ

where $\langle \xi, x \rangle = \sum_i \xi_i x_i$ is the usual i.p.

Prp i) Λ is a lattice in $\mathbb{R}^n \iff \exists \{v_i\}_{i=1}^n$ basis of $\mathbb{R}^n$ s.t. $\forall x \in \Lambda \quad x = \sum a_i v_i$ with $a_i \in \mathbb{Z}$

v_i's are called generators of Λ.

ii) if $\{v_i\}, \{w_i\}$ generate Λ, then $\exists P \in SL_\pm(n, \mathbb{R})$ s.t.

$$w_i = P v_i \quad \forall i = 1, \dots n$$

iii) Λ^* is a lattice in $\mathbb{R}^n$ generated by $\{v_i^*\}_{i=1}^n$ defined as

$$\begin{pmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{pmatrix} = \left(\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^{-1} \right)^T$$

Pf i) ($\Leftarrow$) obvious

($\Rightarrow$) Recall $\exists \phi : \mathbb{Z}^n \to \Lambda$ group isomorphism

$e_i = (0 \dots, \underbrace{1}_{i}, 0 \dots)$ generator of $\mathbb{Z}^n$

$v_i := \phi(e_i)$ claim: $\operatorname{span}_{\mathbb{R}} \{v_i\} = \mathbb{R}^n$

$\therefore \forall x \in \Lambda \quad x = \sum a_i v_i$

$\forall \bar{a} \in \mathbb{Z}_n \,, \; a = \sum a_i e_i$

$$\phi(\bar{a}) = \sum a_i \phi(e_i) = \sum a_i v_i$$

ii) Let $\phi : \mathbb{Z}^n \to \Lambda \quad \phi(\bar{a}) = \sum a_i v_i$

$\psi : \mathbb{Z}^n \to \Lambda \quad \psi(\bar{a}) = \sum a_i w_i$

$\phi^{-1} \psi : \mathbb{Z}^n \to \mathbb{Z}^n$ isomorphism

$$\phi^{-1}\psi \in SL_{\pm}(n, \mathbb{Z})$$

$$\rho := \psi\phi^{-1} : v_i \longmapsto w_i$$

$$\det(\psi\phi^{-1}) = \det(\phi^{-1}\psi) = \pm 1$$

- Let v_i^* be $\left(v_1^* \cdots v_n^*\right)^T = \left(v_1 \cdots v_n\right)^{-1}$

$$\langle v_i^*, v_j \rangle = \delta_{ij} \quad \text{(by construction)}$$

Let $\tilde{\Lambda}$ be the lattice gen'd by $\{v_1^*, \cdots, v_n^*\}$. wtp $\tilde{\Lambda} = \Lambda^*$

Let $\xi \in \tilde{\Lambda}$: $\langle \xi, x \rangle = \left\langle \sum_i a_i v_i^*, \sum_j b_j v_j \right\rangle$

$$= \sum_{ij} a_i b_j \langle v_i^*, v_j \rangle = \sum_i a_i b_i \in \mathbb{Z}$$

$$\tilde{\Lambda} \subset \Lambda^*$$

Assme $\xi \in \Lambda^*$, $\xi \notin \tilde{\Lambda}$. Since $\{v_i^*\}$ spanning

$\xi = \sum_i a_i v_i^*$ with $a_i \notin \mathbb{Z}$ for some i. Then

$$\mathbb{Z} \ni \langle \xi, v_i \rangle = \sum_j a_j \langle v_j^*, v_i \rangle$$

$$= a_i \notin \mathbb{Z} \quad \text{xx}$$

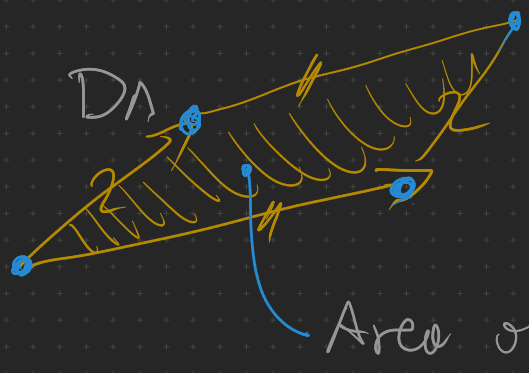
So $\hat{\Lambda} \supset \Lambda^* \Rightarrow \Lambda^*$ is a lattice $\square$

Corollaries: • $\Lambda^{**} = \Lambda$ (from iii)

• $\left| \det \left(v_1 \cdots v_n \right) \right|$ does not depend on the choice of v_i's

$$\text{covol}\,\Lambda := \left| \det \left(v_1 \cdots v_n \right) \right|$$

Example:



$= \Pi_\Lambda$ vol Π_Λ

$=$

Area of $=$ covol Λ

Note: Choosing an orientation for Λ ($\Leftrightarrow \Pi_\Lambda$) fixes the sign of $\det(v_1 \cdots v_n) = \det \phi$

- Harmonic analysis on $\mathbb{T}_\Lambda$

Let us fix Λ, $\rightsquigarrow \Lambda^* \subset \mathbb{R}^n$.

$\forall \xi \in \Lambda^*$ let $e_\xi : \mathbb{R}^n \longrightarrow \mathbb{C}$ defined as

$$\boxed{e_\xi(x) = e^{2\pi i \langle \xi, x \rangle}}$$ Fourier modes.

Prp: i) $\forall \xi \in \Lambda^*$ $e_\xi(x)$ is Λ-periodic

ii) $$\int_{D_\Lambda} e_\xi(x) \overline{e_{\xi'}(x)} \, d\mathrm{Vol} = \begin{cases} \mathrm{Vol}\, D_\Lambda & \text{if } \xi = \xi' \\ 0 & \text{otherwise} \end{cases}$$

iii) $\{e_\xi\}_{\xi \in \Lambda^*}$ spans $L^2(D_\Lambda, \mathbb{C})$

Example: $\Lambda = \mathbb{Z}^2$, then $e^{2\pi i x_1} = e_{e_1}(x_1, x_2)$

$$e_{e_1}(x_1 + a_1, x_2 + a_2) = e_{e_1}(x_1, x_2)$$

$$\forall a = (a_1, a_2) \in \mathbb{Z}^2$$

$$e_{e_1}(x_1 + a_1, x_2 + a_2) = e^{2\pi i (x_1 + a_1)}$$

$$= e^{2\pi i x_1} \underbrace{e^{2\pi i a_1}}_{1} = e_{e_1}(x)$$

$$\shortparallel$$

$$e_{e_1}(x + a)$$

Example: let $n=1$ $\quad \Lambda \subset \mathbb{R}$

$\Lambda = \mathbb{Z}$

$\Lambda^* = \mathbb{Z}$: $\quad e_\xi = e^{2\pi i \xi x}$ Fourier modes

ξ

Pf: i) obvious by def of Λ^*:

$$e_\xi(x+\lambda) = e^{2\pi i \langle \xi, x+\lambda \rangle} = e^{2\pi i \langle \xi, x \rangle} e^{2\pi i \langle \xi, \lambda \rangle}$$

Λ $\qquad$ $\mathbb{Z}$ $\qquad$ $\xi \in \Lambda^*$

$$= e^{2\pi i \langle \xi, x \rangle} = e_\xi(x)$$

ii) $$\int_{D_\Lambda} e_\xi(x) \overline{e_{\xi'}(x)}\, dV =$$

$$= \int_{D_\Lambda} e^{2\pi i \left(\sum_j \xi_j x_j - \sum_j \xi'_j x_j \right)} dV$$

$$= \int_{D_\Lambda} e^{2\pi i \sum_j (\xi_j - \xi'_j) x_j}\, dV$$

$$= \int \cdots \int e^{2\pi i (\xi_1 - \xi'_1) x_1} \cdot e^{2\pi (\xi_2 - \xi'_2) x_2} \cdot$$

$$\cdots \cdot e^{2\pi (\xi_n - \xi'_n) x_n}\, dx_1 \cdots dx_n$$

$$= \int e^{2\pi i (\ ) x_1} \cdot \int \cdots \cdot \int e^{2\pi i (\ ) x_n}$$

$= 0$ if $\xi \neq \xi'$

otherwise $\int_{D_\Lambda} 1 = \text{covol}\,\Lambda$.

iii) Stone Weierstrass implies:

any sub-algebra of continuous functions which contains constants and separates points is dense in the continuous functions.

$$C^0(D_\Lambda)\,,\ \overline{\text{span}_{\mathbb{R}}\{e_\xi\}_{\xi\in\Lambda^*}} = \mathcal{E}$$

- $\xi = 0 \Rightarrow e_0 = 1 \Rightarrow$ constants belong to $\mathcal{E}$
- Recall: $\mathcal{E}$ is said to separate pts if

$$\forall\, x \neq y \in D_\Lambda \quad \exists\, e \in \mathcal{E} \quad \text{s.t.} \quad e(x) \neq e(y)$$

Assume by contradiction $\forall e \in \mathcal{E}\ \ e(x) = e(y)$

$$\text{in particular } \forall \xi \in \Lambda^* \quad e_\xi(x) = e_\xi(y)$$

$$\Leftrightarrow e^{2\pi i \langle \xi, x\rangle} = e^{2\pi i \langle \xi, y\rangle}$$

$$e^{2\pi i \langle \xi, x-y\rangle} = 1$$

$$\Leftrightarrow \langle \xi, x-y\rangle \in \mathbb{Z} \Rightarrow x-y \in \Lambda^{**} = \Lambda$$

$\Leftrightarrow$ $x - y \in \Lambda$ $\forall x$ since $x, y \in D_\Lambda$

- $e_\xi e_{\xi'} = e^{2\pi i (\langle \xi, x \rangle + \langle \xi', x \rangle)}$

$= e^{2\pi i \langle \xi + \xi', x \rangle} = e_{\xi + \xi'}$

(so it is an algebra).

$\Rightarrow$ $\mathcal{E}$ is dense in $C^0(D_\Lambda)$ which is dense in $L^2(D_\Lambda)$ $\square$

Note: $-\Delta e_\xi = -\operatorname{div}\operatorname{grad} e^{2\pi i \langle \xi, x \rangle}$

$= -\sum_j \frac{\partial^2}{\partial x_j^2} e^{2\pi i \sum_i \xi_i x_i}$

$= -(2\pi)^2 (-1) \sum_i \xi_i^2 e_\xi$

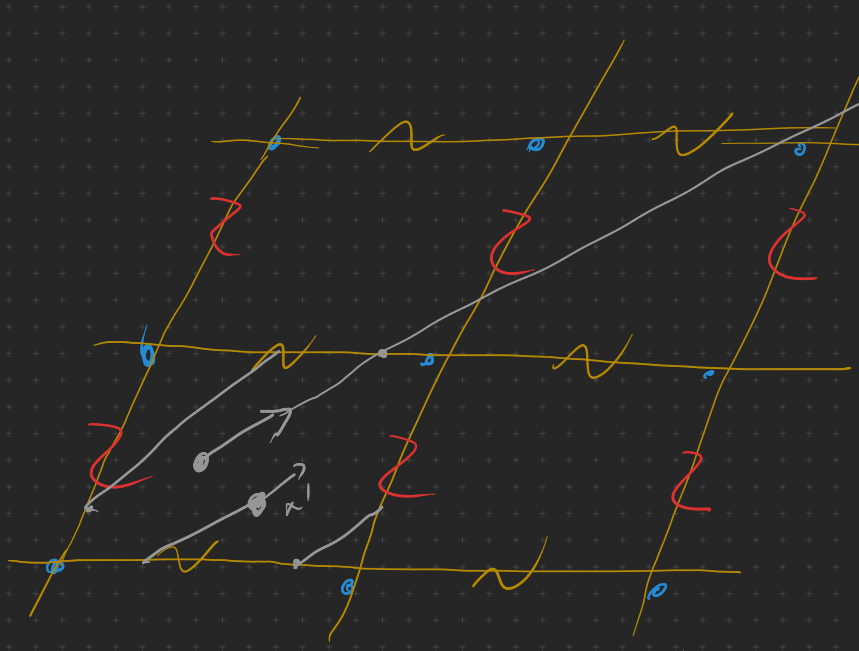
$= (2\pi)^2 \|\xi\|^2 e_\xi$

so e_ξ satisfies $-\Delta u = \lambda u$ $u \in C^2(\pi_\Lambda) \simeq C^2(\mathbb{R}^n)$ Λ-periodic

with $\lambda = (2\pi)^2 \|\xi\|^2 \in Sp_\Delta$

since e_ξ form a basis.

$$Sp_{\Delta_{\pi_\Lambda}} = \{(2\pi)^2 \|\xi\|^2\}_{\xi \in \Lambda^*}$$



$(x,v) \in T^1\Pi_\Lambda$

$\phi_t(x,v) = (x+tv, v)$

(x'', v'')

When is it true that $\phi_t(x,v) = (x,v)$ on Π_Λ?

$\Downarrow$

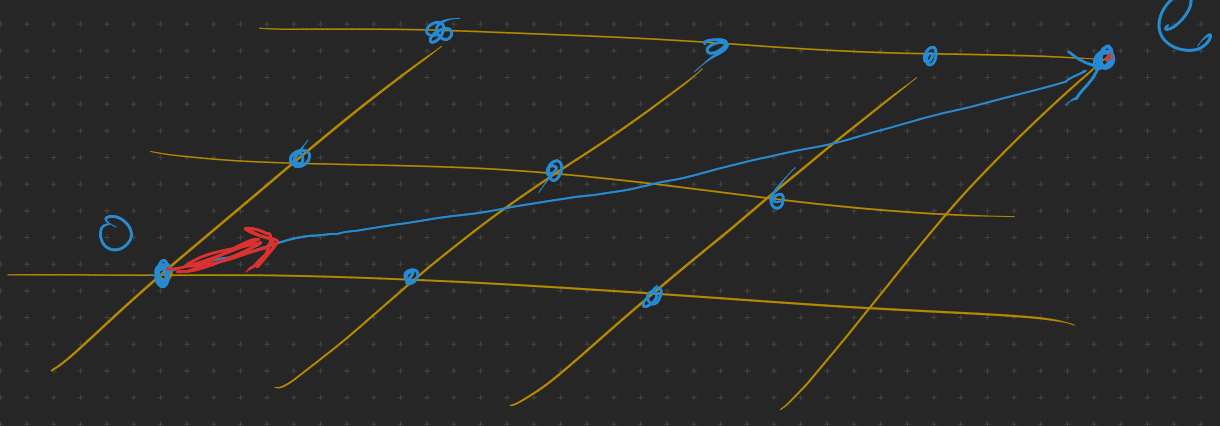
$x' \sim_\Lambda x$ or $x' - x \in \Lambda$

$x + tv - x \in \Lambda$

$tv \in \Lambda$ $\quad T^1M = \left\{ (x,v) \mid \begin{array}{l} x \in M \\ v \in T_xM \\ \|v\| = 1 \end{array} \right\}$

so if $\ell \in \Lambda$, $\ell \neq 0$, then $\left(x, v_\ell = \frac{\ell}{\|\ell\|}\right) \in T^1\Pi_\Lambda$

and $\phi_{\|\ell\|}(x, v_\ell) = (x, v_\ell)$ periodic geodesic



if a closed (periodic) geodesic passes through (x, v) $\exists\, \ell \in \Lambda$ s.t. $v = \frac{\ell}{\|\ell\|}$ of period $\|\ell\|$

$$\boxed{LS(\Pi_\Lambda) = \{ \|\ell\| \mid \ell \in \Lambda \}.}$$

- Poincaré summation.

Example of PS formula:

Recall: $S(\mathbb{R})$ denotes the set of Schwartz functions on $\mathbb{R}$

$$S(\mathbb{R}) \left\{ f \in C^\infty(\mathbb{R}) \,\middle|\, \|x^n D^m f\| < \infty \ \forall n, m \right\}$$

e.g. $\cdot\ C_0^\infty(\mathbb{R}) \subset S(\mathbb{R})$

$\cdot\ e^{-x^2} \in S(\mathbb{R})$

Recall: if $f \in S$

$$\hat{f}(\xi) = \int e^{-i\xi x} f(x)\, dx$$ Fourier transform

$$\check{f}(x) = \frac{1}{2\pi} \int e^{i\xi x} f(\xi) d\xi$$ inverse Fourier transf.

$$\mathcal{F} : f \longmapsto \hat{f} \qquad \mathcal{F}^{-1} : f \longmapsto \check{f}$$

$$\forall f \in S$$

Lemma: $\forall \varphi \in S$; $\forall \omega \in \mathbb{R}_{>0}$ let $T = 2\pi/\omega$

$$\sum_{j \in \mathbb{Z}} \varphi(j\omega) = T \sum_{k \in \mathbb{Z}} \check{\varphi}(kT)$$

Poincaré summation formula in $\mathbb{R}$

Prp: $\forall \varphi \in S(\mathbb{R}^n)$, Λ lattice in $\mathbb{R}^n$

$$\boxed{\sum_{\xi \in \Lambda^*} \varphi(\xi) = \mathrm{Covol}\,\Lambda \sum_{x \in \Lambda} \check{\varphi}(x)}$$

Remark:
Let now consider a function $\varphi \in S$ which is radial (i.e.

$$\varphi(\xi) = \psi(\|\xi\|)$$

Hankel transf.

$$\sum_{\xi \in \Lambda^*} \psi(\|\xi\|) = \operatorname{covol}\Lambda \sum_{x \in \Lambda} \breve{\psi}(\|x\|)$$

$$\|$$

$$\sum_{\lambda \in Sp_\Delta} \psi(\sqrt{\lambda}) = \operatorname{covol}\Lambda \sum_{\ell \in LS} \breve{\psi}(\ell)$$

example: $\varphi(t,\xi) = e^{-\|\xi\|^2 t}$

$$\sum_{\lambda \in Sp_\Delta} e^{-\lambda t} = \operatorname{covol}\Lambda \sum_{\ell \in LS} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\ell}{t}}$$

$\dim \mathbb{T}_n$

$\operatorname{vol} \mathbb{T}_n$

Trace of the Heat operator

$\boxed{\text{Heat trace}}$