ASSIGNMENT 4
DUE THURSDAY FEBRUARY 9

(1) Let G be a finite group. Prove that there exists a Galois extension $F \subset K$ such that $Gal(K/F) = G$. (Note: F and K can depend on G.)

(2) Let p, q be distinct odd prime numbers. Prove that the polynomial $x^4 - px^2 + q \in \mathbb{Q}[x]$ is irreducible and that its Galois group is the dihedral group of order 8.

(3) Let $f(x) \in \mathbb{Q}[x]$ be a degree 5 irreducible polynomial with exactly 3 real roots. Show that the Galois group of $f(x)$ is S_5. (Hint: first show that it contains a transposition, then show that it contains a 5-cycle.)

(4) Let p be an odd prime. Prove that the discriminant of the cyclotomic polynomial $\Phi_p(x)$ is $(-1)^{(p-1)/2}p^{p-2}$. Use this to show that

$$\mathbb{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right) \subset Q(\zeta_p).$$