ASSIGNMENT 6
DUE THURSDAY MARCH 8

(1) Let $f \in k[x]$ and consider $A = k[x, y]/\langle y^2 - f\rangle$. Assume that f is not a square (note that this ensures that A is a domain). Show that A is integrally closed if and only if f has no square factors. If A is not integrally closed, explain how to find its normalization using f.

(2) Let $k \subset L$ be a finite Galois extension and let $G = Gal(L/k)$. Let $I \subset k[x_1, \dots, x_n]$ be a radical ideal. Let $R = k[x_1, \dots, x_n]/I$.

Let $Z(I)_L$ denote the zero set of I inside $\mathbb{A}^n_L$. Prove that G acts on $Z(I)_L$ and prove that the orbits of this action are in bijection with the maximal ideals m of R such that R/m is isomorphic to a subfield of L. (Hint: start with $n = 1$.)

(3) Define X to be the set of pairs of $n \times n$ matrices whose product in either direction is 0, ie.

$$X := \{(A, B) \in Mat_{n,n}(k) \times Mat_{n,n}(k) : AB = BA = 0\}$$

Show that X is an affine variety. Find (without proof) the irreducible components of X. (Hint: start with $n = 1$.)