Bilinear forms and their matrices

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0.1 Definitions

A *bilinear form* on a vector space V over a field $\mathbb{F}$ is a map

$$H : V \times V \to \mathbb{F}$$

such that

(i) $H(v_1 + v_2, w) = H(v_1, w) + H(v_2, w),$ for all $v_1, v_2, w \in V$

(ii) $H(v, w_1 + w_2) = H(v, w_1) + H(v, w_2),$ for all $v, w_1, w_2 \in V$

(iii) $H(av, w) = aH(v, w),$ for all $v, w \in V, a \in \mathbb{F}$

(iv) $H(v, aw) = aH(v, w),$ for all $v, w \in V, a \in \mathbb{F}$

A bilinear form H is called *symmetric* if $H(v, w) = H(w, v)$ for all $v, w \in V$.

A bilinear form H is called *skew-symmetric* if $H(v, w) = -H(w, v)$ for all $v, w \in V$.

A bilinear form H is called *non-degenerate* if for all $v \in V$, there exists $w \in V$, such that $H(w, v) \neq 0$.

A bilinear form H defines a map $H^\# : V \to V^*$ which takes w to the linear map $v \mapsto H(v, w)$. In other words, $H^\#(w)(v) = H(v, w)$.

Note that H is non-degenerate if and only if the map $H^\# : V \to V^*$ is injective. Since V and V^* are finite-dimensional vector spaces of the same dimension, this map is injective if and only if it is invertible.

0.2 Matrices of bilinear forms

If we take $V = \mathbb{F}^n$, then every $n \times n$ matrix A gives rise to a bilinear form by the formula

$$H_A(v, w) = v^t A w$$

Example 0.1. Take $V = \mathbb{R}^2$. Some nice examples of bilinear forms are the ones coming from the matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Conversely, if V is any vector space and if $v_1, \dots, v_n$ is a basis for V, then we define the matrix $[H]_{v_1,\dots,v_n}$ for H with respect to this basis to be the matrix whose i, j entry is $H(v_i, v_j)$.

Proposition 0.2. *Take $V = \mathbb{F}^n$. The matrix for H_A with respect to the standard basis is A itself.*

Proof. By definition,

$$H_A(e_i, e_j) = e_i^t A e_j = A_{ij}.$$

□

Recall that if V is a vector space with basis $v_1, \dots, v_n$, then its dual space V^* has a dual basis $\alpha_1, \dots, \alpha_n$. The element α_j of the dual basis is defined as the unique linear map from V to $\mathbb{F}$ such that

$$\alpha_j(v_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Proposition 0.3. *The matrix for H with respect to $v_1, \dots, v_n$ is the same as the matrix for $v_1, \dots, v_n$ and $\alpha_1, \dots, \alpha_n$ with respect to the map $H^\# : V \to V^*$.*

Proof. Let $A = [H^\#]_{v_1,\dots,v_n}^{\alpha_1,\dots,\alpha_n}$. Then

$$H^\# v_j = \sum_{k=1}^{n} A_{kj} \alpha_k$$

Hence, $H(v_i, v_j) = H^\#(v_j)(v_i) = A_{ij}$ as desired. □

From this proposition, we deduce the following corollary.

Corollary 0.4. *H is non-degenerate if and only if the matrix $[H]_{v_1,\dots,v_n}$ is invertible.*

It is interesting to see how the matrix for a bilinear form changes when we changes the basis.

Theorem 0.5. *Let V be a vector space with two bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$. Let Q be the change of basis matrix. Let H be a bilinear form on V.*

Then

$$Q^t \, [H]_{v_1,\dots,v_n} \, Q = [H]_{w_1,\dots,w_n}$$

Proof. Choosing the basis $v_1, \dots, v_n$ means that we can consider the case where $V = \mathbb{F}^n$, and $v_1, \dots, v_n$ denotes the standard basis. Then $w_1, \dots, w_n$ are the columns of Q and $w_i = Qv_i$.

Let $A = [H]_{v_1,\dots,v_n}$.

So we have

$$H(w_i, w_j) = w_i^t A w_j = (Qv_i)^t A Q v_j = v_i^t Q^t A Q v_j$$

as desired. □

You can think of this operation $A \mapsto Q^t A Q$ as simultaneous row and column operations.

Example 0.6. Consider

$$A = \begin{bmatrix} 0 & 4 \\ 4 & 2 \end{bmatrix}$$

After doing simultaneous row and column operations we reach

$$Q^t A Q = \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix}$$

The new basis is $(1, -2), (0, 1)$.

0.3 Isotropic vectors and perp spaces

A vector v is called *isotropic* if $H(v, v) = 0$.

If H is skew-symmetric, then $H(v, v) = -H(v, v)$, so every vector is isotropic.

Let H be a non-degenerate bilinear form on a vector space V and let $W \subset V$ be a subspace. We define the *perp space* to W as

$$W^\perp = \{v \in V : H(w, v) = 0 \text{ for all } w \in W\}$$

Notice that $W^\perp$ may intersect W. For example if W is the span of a vector v, then $W \subset W^\perp$ if and only if v is isotropic.

Example 0.7. If we take $\mathbb{R}^2$ with the bilinear form $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $(1, 1)$ is an isotropic vector and $span(1, 1)^\perp = span(1, 1)$.

So in general, V is not the direct sum of W and $W^\perp$. However, we have the following result which says that they have complementary dimension.

Proposition 0.8. $\dim W^\perp = \dim V - \dim W$

Proof. We have defined $H^\# : V \to V^*$. The inclusion of $W \subset V$ gives us a surjective linear map $\pi : V^* \to W^*$, and so by composition we get $T = \pi \circ H^\# : V \to W^*$. This map T is surjective since $H^\#$ is an isomorphism. Thus

$$\dim \operatorname{null}(T) = \dim V - \dim W^* = \dim V - \dim W$$

Checking through the definitions, we see that

$$v \in \operatorname{null}(T) \text{ if and only if } H^\#(v)(w) = 0 \text{ for all } w \in W$$

Since $H^\#(v)(w) = H(w, v)$, this shows that $v \in \operatorname{null}(T)$ if and only if $v \in W^\perp$. Thus $W^\perp = \operatorname{null}(T)$ and so the result follows. □