Symplectic forms

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March 24, 2011

1 Symplectic forms

We assume that the characteristic of our field is not 2 (so $1 + 1 \neq 0$).

1.1 Definition and examples

Recall that a *skew-symmetric bilinear form* is a bilinear form Ω such that $\Omega(v, w) = -\Omega(w, v)$ for all $v, w \in V$. Note that if Ω is a skew-symmetric bilinear form, then $\Omega(v, v) = 0$ for all $v \in V$. In other words, every vector is isotropic.

A *symplectic form* is a non-degenerate skew-symmetric bilinear form. Recall that non-degenerate means that for all $v \in V$ such that $v \neq 0$, there exists $w \in V$ such that $\Omega(v, w) \neq 0$.

Example 1.1. Consider $V = \mathbb{F}^2$ and take the bilinear form given by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Here is a more general example. Let W be a vector space. Define a vector space $V = W \oplus W^*$. We define a bilinear form on W by the rule

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2)$$

This is clearly skew-symmetric. It is also non-degenerate since if $(v_1, \alpha_1) \in V$ is non-zero, then either $v_1 \neq 0$ or $\alpha_1 \neq 0$. Assume that $v_1 \neq 0$. Then we can choose $\alpha_2 \in V^*$ such that $\alpha_2(v_1) \neq 0$ and so $\Omega((v_1, \alpha_1), (0, \alpha_2)) \neq 0$. So Ω is a symplectic form.

1.2 Symplectic bases

We cannot hope to diagonalize a symplectic form since every vector is isotropic. We will instead introduce a different goal.

Let V, Ω be a vector space with a symplectic form. Suppose that $\dim V = 2n$. A *symplectic basis* for V is a basis $q_1, \dots, q_n, p_1, \dots, p_n$ for V such that

$$\begin{aligned}
\Omega(p_i, q_i) &= 1 \\
\Omega(q_i, p_i) &= -1 \\
\Omega(p_i, q_j) &= 0 \text{ if } i \neq j \\
\Omega(p_i, p_j) &= 0 \\
\Omega(q_i, q_j) &= 0
\end{aligned}$$

In other words the matrix for Ω with respect to this basis is the $2n \times 2n$ matrix

$$\begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

Theorem 1.2. *Let V be a vector space and let Ω be a symplectic form. Then $\dim V$ is even and there exists a symplectic basis for V.*

Proof. We proceed by induction on $\dim V$. The base cases are $\dim V = 0$ and $\dim V = 1$.

Now assume $\dim V = n$ and assume the result holds for all vector spaces of dimension $n - 2$. Let $q \in V$, $q \neq 0$. Since Ω is non-degenerate, there exists $p \in V$ such that $\Omega(p, q) \neq 0$. By scaling q, we can ensure that $\Omega(p, q) = 1$. Then $\Omega(q, p) = -1$ by skew symmetry.

Let $W = span(p, q)^{\perp}$. So

$$W = \{v \in V : \Omega(v, p) = 0 \text{ and } \Omega(v, q) = 0\}.$$

We claim that $W \cap span(p, q) = 0$. Let $v \in W \cap span(p, q)$. Then $v = ap + bq$ for some $a, b \in \mathbb{F}$. Since $v \in W$, we see that $\Omega(v, p) = 0$. But $\Omega(v, p) = -b$, so $b = 0$. Similarly, $\Omega(v, q) = 0$, which implies that $a = 0$. Hence $v = 0$.

Since Ω is non-degenerate, we know that $\dim W + \dim span(p, q) = \dim V$. Thus $W \oplus span(p, q) = V$.

To apply the inductive hypothesis, we need to check now that the restriction of Ω to W is a symplectic form. It is clearly skew-symmetric, so we just need to check that it is non-degenerate. To see this, pick $w \in W$, $w \neq 0$. Then there exists $v \in V$ such that $\Omega(w, v) \neq 0$. We can write $w = u + u'$, where $u \in W$ and $u' \in span(p, q)$. By the definition of $span(p, q)$, $\Omega(u', w) = 0$. Hence $\Omega(u, w) \neq 0$ and so the restriction of Ω to W is non-degenerate.

We now apply the inductive hypothesis to W. Note that $\dim W = \dim V - 2$. By the inductive hypothesis, $\dim W$ is even and we have a symplectic basis $q_1, \dots, q_m, p_1, \dots p_m$ where $2m = n - 2$. Hence $\dim V$ is even. We claim that $q_1, \dots, q_m, q, p_1, \dots, p_m, p$ is a symplectic basis for W. This follows from the definitions. $\square$

1.3 Lagrangians

Let V be a vector space of dimension $2n$ and Ω be a symplectic form on V. Recall that a subspace W of V is called *isotropic* if $\Omega(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. This is equivalent to the condition that $W \subset W^{\perp}$. Since Ω is non-degenerate, $\dim W^{\perp} + \dim W = \dim V$. Hence the maximum possible dimension of an isotropic subspace is n. An isotropic subspace L of dimension n is called a *Lagrangian*.

Example 1.3. Let V be a 2 dimensional vector space. Then any 1-dimensional subspace is Lagrangian.

We can also produce Lagrangian subspaces from symplectic bases as follows.

Proposition 1.4. *Let $q_1, \dots, q_n, p_1, \dots, p_n$ be a symplectic basis for V. Then $span(q_1, \dots, q_n), span(p_1, \dots, p_n)$ are both Lagrangian subspaces of V.*

Proof. From the fact that $\Omega(q_i, q_j) = 0$ for all i, j, we see that $\Omega(v, w) = 0$ for all $v, w \in span(q_1, \dots, q_n)$. Hence $span(q_1, \dots, q_n)$ is isotropic. Since it has dimension n, it is Lagrangian. The result for $p_1, \dots, p_n$ is similar. □

Now suppose that $V = W \oplus W^*$ for some vector space W and we define a symplectic form Ω on W as above. Then it is easy to see that W and W^* are both Lagrangian subspaces of V.