MAT 247
ASSIGNMENT 1
DUE THURSDAY JANUARY 20

Throughout the assignment, use the convention that $\langle av, w\rangle = \bar{a}\langle v, w\rangle$ and $\langle v, aw\rangle = a\langle v, w\rangle$.

(1) Let V an inner product space and let $v, w \in V$. Prove that $\langle v, w\rangle = 0$ if and only if $||v|| \leq ||v + aw||$ for all $a \in \mathbb{F}$ (here $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$).

(2) On $\mathcal{P}_2(\mathbb{R})$ (the vector space of polynomials of degree less than or equal to 2), consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx.$$

Apply the Gram-Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

(3) What happens if the Gram-Schmidt procedure is applied to a list of vectors that is not linearly independent?

(4) Let $V, \langle , \rangle$ be a finite dimensional real inner product space.

(a) For each $v \in V$, define a map $T_v : V \to \mathbb{R}$ by

$$T_v(w) = \langle w, v\rangle.$$

Show that for each $v \in V$, T_v is a linear map.

(b) Now define a map $F : V \to V^*$ by $F(v) = T_v$. Here $V^* = L(V, \mathbb{R})$ denotes the dual space of V. Show that F is a linear map.

(c) Prove that F is invertible. (First prove that it is injective and then use that $\dim V = \dim V^*$).

(d) What happens if $V, \langle , \rangle$ is a complex inner product space?