MAT 247
ASSIGNMENT 6
DUE THURSDAY MARCH 10

(1) (Axler 6.29) Let T be a linear operator on an inner product space. Prove that the dimension of the range of T equals the number of non-zero singular values of T.

(2) (Axler 6.33) Let T be a linear operator on an inner product space V. Let $\hat{s}$ denote the smallest singular value of T and let s denote the largest singular value of T. Show that for all $v \in V$,

$$\hat{s}||v|| \leq ||Tv|| \leq s||v||.$$

(3) (a) Let H_1, H_2 be two bilinear forms on a vector space V. Define a map $H_1 + H_2 : V \times V \to \mathbb{F}$ by

$$(H_1 + H_2)(v, w) = H_1(v, w) + H_2(v, w).$$

Show that $H_1 + H_2$ is a bilinear form.

(b) Let H be a bilinear form on V and let $a \in \mathbb{F}$. Define $aH : V \to \mathbb{F}$ by

$$(aH)(v, w) = aH(v, w).$$

Show that aH is a bilinear form.

Thus the set of all bilinear forms on a vector space forms a vector space.

(4) Let $V = M_2(\mathbb{F})$ be the vector space of 2×2 matrices over the field $\mathbb{F}$. Define a bilinear form H on V by

$$H(A, B) = \text{trace}(AB).$$

Compute the matrix for H with respect to the usual basis for $M_2(\mathbb{F})$. Find a basis for $M_2(\mathbb{F})$ for which matrix of H is a diagonal matrix. (You may assume that $1 + 1 \neq 0$ in $\mathbb{F}$.)