Multilinear forms

Joel Kamnitzer

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Assume that all fields are characteristic 0 (i.e. $1 + \cdots + 1 \neq 0$), for example $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Assume also that all vector spaces are finite dimensional.

1 Dual spaces

If V is a vector space, then $V^* = L(V, \mathbb{F})$ is defined to be the space of linear maps from V to $\mathbb{F}$.

If $v_1, \dots, v_n$ is a basis for V, then we define $\alpha_i \in V^*$ for $i = 1, \dots, n$, by setting

$$\alpha_i(v_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{otherwise} \end{cases}$$

Proposition 1.1. *$\alpha_1, \dots, \alpha_n$ forms a basis for V^* (called the dual basis).*

In particular, this shows that V and V^* are vector spaces of the same dimension. However, there is no natural way to choose an isomorphism between them, unless we pick some additional structure on V (such as a basis or a non-degenerate bilinear form).

On the other hand, we can construct an isomorphism ψ from V to $(V^*)^*$ as follows. If $v \in V$, then we define $\psi(v)$ to be the element of V^* given by

$$(\psi(v))(\alpha) = \alpha(v)$$

for all $\alpha \in V^*$. In other words, given a guy in V, we tell him to eat elements in V^* by allowing himself to be eaten.

Proposition 1.2. *ψ is an isomorphism.*

Proof. Since V and $(V^*)^*$ have the same dimension, it is enough to show that ψ is injective.

Suppose that $v \in V$, $v \neq 0$, and $\psi(v) = 0$. We wish to derive a contradiction.

Since $v \neq 0$, we can extend v to a basis $v_1 = v, v_2, \dots, v_n$ for V. Then let α_1 defined as above. Then $\alpha_1(v) = 1 \neq 0$ and so we have a contradiction. Thus ψ is injective as desired. □

From this proposition, we derive the following useful result.

Corollary 1.3. *Let $\alpha_1, \ldots, \alpha_n$ be a basis for V^*. Then there exists a basis $v_1, \ldots, v_n$ for V such that*

$$\alpha_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

for all i, j.

Proof. Let $w_1, \ldots, w_n$ be the dual basis to $\alpha_1, \ldots, \alpha_n$ in $(V^*)^*$. Since ψ is invertible, ψ^{-1} exists. Define $v_i = \psi^{-1}(w_i)$. Since $w_1, \ldots, w_n$ is a basis, so is $v_1, \ldots, v_n$. Checking through the definitions shows that $v_1, \ldots, v_n$ have the desired properties. □

2 Bilinear forms

Let V be a vector space. We denote the set of all bilinear forms on V by $(V^*)^{\otimes 2}$. We have already seen that this set is a vector space.

Similarly, we have the subspaces $Sym^2 V^*$ and $\Lambda^2 V^*$ of symmetric and skew-symmetric bilinear forms.

Proposition 2.1. $(V^*)^{\otimes 2} = Sym^2 V^* \oplus \Lambda^2 V^*$

Proof. Clearly, $Sym^2 V^* \cap \Lambda^2 V^* = 0$, so it suffices to show that any bilinear form is the sum of a symmetric and skew-symmetric bilinear form. Let H be a bilinear form. Let $\hat{H}$ be the bilinear form defined by

$$\hat{H}(v_1, v_2) = H(v_2, v_1)$$

Then $(H + \hat{H})/2$ is symmetric and $(H - \hat{H})/2$ is skew-symmetric. Hence $H = (H+\hat{H})/2 + (H-\hat{H})/2$ is the sum of a symmetric and skew-symmetric form. □

If $\alpha, \beta \in V^*$, then we can define a bilinear form $\alpha \otimes \beta$ as follows.

$$(\alpha \otimes \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2)$$

for $v_1, v_2 \in V$.

We can also define a symmetric bilinear form $\alpha \cdot \beta$ by

$$(\alpha \cdot \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) + \alpha(v_2)\beta(v_1)$$

and a skew-symmetric bilinear from $\alpha \wedge \beta$ by

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

These operations are linear in each variable. In other words

$$\alpha \otimes (\beta + \gamma) = \alpha \otimes \beta + \alpha \otimes \gamma$$

and similar for the other operations.

Example 2.2. Take $V = \mathbb{R}^2$. Let α_1, α_2 be the standard dual basis for V^*, so that

$$\alpha_1(x_1, x_2) = x_1, \ \ \alpha_2(x_1, x_2) = x_2$$

Then $\alpha_1 \otimes \alpha_2$ is given by

$$(\alpha_1 \otimes \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2$$

Similarly $\alpha_1 \wedge \alpha_2$ is the standard symplectic form on $\mathbb{R}^2$, given by

$$(\alpha_1 \wedge \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$$

$\alpha_1 \cdot \alpha_2$ is the symmetric bilinear form of signature $(1, 1)$ on $\mathbb{R}^2$ given by

$$(\alpha_1 \cdot \alpha_2)((x_1, x_2), (y_1, y_2)) = x_1 y_2 + x_2 y_1$$

The standard positive definite bilinear form on $\mathbb{R}^2$ (the dot product) is given by $\alpha_1 \cdot \alpha_1 + \alpha_2 \cdot \alpha_2$.

3 Multilinear forms

Let V be a vector space.

We can consider k-forms on V, which are maps

$$H : V \times \cdots \times V \to \mathbb{F}$$

which are linear in each argument. In other words

$$\begin{aligned} H(av_1, \dots, v_k) &= aH(v_1, \dots, v_k) \\ H(v + w, v_2, \dots, v_k) &= H(v, v_2, \dots, v_k) + H(w, v_2, \dots, v_k) \end{aligned}$$

for $a \in \mathbb{F}$ and $v, w, v_1, \dots, v_k \in V$, and similarly in all other arguments.

H is called symmetric if for each i, and all $v_1, \dots, v_k$,

$$H(v_1, \dots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \dots, v_n) = H(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n)$$

H is called skew-symmetric (or alternating) if for each i, and all $v_1, \dots, v_k$,

$$H(v_1, \dots, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \dots, v_n) = -H(v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n)$$

The vector space of all k-forms is denoted $(V^*)^{\otimes k}$ and the subspaces of symmetric and skew-symmetric forms are denote $Sym^k V^*$ and $\Lambda^k V^*$.

3.1 Permutations

Let S_k denote the set of bijections from $\{1, \dots, k\}$ to itself (called a permutation). S_k is also called the symmetric group. It has $k!$ elements. The permutations occuring in the definition of symmetric and skew-symmetric forms are called simple transpositions (they just swap i and $i+1$). Every permutation can be written as a composition of simple transpositions.

From this it immediately follows that if H is symmetric and if $\sigma \in S_k$, then

$$H(v_1, \ldots, v_k) = H(v_{\sigma(1)}, \cdots, v_{\sigma(n)})$$

There is a function $\varepsilon : S_k \to \{1, -1\}$ called the sign of a permutation, which is defined by the conditions that $\varepsilon(\sigma) = -1$ if σ is a simple transposition and

$$\varepsilon(\sigma_1 \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$$

for all $\sigma_1, \sigma_2 \in S_k$.

The sign of a permutation gives us the behaviour of skew-symmetric k-forms under permuting the arguments. If H is skew-symmetric and if $\sigma \in S_k$, then

$$H(v_1, \ldots, v_k) = \varepsilon(\sigma) H(v_{\sigma(1)}, \cdots, v_{\sigma(n)})$$

3.2 Iterated tensors, dots, and wedges

If H is a $k-1$-form and $\alpha \in V^*$, then we define $H \otimes \alpha$ to be the k-form defined by

$$(H \otimes \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k)$$

Similarly, if H is a symmetric $k-1$-form and $\alpha \in V^*$, then we define $H \cdot \alpha$ to be the k-form defined by

$$(H \otimes \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k) + \cdots + H(v_2, \ldots, v_k)\alpha(v_1)$$

It is easy to see that $H \cdot \alpha$ is a symmetric k-form.

Similarly, if H is a skew-symmetric $k-1$-form and $\alpha \in V^*$, then we define $H \wedge \alpha$ to be the k-form defined by

$$(H \otimes \alpha)(v_1, \ldots, v_k) = H(v_1, \ldots, v_{k-1})\alpha(v_k) - \cdots \pm H(v_2, \ldots, v_k)\alpha(v_1)$$

It is easy to see that $H \wedge \alpha$ is a skew-symmetric k-form.

From these definitions, we see that if $\alpha_1, \ldots, \alpha_k \in V^*$, then we can iteratively define

$$\alpha_1 \otimes \cdots \otimes \alpha_k := ((\alpha_1 \otimes \alpha_2) \otimes \alpha_3) \otimes \cdots \otimes \alpha_k$$

and similar definitions for $\alpha_1 \cdots \alpha_k$ and $\alpha_1 \wedge \cdots \wedge \alpha_k$.

When we expand out the definitions of $\alpha_1 \cdots \alpha_k$ and $\alpha_1 \wedge \cdots \wedge \alpha_k$ there will be $k!$ terms, one for each element of S_k.

For any $\sigma \in S_k$, we have

$$\alpha_1 \cdots \alpha_k = \alpha_{\sigma(1)} \cdots \alpha_{\sigma(k)}$$

and

$$\alpha_1 \wedge \cdots \wedge \alpha_k = \varepsilon(\sigma) \alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)}$$

The later property implies that $\alpha_1 \wedge \cdots \wedge \alpha_k = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$.

The following result is helpful in understanding these iterated wedges.

Theorem 3.1. *Let* $\alpha_1, \dots, \alpha_k \in V^*$.

$$\alpha_1 \wedge \cdots \wedge \alpha_k = 0 \text{ if and only if } \alpha_1, \dots, \alpha_k \text{ are linearly dependent}$$

Proof. Suppose that $\alpha_1, \dots, \alpha_k$ is linearly dependent. Then there exists $x_1, \dots, x_k$ such that

$$x_1\alpha_1 + \cdots + x_k\alpha_k = 0$$

and not all $x_1, \dots, x_k$ are zero. Assume that $x_k \neq 0$. Let $H = \alpha_1 \wedge \cdots \wedge \alpha_{k-1}$ and let us apply $H\wedge$ to both sides of this equation. Using the above results and the linearity of $\wedge$, we deduce that

$$x_k\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \alpha_k = 0$$

which implies that $\alpha_1 \wedge \cdots \wedge \alpha_k = 0$ as desired.

For the converse, suppose that $\alpha_1, \dots, \alpha_k$ are linearly independent. Then we can extend $\alpha_1, \dots, \alpha_k$ to a basis $\alpha_1, \dots, \alpha_n$ for V^*. Let $v_1, \dots, v_n$ be the dual basis for V. Then

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \dots, v_k) = 1$$

and so $\alpha_1 \wedge \cdots \wedge \alpha_k \neq 0$. □

The same method of proof can be used to prove the following result.

Theorem 3.2. *Let* $v_1, \dots, v_k \in V$. *Then there exists* $H \in \Lambda^k V^*$ *such that* $H(v_1, \dots, v_k) \neq 0$ *if and only if* $v_1, \dots, v_k$ *are linearly independent.*

In particular this theorem shows that $\Lambda^k V^* = 0$ if $k > dimV$.

3.3 Bases and dimension

We will now describe bases for our vector spaces of k-forms.

Theorem 3.3. *Let* $\alpha_1, \dots, \alpha_n$ *be a basis for* V^*.

(i) $\{\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_k}\}_{1 \leq i_1, \dots, i_k \leq n}$ *is a basis for* $(V^*)^{\otimes k}$.

(ii) $\{\alpha_{i_1} \cdots \alpha_{i_k}\}_{1 \leq i_1 \leq \cdots \leq i_k \leq n}$ *is a basis for* $Sym^k V^*$.

(iii) $\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq n}$ *is a basis for* $\Lambda^k V^*$.

Proof. We give the proof for the case of $(V^*)^{\otimes k}$ as the other cases are similar. So simplify the notation, let us assume that $k = 2$.

Let us first show that every bilinear form is a linear combination of $\{\alpha_i \otimes \alpha_j\}$. Let H be a bilinear form. Let $v_1, \dots, v_n$ be the basis of V dual to $\alpha_1, \dots, \alpha_n$. Let $c_{ij} = H(v_i, v_j)$ for each i, j. We claim that

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\alpha_i \otimes \alpha_j$$

Since both sides are bilinear forms, it suffices to check that they agree on all pairs (v_k, v_l) of basis vectors. By definition $H(v_k, v_l) = c_{kl}$. On the other hand,

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\alpha_i \otimes \alpha_j\right)(v_k, v_l) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\alpha_i(v_k)\alpha_j(v_l) = c_{kl}$$

and so the claim follows.

Now to see that $\{\alpha_i \otimes \alpha_j\}$ is a linearly independent set, just note that if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\alpha_i \otimes \alpha_j = 0,$$

then by evaluating both sides on (v_i, v_j), we see that $c_{ij} = 0$ for all i, j. $\square$

Example 3.4. Take $n = 2, k = 2$. Then our bases are

$$\alpha_1 \otimes \alpha_1, \alpha_1 \otimes \alpha_2, \alpha_2 \otimes \alpha_1, \alpha_2 \otimes \alpha_2$$

and

$$\alpha_1 \cdot \alpha_1, \alpha_1 \cdot \alpha_2, \alpha_2 \cdot \alpha_2$$

and

$$\alpha_1 \wedge \alpha_2$$

Corollary 3.5. *The dimension of $(V^*)^{\otimes k}$ is n^k, the dimension of $Sym^k V^*$ is $\binom{n+k-1}{k}$ and the dimension of $\Lambda^k V^*$ is $\binom{n}{k}$.*