Tensor products

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1 The definition

Let V, W, X be three vector spaces. A *bilinear map* from $V \times W$ to X is a function $H : V \times W \to X$ such that

$$H(av_1 + v_2, w) = aH(v_1, w) + H(v_2, w) \text{ for } v_1, v_2 \in V, w \in W, a \in \mathbb{F}$$
$$H(v, aw_1 + w_2) = aH(v, w_1) + H(v, w_2) \text{ for } v \in V, w_1, w_2 \in W, a \in \mathbb{F}$$

Let V and W be vector spaces. A *tensor product* of V and W is a vector space $V \otimes W$ along with a bilinear map $\phi : V \times W \to V \otimes W$, such that for every vector space X and every bilinear map $H : V \times W \to X$, there exists a unique linear map $T : V \otimes W \to X$ such that $H = T \circ \phi$.

In other words, giving a linear map from $V \otimes W$ to X is the same thing as giving a bilinear map from $V \times W$ to X.

If $V \otimes W$ is a tensor product, then we write $v \otimes w := \phi(v \otimes w)$. Note that there are two pieces of data in a tensor product: a vector space $V \otimes W$ and a bilinear map $\phi : V \times W \to V \otimes W$.

Here are the main results about tensor products summarized in one theorem.

Theorem 1.1. *(i) Any two tensor products of V, W are isomorphic.*

(ii) V, W has a tensor product.

(iii) If $v_1, \dots, v_n$ is a basis for V and $w_1, \dots, w_m$ is a basis for W, then

$$\{v_i \otimes w_j\}_{1 \le i \le n, 1 \le j \le m}$$

is a basis for $V \otimes W$.

In particular, the last part of the theorem shows we can think of elements of $V \otimes W$ as $n \times m$ matrices with entries in $\mathbb{F}$.

2 Existence

We will start by proving that the tensor product exists. To do so, we will construct an explicit tensor product. This construction only works if V, W are finite-dimensional.

Let $B(V^*, W^*; \mathbb{F})$ be the vector space of bilinear maps $H : V^* \times W^* \to \mathbb{F}$. If $v \in V$ and $w \in W$, then we can define a bilinear map $v \otimes w$ by

$$(v \otimes w)(\alpha, \beta) = \alpha(v)\beta(w).$$

Just as we saw before, we have the following result.

Theorem 2.1. *Let $v_1, \ldots, v_n$ be a basis for V and let $w_1, \ldots, w_m$ be a basis for W. Then $\{v_i \otimes w_j\}_{1 \le i \le n, 1 \le j \le m}$ is a basis for $B(V^*, W^*; \mathbb{F})$*

Now, we define a map $\phi : V \times W \to B(V^*, W^*; \mathbb{F})$ by $\phi(v, w) = v \otimes w$.

Theorem 2.2. *$B(V^*, W^*; \mathbb{F})$ along with ϕ is a tensor product for V, W.*

Note that this proves parts (ii) and (iii) of our main theorem.

Proof. Fix bases $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W.

Let X be a vector space and let $H : V \times W \to X$ be a bilinear map. We define a linear map $T : V \otimes W \to X$ by defining it on our basis as $T(v_i \otimes w_j) = H(v_i, w_j)$. Then $T \circ \phi$ and H are two bilinear maps from $V \times W$ to X which agree on basis vectors, hence they are equal. (Note that it is easy to show that for any $(v, w) \in V \times W$, $T(v \otimes w) = H(v, w)$.)

Finally, note that T is the unique linear map with this property, since it is determined on the basis for $B(V^*, W^*, \mathbb{F})$. □

Using the same ideas, it is easy to see that $L(V^*, W)$, $L(W^*, V)$, and $B(V, W; \mathbb{F})^*$ are all also tensor products of V, W.

3 Uniqueness

Now we prove uniqueness. Here is the precise statement.

Theorem 3.1. *Let $(V \otimes W)_1, \phi_1$ and $(V \otimes W)_2, \phi_2$ be two tensor products of V, W. Then there exists a unique isomorphism $T : (V \otimes W)_1 \to (V \otimes W)_2$ such that $\phi_2 = T \circ \phi_1$.*

Proof. Let us apply the definition of tensor product to $(V \otimes W)_1, \phi_1$ with the role of X, H taken by $(V \otimes W)_2, \phi_2$. By the definition, we obtain a (unique) linear map $T : (V \otimes W)_1 \to (V \otimes W)_2$ such that $\phi_2 = T \circ \phi_1$.

Reversing the roles of $(V \otimes W)_1, \phi_1$ and $(V \otimes W)_2, \phi_2$, we find a linear map $S : (V \otimes W)_2 \to (V \otimes W)_1$ such that $\phi_1 = S \circ \phi_2$.

We claim that $T \circ S = I_{(V \otimes W)_2}$ and $S \circ T = I_{(V \otimes W)_1}$ and hence T is an isomorphism. We will now prove $S \circ T = I_{(V \otimes W)_1}$.

Note that $(S \circ T) \circ \phi_1 = S \circ \phi_2 = \phi_1$ by the above equations. Now, apply the definition of tensor product to $(V \otimes W)_1, \phi_1$ with the role of X, H taken by $(V \otimes W)_1, \phi_1$. Then both $S \circ T$ and $I_{(V \otimes W)_1}$ can play the role of T. So by the uniqueness of "T" in the definition, we conclude that $S \circ T = I_{(V \otimes W)_1}$ as desired.

Finally to see that the T that appears in the statement of the theorem is unique, we just note from the first paragraph of this proof, it follows that there is only one linear map T such that $\phi_2 = T \circ \phi_1$. $\square$