Symmetric Bilinear Forms: Definitions

Eric Bannatyne

April 18, 2014

Definition 1. If V_1, V_2, W are vector spaces, a map $B : V_1 \times V_2 \to W$ is bilinear if, for every $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$, $a \in \mathbb{F}$,

1. $B(av_1, v_2) = aB(v_1, v_2) = B(v_1, av_2)$,
2. $B(v_1 + v_1', v_2) = B(v_1, v_2) + B(v_1', v_2)$, and
3. $B(v_1, v_2 + v_2') = B(v_1, v_2) + B(v_1, v_2')$.

Definition 2. A bilinear pairing is a map $B : V \times W \to \mathbb{F}$ that is bilinear.

Definition 3. A bilinear pairing $B : V \times W \to F$ is non-degenerate if

1. for every nonzero $v \in V$, there is some $w \in W$ such that $B(v, w) \neq 0$, and
2. for every nonzero $w \in W$, there is some $v \in V$ such that $B(v, w) \neq 0$.

Definition 4. If $B : V \times W \to \mathbb{F}$ is a bilinear pairing, we have two maps

1. $\tilde{B} : V \to W^*$ defined by $\tilde{B}(v)(w) = B(v, w)$, and
2. $\tilde{B} : W \to V^*$ defined by $\tilde{B}(w)(v) = B(v, w)$.

Definition 5. A bilinear form is a bilinear pairing $H : V \times V \to \mathbb{F}$.

Definition 6. A bilinear form H is symmetric if $H(v, w) = H(w, v)$ for all $v, w \in V$.

Definition 7. Given a symmetric bilinear form H on V, we can define a quadratic form $Q : V \to \mathbb{F}$ by $Q(v) = B(v, v)$ for all $v \in V$.

Definition 8. If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and H is a bilinear form on V, the matrix of H with respect to β is

$$\begin{bmatrix} H(v_1, v_1) & \cdots & H(v_1, v_n) \\ \vdots & \ddots & \vdots \\ H(v_n, v_1) & \cdots & H(v_n, v_n) \end{bmatrix}.$$

Definition 9. A matrix A is symmetric if $A^t = A$.

Definition 10. Two matrices A and B are congruent if there is an invertible matrix P such that $P^t AP = B$.

Definition 11. If H is a symmetric bilinear form over a vector space V, then a vector $v \in V$ is a null vector if $H(v, v) = 0$.

Definition 12. If $W \subseteq V$ is a subspace of V, then $W^{\perp_H} = \{v \in V : H(v, w) = 0 \text{ for all } w \in W\}$.

Definition 13. If V is a vector space over $\mathbb{R}$, and H is a symmetric bilinear form over V, then the signature of H is (p, q) where p and q are the numbers of 1's and -1's respectively in the diagonal form of H derived from Sylvester's law of inertia.

Definition 14. Let H be a symmetric bilinear form over a real vector space. H is positive-definite if $H(v, v) > 0$ for all nonzero $v \in V$. Similarly, H is negative-definite if $H(v, v) < 0$ for all nonzero $v \in V$.

Definition 15. Let H be a symmetric bilinear form on V. The radical of H is

$$\text{rad}(H) = \{v \in V : H(v, w) = 0 \text{ for all } w \in V\}.$$

Definition 16. The rank of a symmetric bilinear form H over V is

$$\text{rank}(H) = \dim(V) - \dim \text{rad}(H).$$

Definition 17. Let Q be a non-degenerate quadratic form on $\mathbb{R}^3$. Given any $c \in \mathbb{R}$, the set of solutions to the equation $Q(x, y, z) = c$ is a non-degenerate quadric surface.