MAT 247, Winter 2014
Assignment 2
Due Jan 21

January 13, 2014

1. This exercise concerns the matrix expontential. We work with $\mathbb{F} = \mathbb{R}$. Let A be a square matrix, we define
$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$
You may assume that e^A converges (in some appropriate sense) and that $e^{A+B} = e^A e^B$ whenever A, B commute.

   (a) Let
$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$
   Compute e^A.

   (b) Let A be an $n \times n$ matrix and let $v \in \mathbb{R}^n$. Consider the differential equation $f'(x) = Af(x)$ with initial condition $f(0) = v$, where $f : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function. Prove that this differential equation has solution $f(x) = e^{xA}v$. (You may use that $\frac{d}{dx}e^{xA} = Ae^{xA}$.)

   (c) Solve the system of differential equations
$$f_1'(x) = 2f_1(x) + f_2(x) \quad f_2'(x) = 2f_2(x)$$
   with initial condition $f_1(0) = a$, $f_2(0) = b$.

2. Show that the two matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   are not similar. Do not use any theorems from class which we have not proved. (Hint: consider the null space of powers of these matrices.)

3. Let $\mathbb{F} = \mathbb{R}$. Consider the vector space V of polynomials of degree at most 3. Let $T : V \to V$ be the linear operator of differentiation. Find a basis β for V for which $[T]_\beta$ is a Jordan form matrix.

4. Let $\mathbb{F} = \mathbb{R}$. Consider the vector space $V = \{p(x)e^x\}$ such that $p(x)$ is a polynomial of degree at most 3. Let $T : V \to V$ be the linear operator of differentiation. Find a basis β for V for which $[T]_\beta$ is a Jordan form matrix.

5. Find two 5×5 Jordan form matrices which are not similar but which both have minimal polynomial $(x-2)^2(x-3)$.