TORONTO REPRESENTATION THEORY RETREAT, JULY 2013

1. Quantum groups and crystals

- Definition of $U_q(\mathfrak{g})$ (don't spend too much time on this; perhaps give the example of $U_q(\mathfrak{sl}_3)$).
- Definition of a crystal basis and of the corresponding crystal for a representation $V(\lambda)$ of $U_q(\mathfrak{g})$.
- The crystal $B(-\infty)$ as a limit of $B(\lambda)$ for large λ, or as the crystal for $U^-(\mathfrak{g})$.
- Combinatorial models for crystals (maybe just by example in $\mathfrak{sl}_n$).
- Some reasons one might care about crystals: relate the character formula to generating functions; tensor product rule for crystals allows one to understand tensor product multiplicities for the representations; globalization and relationship with canonical bases.

The purpose is to explain what crystals are and why they are interesting. For what they are, I think it would be good to discuss the definition in terms of the Kashiwara operators $\widetilde{F}_i$, and actually define crystal lattice and crystal basis. Probably you should also draw some pictures. For "why they are interesting," it is up to you exactly what you cover; the exact points listed above won't be too crucial later on, but you should say something about some of them. Or, if you find other nice facts about crystals, that is fine as well!

Either Kashiwara's survey paper [Kas] or Hong and Kang's book [HK] should be a good reference. Lectures 2 and 6 of my notes [Notes] may also be helpful.

2. Crystals and PBW bases; a definition of MV polytopes.

- Explain the braid group action on $U_q(\mathfrak{g})$; use this to define root vectors and PBW type bases.
- State that these are crystal bases – probably you won't have time to discuss the proof; explain how this gives many different parameterizations of $B(-\infty)$.
- Explain how one can record the exponents in one PBW monomial as a path in weight space, and that by recording all paths you get the 1-skeleton of a polytope.
- Discuss Lusztig's piecewise linear functions (see [Lus1, Section 2]), and how they let you recover the whole polytope from a single path.
- Explain how the braid group operator T_i gives some sort of operation on the crystal (this is Saito's crystal reflection, but you need not get into any details as this will be done in a later talk).

For the definition of the PBW bases there are many sources: Lusztig's original work [Lus1] or Lusztig's book [Lus2], or I find Chari-Pressley [CP] a bit more friendly. You can also look at my notes [Notes, lecture 3].

For the relationship between the various parameterizations and MV polytopes: I think the original source is [Kam2], but it would be a good idea to look at my notes [Notes, lecture 12, part 1].

For the operation on a crystal corresponding to T_i see [Sai], but you really don't need to say much about this. Just observe that applying T_i to a PBW bases vector sometimes gives you something that still looks like a PBW basis vector.

One warning: you will not find mention of the word "polytope" in most of these sources. That was added later; I think the fact that these paths form the 1-skeleton of a polytope was only really established in [Kam2], which involves the affine grassmannian. I'm confident one can prove this using only the fact that the orders on positive roots coming from reduced expressions of w_0 are convex, and looking at the form of Lusztig's piecewise linear functions. I would love to see this done, but you would have to figure out some of the details... .

3. Recognizing and using MV polytopes.

- Define GGMS polytope/Pseudo-Weyl polytope (both terms mean the same thing), and explain how to read off Lusztig data for a charge or more generally for a convex order.
- Explain the basic properties of MV polytopes (a single path determines the polytope; how crystal operators change well-chosen paths; what 2-faces look like, and, crucially, how the operation on crystals corresponding to T_i acts on nice paths), and why they hold for the polytopes as defined from Lusztig's PBW bases.
- Give Saito's description of crystal reflection (i.e. the operation corresponding to T_i from lecture 2) in terms of the crystal operators.
- Explain the characterization of the map that sends $b \in B(-\infty)$ to its MV polytope ([TW, Proposition 1.20]).
- Discuss some information that is readily available from MV polytope. For instance: how you can see which elements of $B(-\infty)$ survive in a given $B(\lambda)$; Kashiwara involution; which MV polytopes give highest weight elements of $B(\lambda) \otimes B(\mu)$... . You may not have time for a lot of this, but it would be good to mention something.

I think you should mainly follow [TW, Section 1.2-1.4]. Some parts of this are written to work in greater generality then finite type, but try to ignore such difficulties.

For the original results on Saito reflection see Saito's paper [Sai], although I think everything you need should be summarized in [TW].

You can find some theorems about reading off information from MV polytopes in [Kam1, §6].

4. Crystals and quiver varieties.

- Define Lusztig's nilpotent variety (try to avoid too many technicalities).
- Define the crystal operators.
- State and discuss Kashiwara and Saito's characterization of $B(-\infty)$ (either [KS, Proposition 3.2.3] or [TW, Proposition 1.4], or both).
- Discuss why the operators on irreducible components of Lusztig's nilpotent varieties satisfy the characterization theorem and hence realize $B(-\infty)$.

- Define HN-polytopes as the convex hull of the dimension vectors of the sub-representations of a given representation. State but don't prove that for a generic point in an irreducible component, this gives the corresponding MV polytope; proof is the subject of lecture 5 (statement of this is in [BKT, end of section 1.3].
- Time permitting, you could say what Nakajima's Lagrangian varieties are, how they realize the crystals $B(\lambda)$, and how this gives a geometric interpretation of the embedding $B(\lambda) \subset B(-\infty)$.

I think you can essentially follow [KS, roughly p7-16]. If you want to see it, the original reference for Lusztig's nilpotent variety is [Lus0]. In the unlikely event that you do get to the part about Lagrangian quiver varieties and the highest weight crystals $B(\lambda)$, the reference is [Sai2].

DAY 2

5. Harder-Narasimhan filtrations and Reflection functors on Lusztig's quiver varieties.

- Recall from talk 4 that quiver varieties are representation varieties for the preprojective algebra.
- Describe the filtration of a representation defined by either a linear functional on weight space or a reduced expression for w_0.
- Explain that the dimension vectors of the sub-representations in this filtration are vertices of the HN polytope from lecture 4.
- Describe reflection functors on the category of representations, and their effect on HN filtrations (for this, I think it is best to just give the explicit form of the sub-representations).
- Explain why the map which sends a component to its HN polytope satisfies the conditions of the recognition theorem from Lecture 3 (see [TW, Proposition 1.20]), and hence HN polytopes and MV polytopes are the same. In particular, discuss the action of the reflection functors and how this is really Saito reflection.

For a short explanation which is also short on details, see my notes [Notes, Lecture 12, part 2]. For more details on the reflection functors on quiver varieties, see [BK]. For more details on the relationship with filtrations, see [BKT] (although that paper has many technicalities you can ignore).

For the relationship with MV polytopes: one statement is [BK, Theorem 19], but the statement that is a bit more relevant for us is given in [BKT] (see the second last paragraph of §1.3). Unfortunately, there is no source that does this using the characterization theorem [TW, Proposition 1.20]. I think this would be the nicest way, but you would have to work out some of the details...

6. KLR/Quiver-Hecke algebra.

- Define KLR algebras.
- State the categorification theorem ([KL, Theorem 8]); if possible briefly discuss the proof, but this is not strictly needed.

- State the results from Lauda-Vazirani showing that the gradable simple representations index the vertices of $B(-\infty)$ and describing the crystal operators.
- Time permitting, discuss some of the ideas in the proof of Lauda-Vazirani's result; this should rely on Kashiwara and Saito's characterization of $B(-\infty)$ from [KS, Proposition 3.2.3], and I think the important technical step is what they call the "jump Lemma" ([LV, Lemma 6.5]).

7. Polytopes from representations of KLR algebras.

- Define the character polytope of a representation of a KLR algebra ([TW, Definition 3.1]).
- Define the notion of charge and the corresponding cuspidal decomposition for a representation of a KLR algebra.
- Discuss why the ν which show up as the weights of initial segments in a cuspidal decomposition are vertices of the character polytope.
- Discuss Saito reflection in terms of cuspidal decompositions ([TW, Proposition 2.23])
- Explain how the previous point, along with the characterization theorem, shows that character polytopes are MV polytopes.

Everything you need should be in [TW], and for the statements involving MV polytopes and Saito reflection I think this is the only source. You should try to ignore any complications coming from the affine case (which is the subject of a lot of that paper, and will be discussed in lecture 8). For the notion of cuspidal decomposition, [McN] may be helpful as well (see especially Theorem 3.1 and the explanation of the cuspidal decomposition). You should probably also mention [KR], at least for historical context; that paper essentially explains cuspidal decompositions but only for some very special convex orders, and using quite different methods.

8. More properties of KLR-polytopes: face crystals, affine type, and beyond.

- Definition of "KLR polytope" in general type
- Simplified definition in affine type.
- The crystal corresponding to a face, and what the KLR representations corresponding to the partitions in the definition of affine MV polytope are.
- If there is time, a discussion of pathologies outside of affine type OR a combinatorial discussion of the $\mathfrak{sl}_2$ type faces.

The main reference is [TW]. The definition of general type MV polytopes is explained in Theorem D and Corollaries 3.10, 3.11. The simplified definition in affine type is Theorem C, but of course in the introduction it is not explained what the partitions λ_γ are. This is the subject of Section 3.4. The key ideas are the face crystal from section 3.2, and the construction of the distinguished cuspidal representations in Definition 3.32 and Equation (11).

If you want to discuss $\mathfrak{sl}_2$ faces combinatorially, the combinatorics is in [BDKT] and the proof that this gives the same polytopes as the geometry is in [MT].

DAY 3

9. The affine Grassmannian, and MV cycles

- Define the affine Grassmannian.
- Define MV cycles
- Explain how to construct polytopes as moment-map images of MV cycles.
- Discuss the Lusztig data parameterization of MV cycles ([Kam2, Theorem 4.5]).

This material is from [And] and [Kam2]. Probably [And] is easier to follow for the most part. It would be nice to explain Lustzig data though, since this is such an important part of the rest of this workshop, and for that you will need to look in [Kam2].

10. Crystal structure on MV cycles and polytopes

- The Braverman-Finkelberg-Gaitsgory crystal structure on MV cycles; both the definition in terms of parabolic restriction and the more recent explicit descriptions (e.g. [BG, propositions 12 and 14]).
- How to extract the combinatorial crystal structure on MV polytopes from the geometric crystal structure on MV cycles (by comparing the effect on Lusztig data).

You should look at [BG, §4] and [Kam1, §4]. The crystal structure on MV cycles is actually from [BFG], but I think [BG] or [Kam1] have everything you need. Probably [BG] is the best place to begin.

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