Perverse sheaves learning seminar: Smooth maps and constructible sheaves

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1 Introduction

In these notes, we will study constructible sheaves on algebraic varieties. Our main reference is [Ach], and we will focus on varieties over $\mathbb{C}$ with the complex topology. So our spaces will be Hausdorff, locally compact and locally simply connected. Constructible sheaves are sheaves that are "glued from" local systems. In this and later talks, we will see that the derived category of constructible sheaves on a variety contains local systems and is closed under the six functors introduced in [Bai18]. We start by studying the behavior of the functors for various class of morphisms.

2 Pullbacks and pushforwards under algebraic morphisms

2.1 (Quasi-)finite morphisms

Definition 2.1 ([Ach], p190.). *A morphism of varieties $f : X \to Y$ is* ***quasi-finite*** *if $f^{-1}(y)$ is a finite set for each point $y \in Y$, and it is* ***finite*** *if it is quasi-finite and proper.*

Example 2.2. *Examples of finite morphisms include:*

1. *Closed immersions.*
2. *Any polynomial morphism $\mathbb{C} \to \mathbb{C}$.*

Example 2.3. *An open inclusion $U \hookrightarrow X$ is quasi-finite but not finite (as it is not a closed morphism).*

Lemma 2.4 ([Ach], Lemma 3.1.5). *Let $f : X \to Y$ be a finite morphism. Then ${}^o f_* : Sh(X) \to Sh(Y)$ is exact. Moreover, for any sheaf $\mathcal{F} \in Sh(X)$, we have* $\operatorname{supp} {}^o f_*(\mathcal{F}) = f(\operatorname{supp} \mathcal{F})$.

Proof. We already know that ${}^o f_*$ is left-exact, we just need to show that $f_*(\mathcal{F}) \in Sh(Y)$ (i.e. it has no higher direct images). We will show that the stalks $f_*(\mathcal{F})_y$ are concentrated in degree 0. Consider the diagram

$$\begin{tikzcd} X \times_Y y = f^{-1}(y) \arrow[r, hook, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ y \arrow[r, hook, "g"] & Y \end{tikzcd}$$

Note that the stalk $f_*(\mathcal{F})_y$ is $g^* f_*(\mathcal{F})$. Since f is proper, $f_! = f_*$ and by proper base change ([Bai18], Theorem 3.25) we have $g^* f_* \cong f'_* g'^*$. Therefore $(f_*\mathcal{F})_y = f'_* g'^* = R\Gamma\left(\mathcal{F}\big|_{f^{-1}(y)}\right)$, as f' is the map to a point. Since f is finite, $f^{-1}(y)$ is a finite set, so

$$R\Gamma\left(\mathcal{F}\big|_{f^{-1}(y)}\right) \cong \Gamma\left(\mathcal{F}\big|_{f^{-1}(y)}\right) \cong \prod_{x \in f^{-1}(y)} \mathcal{F}_x,$$

as desired.

Now we show that $\operatorname{supp} {}^o f_*(\mathcal{F}) = f(\operatorname{supp} \mathcal{F})$. Note that $\operatorname{supp} {}^o f_*(\mathcal{F}) \subseteq f(\operatorname{supp}(\mathcal{F}))$, as $f(\operatorname{supp}(\mathcal{F}))$ is closed since f is proper (and hence a closed morphism). For the opposite inclusion, let $S = \{x \in X | \mathcal{F}_x \neq 0\}$. By definition, S is dense in $\operatorname{supp} \mathcal{F}$. We just computed ${}^o f_* \mathcal{F}$ at all points of $f(S)$ above and we found that they were nonzero. As $f(S)$ is dense in $f(\operatorname{supp} \mathcal{F})$, and all the stalks were nonzero, $f(\operatorname{supp}(\mathcal{F})) \subseteq \operatorname{supp} {}^o f_*(\mathcal{F})$.

Q.E.D.

Remark 2.5. *To see that finiteness is necessary for the statement about supports, consider an open inclusion $j : U \hookrightarrow X$ and $\mathcal{F} = \underline{\mathbb{C}}_U$. Then $j(\operatorname{supp}(\mathcal{F})) = U$ but $\operatorname{supp}({}^o j_*(\mathcal{F})) = \operatorname{supp}(\underline{\mathbb{C}}_X)$.*

3 Smooth and Étale morphisms

Definition 3.1. *[[Ara10], Definition 18.1.7] A morphism $f : X \to Y$ of schemes is said to be* ***smooth of relative dimension*** *d* ***at*** *$p \in X$ if there is*

- *Am affine Zariski-open neighborhood Y^0 of $f(p)$,*
- *An affine Zariski-open neighborhood of p in $f^{-1}(Y^0) \subseteq X$*
- *A commutative diagram*
- *and*
$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right) = n.$$

The morphism is ***smooth*** *if it is smooth at all $p \in X$.*

Remark 3.2. *This is not the standard definition of a smooth morphism, to see the comparison to the more standard definition ([Har77], Definition 10.0), see [Ara10], Theorem 18.1.10.*

A smooth morphism an the algebraic analogue of a submersion.

Proposition 3.3 ([Har77], Proposition 10.4)**.** *Let $f : X \to Y$ be a morphism of nonsingular varieties and $m = \dim X - \dim Y$. Then f is smooth of relative dimension m if and only if for every closed point $x \in X$, the induced map on Zariski tangent spaces $T_f : T_x X \to T_{f(x)} Y$ is surjective.*

Example 3.4. *Examples of smooth morphisms include:*

1. *The projection $pr_2 : X \times Y \to Y$ for any smooth variety X.*

2. *The map $X \to pt$ is smooth if and only if X is a smooth variety.*

3. *Any open embedding is smooth (of relative dimension 0).*

Example 3.5. *Non-examples of smooth morphisms include:*

1. *Families with singular fibers, for example $\mathbb{C}^2 \to \mathbb{C}$ given by $(x, y) \mapsto xy$ is not smooth at $(0, 0)$.*

2. *If $\dim X < \dim Y$, then no morphism $X \to Y$ is smooth.*

Exercise 3.6. *Show that smoothness is preserved under base change. As a corollary, deduce that fibers of a smooth morphism are smooth varieties.*

A morphism of smooth varieties is smooth *generically* in the following sense

Theorem 3.7 (Generic smoothness, [Ach], Theorem 3.1.3, [Har77], Corollary 10.7)**.** *Let $f : X \to Y$ be a morphism of varieties and assume that X is smooth. Then there is a nonempty Zariski-open subset $U \subseteq Y$ such that $f\big|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a smooth morphism.*

Example 3.8. *The family in example 3.5, $\mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto xy$ is smooth away from $(0, 0)$, as*

$$\operatorname{rank} \begin{pmatrix} \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} y & x \end{pmatrix}$$

Definition 3.9. *An **étale morphism** is a smooth morphism of relative dimension 0.*

Remark 3.10. *Any étale morphism is quasi-finite.*

Lemma 3.11 ([Ach] Lemma 3.1.4)**.** *An étale morphism is a local homeomorphism, in particular, a finite étale morphism is a covering map.*

Pullbacks by smooth morphisms are really well-behaved:

Theorem 3.12. *[Smooth base change, [Ach] Theorem 3.3.4] Suppose we have a cartesian square of varieties*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that g and g' are smooth. Then, for $\mathcal{F} \in D^+(X)$, the base change map $g^* f_* \mathcal{F} \to f'_*(g')^* \mathcal{F}$ is an isomorphism.

Compare this to [Bai18], Theorem 6.3 (base change for locally trivial fibrations). The base change map is an isomorphism if we are pushing forward a complex whose cohomology groups are local systems by a locally trivial fibration or if we are pulling back any complex by a smooth morphism.

Remark 3.13. *To see how smoothness is necessary, consider $X = \mathbb{C}^\times$, with the map $\iota : X \to Y = \mathbb{C}$. Then we have the pullback diagram*

$$\begin{tikzcd} \emptyset = \iota^{-1}(0) \arrow[r, hook, "g'"] \arrow[d, "\iota'"'] & X \arrow[d, "\iota"] \\ 0 \arrow[r, hook, "g"] & Y \end{tikzcd}$$

Consider the constant sheaf $\underline{\mathbb{C}}_X$ on X. Then $\iota_* \underline{\mathbb{C}}_X \cong \underline{\mathbb{C}}_Y$, as the preimage of each open connected set is connected. Therefore

$$g^* \left(\iota_* \underline{\mathbb{C}}_X\right) = \left(\iota_* \underline{\mathbb{C}}_X\right)_0 \cong \mathbb{C}.$$

Going the other way, since $\iota^{-1}(0) = \emptyset$, we get that $\iota'_* g^*(\underline{\mathbb{C}}_X) = 0$.

To prove the theorem, we need a technical lemma first, which is a concrete demonstration that smooth maps are like submersions in differential geometry. Submersions can locally be written as a projection map from a vector space to a subspace using an implicit function theorem type argument, which is what the following lemma does:

Lemma 3.14. *[[Ach], Lemma 3.3.1] Let $f : X \to Y$ be a smooth morphism of relative dimension d. For any point $p \in X$, there is a neighborhood U of p, a neighborhood V of $f(p)$ (in the complex topology), a small disk $D \subseteq \mathbb{C}^d$, and a biholomorphism (invertible holomorphic map) $b : U \to V \times D$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "b"] \arrow[d, "f"'] & V \times D \arrow[dl, "pr_1"] \\ V & \end{tikzcd}$$

Proof. Since the statement is local, by definition 3.1, we may assume that X and Y are affine varieties, and that f is given by $pr_1 : \operatorname{Spec}(\mathbb{C}[Y][x_1, \dots x_{d+n}]/(g_1, \dots g_n)) \to \operatorname{Spec}\mathbb{C}[Y]$. We may also assume that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right)_{1 \le i \le n, k+d+1 \le j \le k+d+n} (p)$$

(formed by the last columns of the Jacobian matrix) is nonsingular. Let $p = (q, p', p'')$, where $q \in \mathbb{C}^k$, $p' \in \mathbb{C}^d$, and $p'' \in \mathbb{C}^n$. The implicit function theorem tells us that there is an open subset $W_1 \subseteq \mathbb{C}^{k+d}$ containing (q, p'), an open subset $W_2 \subseteq \mathbb{C}^n$ containing p'' and a holomorphic map $h_0 : W_1 \to W_2$ such that

$$h : W_1 \to g^{-1}(0) \cap (W_1 \times W_2) \quad \text{given by} \quad h(u) = (u, h_0(u))$$

(where $g = (g_1, \dots, g_n)$) is a bijection. Moreover h has an inverse b given by the projection map $\mathbb{C}^{k+d+n} \to \mathbb{C}^{k+d}$ onto the first $k + d$ coordinates.

Now replace W_1 by a smaller open set of the form $V_0 \times D$ where $V_0 \subseteq \mathbb{C}^k$ contains q and $D \subseteq \mathbb{C}^d$ is an open set containing p' and is homeomorphic to $\mathbb{C}^d$. Let $U_0 = V_0 \times D \times W_2$. Then we have

$$\begin{tikzcd} g^{-1}(0) \times U_0 \arrow[d, "pr_1"'] & V_0 \times D \arrow[l, "h"'] \arrow[dl, "pr_1"] \\ V_0 & \end{tikzcd}$$

(note that h is a biholomorphism). Letting $U = U_0 \cap X$ and $V = V_0 \cap Y$ finishes the proof.

Q.E.D.

Now we are ready to give a proof of the smooth base change theorem

Proof. (of Thm. 3.12) Let $p \in Y'$, and pick $U \subseteq Y'$ containing p ,$V \subseteq Y$ containing $g(p)$, and $M \subseteq \mathbb{C}^d$ as in lemma 3.14 above. Since p is arbitrary, is enough to show that the map ([Daw18], Lemma 4.1.2) $g^* f_* \mathcal{F} \to f'_*(g')^* \mathcal{F}$ is an isomorphism after restriction to U. This means that we have the diagram (slight abuse of notation)

$$\begin{tikzcd} (f')^{-1}(U) \arrow[r, "g'"] \arrow[d, "f'"'] & f^{-1}(V) \arrow[d, "f"] \\ U \arrow[r, "g"'] & V \end{tikzcd}$$

which, by lemma 3.14 can be replaced by

$$\begin{tikzcd} (f)^{-1}(V) \times M \arrow[r, "g'"] \arrow[d, "f'"'] & f^{-1}(V) \arrow[d, "f"] \\ V \times M \arrow[r, "pr_1"'] & V \end{tikzcd}$$

and the map $pr_1^* f_* \mathcal{F} \to f'_* pr_1^* \mathcal{F}$ is an isomorphism by [Ach], Proposition 2.12.1.

Q.E.D.

The next result involving smooth morphisms is the "relative version of Poincaré duality":

Theorem 3.15. *[[Ach], Theorem 3.3.8] Let $f : X \to Y$ be a smooth morphism of relative dimension d. There is a natural isomorphism of functors $f^! \cong f^*[2d]$.*

Proof. We will prove the theorem in the special case where Y is a point and $d = \dim_{\mathbb{C}} X$. We proceed in multiple steps.

Step 1: We show that $f^! \underline{\mathbb{C}}_{pt}$ is a local system of rank 1 in degree $2d$. Let $j : U \hookrightarrow X$ be an inclusion of a disk. Then as j is an open immersion, $j^* = j^!$. Therefore $j^! f^! \underline{\mathbb{C}}_{pt} = (f \circ j)^! \underline{\mathbb{C}}_{pt} = H_c^*(U) \cong \underline{\mathbb{C}}_U[2d]$. Also, since j is an open inclusion, this means that $f^! \underline{\mathbb{C}}_{pt}$ is a rank 1 local system on X in cohomological degree $2d$.

Step 2: We recall Poincaré duality with coefficients in a local system $\mathcal{L}$. This is the following statement:

$$\left(H_c^k(X, \mathcal{L}^*)\right)^* \cong H^{2d-k}(X, \mathcal{L}), \tag{1}$$

where $\mathcal{L}^*$ is the dual local system . The proof is the same as for the usual Poincaré duality (which is the case where $\mathcal{L} = \underline{\mathbb{C}}_X$).

Step 3: Since rank 1 local systems on X form a full subcategory, to show that $f^! \underline{\mathbb{C}}_{pt} \cong f^* \underline{\mathbb{C}}_{pt}[2d]$, by the Yoneda lemma it suffices to check that

$$R\mathrm{Hom}_{D(X)}(\mathcal{L}, f^! \underline{\mathbb{C}}_{pt}) \cong R\mathrm{Hom}_{D(X)}(\mathcal{L}, f^* \underline{\mathbb{C}}_{pt}[2d])$$

for any rank 1 local system $\mathcal{L}$ on X. By adjunction between $f_!$ and $f^!$, we have

$$R\mathrm{Hom}_{D(X)}(\mathcal{L}, f^!\underline{\mathbb{C}}_{pt}) \cong R\mathrm{Hom}_{D(pt)}(f_!\mathcal{L}, \mathbb{C}_{pt}) \tag{2}$$

The right hand side of equation 2 is $(R\Gamma_c(X,\mathcal{L}))^*$ and taking $-k$-th cohomology groups yields $H_c^k(X,\mathcal{L}^*)$. The left hand side of equation 2 is $R\mathrm{Hom}_{D(X)}(\mathcal{L}, \underline{\mathbb{C}}_X[2d]) = R\Gamma(X, \mathcal{L}[2d])$. Taking $-k$-th cohomology groups yields $H^{2n-k}(X,\mathcal{L})$. These are isomorphic by 1. So the two objects have isomorphic cohomology, and since they are objects in $D(pt) \cong D(\mathrm{Vec}_{\mathbb{C}})$, this means that the complexes are isomorphic. Since $\mathcal{L}$ was arbitrary, this completes our proof.

For a full proof, see [Ach], Theorem 3.3.8.

Q.E.D.

The next result isn't about smooth morphisms, but it will be very useful for us when we start pushing around local systems:

Theorem 3.16 ([Ach], Theorem 3.3.10)**.** *Let X be a smooth, equidimensional variety and let Y be a smooth locally closed equidimensional subvariety. Let $h : Y \hookrightarrow X$ be the inclusion map, and let $d = \dim X - \dim Y$. For $\mathcal{L} \in \mathrm{Loc}(X)$, there is a natural isomorphism $h^!\mathcal{L} \cong h^*\mathcal{L}[-2d]$.*

Proof. Let $D \subseteq X$ be an open disk. Let $V = D \cap Y$ and let $h' = h\big|_D$. Let $j : D \hookrightarrow X$, $j' : V \hookrightarrow Y$ be the inclusion maps. Then we have the diagram

$$\begin{tikzcd} V \arrow[r, hook, "h'"] \arrow[d, hook, "j'"'] & D \arrow[d, hook, "j"] \\ Y \arrow[r, hook, "h"'] & X \end{tikzcd}$$

As j is an open inclusion, $j^! = j^*$ by [Ach], Lemma 2.4.7. We want to show $(h')^!\left(\mathcal{L}\big|_D\right) = (h^!\mathcal{L})\big|_V$.

$$\begin{aligned}
(h')^!\left(\mathcal{L}\big|_D\right) &= (h')^! j^*\mathcal{L} \\
&= (h')^! j^!\mathcal{L} && \text{as } j^! = j^* \\
&= (j \circ h')^!\mathcal{L} && \text{by [Bai18], Proposition 3.29} \\
&= (h \circ j')^!\mathcal{L} && \text{as the diagram is Cartesian} \\
&= (j')^* h^!\mathcal{L} \\
&= (h^!\mathcal{L})\big|_V
\end{aligned}$$

Let $M = \mathcal{L}\big|_D = j^*\mathcal{L} = j^!\mathcal{L}$. Then M must be a constant sheaf $\underline{M}_D$. Let $n = \dim X$. Then by theorem 3.15, we have

$$\underline{M}_D \cong \underline{M}_X\big|_D \cong (a_X^! \underline{M}_{pt}[-2n])\big|_D,$$

where $a_X : X \to pt$. By the same argument, if $m = \dim Y$ (so $d = m - n$), and $a_Y : Y \to pt$,

$$\begin{aligned}
(h^!\mathcal{L})\big|_V &= (j')^! h^!(\mathcal{L}) \\
&\cong (h')^! j^!(\mathcal{L}) \\
&\cong (h')^!(a_X^! \underline{M}_{pt}[-2n])\big|_D \\
&\cong (a_Y^! \underline{M}_{pt})\big|_V[-2n]) \\
&\cong \underline{M}_Y\big|_V[2m-2n] \\
&\cong (h^*\mathcal{L})\big|_V[-2d]
\end{aligned}$$

We can cover Y by open sets V as above. On each of these, the two sheaves $h^!\mathcal{L}$ and $h^*\mathcal{L}[-2d]$ agree, and the sheaves glue the same way (the isomorphism commutes with restriction maps). Therefore the two sheaves must be isomorphic.

Q.E.D.

Example 3.17. *Let $Y = \{0\}, X = \mathbb{C}$, $f : Y \hookrightarrow X$. Then as we computed during Roger's talk, $f^!\underline{\mathbb{C}}_X = \underline{\mathbb{C}}_{pt}[-2]$.*

4 Stratifications

Definition 4.1 (Definition 3.4.1 in [Ach]). *Let X be a variety. A **stratification** on X is a finite collection $(X_s)_{s\in\mathcal{S}}$ of disjoint smooth, connected, locally closed subvarieties such that $X = \bigcup_{s\in\mathcal{S}} X_s$, and such that the closure of each stratum is a union of strata $\overline{X_t} = \bigcup_{s\leq t} X_s$ for some partial order on S.*

Definition 4.2. *The subvarieties X_s are called the **strata** of the stratification.*

Remark 4.3. *The set $\mathcal{S}$ carries a natural partial order called the **closure partial order** given by*

$$t \leq s \text{ if } X_t \subseteq \overline{X_s}.$$

Example 4.4. *If an algebraic group G acts on a variety X with finitely many orbits then the orbits constitute a stratification of X, for example:*

1. *The cell decomposition*

$$\mathbb{CP}^n = \bigsqcup_{i=0}^{n} \mathbb{C}^i$$

   defines a stratification on $\mathbb{CP}^n$. In this case, the closure partial order is total order.

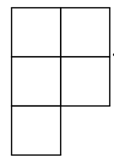
2. *Let $Gr(k,n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$. Let's represent a k-plane by the $k \times n$ matrix in reduced row echelon form whose row span is the k-plane. Each of the non-pivot columns of the matrix have some entries that must be zero, and some that can be arbitrary, e.g. $(k = 3, n = 6)$:*

$$\begin{pmatrix} 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}.$$

Forgetting the pivot columns then gives us a partition that fits into a $k \times (n-k)$ rectangle:

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

corresponds to



This defines a stratification (in fact, a cell decomposition) of $Gr(k,n)$ into affine spaces $X_\lambda \cong \mathbb{C}^{|\lambda|}$. The closure partial order is given by containment of Young diagrams.

3. *Let G be semisimple with Borel subgroup B. Then by the Bruhat decomposition, B acts on the flag variety G/B with finitely many orbits. The strata $BwB/B \cong \mathbb{C}^{l(w)}$ are known as Bruhat cells. Here $\mathcal{S} = W$, the Weyl group of G, and the closure partial order is Bruhat order.*

If all of the strata in a stratification are affine spaces, the stratification $\mathcal{S}$ is also known as an **affine paving** of X.

Definition 4.5 (Definition 3.4.4 in [Ach])**.** *Let X be a variety. A **filtration of X by smooth varieties** is a finite collection $(X_s)_{s \in \mathcal{S}}$ of disjoint smooth, connected, locally closed subvarieties such that $X = \bigcup_{s \in \mathcal{S}} X_s$, and such that the elements of $\mathcal{S}$ can be ordered as $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$ in such a way that for each i, the subset*

$$X_{s_1} \cup X_{s_2} \cup \ldots \cup X_{s_i}$$

is a closed subset of X.

Remark 4.6. *Any stratification can be made into a filtration by smooth varieties, by picking a linear extension (order-preserving map to a total order) of $\mathcal{S}$.*

Filtrations by smooth varieties can be used to study singular varieties in the following way: let X be an irreducible variety, and X^{sing} be the singular locus. Then $Y = X \setminus X^{sing}$ is an open connected smooth variety. Iterating taking smooth locus, we can produce a filtration of X by smooth varieties ([Ach], Remark 3.4.5).

Remark 4.7. *A filtration by smooth varieties is a weaker notion than a stratification, for example, let $[x,y,z]$ be homogeneous coordinates on $\mathbb{CP}^2$. Then*

$$\{y=0\}, \{z=0, y \neq 0\}, \{y \neq 0, z \neq 0\}$$

is a filtration by smooth varieties that is not a stratification.

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