SEMISMALL MORPHISMS

YEHAO ZHOU

1. Motivation

Let's recall some results concerning t-exactness of pushforward from last time:

Proposition 1.1. *Let $f : X \to Y$ be a morphism between varieties, then*

(1) *If f is quasi-finite, then f_* is left t-exact and $f_!$ is right t-exact;*
(2) *If f is affine, then f_* is right t-exact and $f_!$ is left t-exact;*
(3) *As a result, if f is quasi-finite affine, then f_* and $f_!$ are t-exact.*

Unfortunately, proper morphisms, which commonly show up in applications (e.g. resolutions of singularities), are not in this list (except for finite morphisms). Heuristically, this comes from the fact that for a proper variety Z of dimension d, the top degree cohomology $\mathrm{H}^{2d}(Z, \mathbb{C}) \neq 0$ (Lemma 2.3), so if there is a proper morphism $f : X \to Y$ such that there is a subvariety $W \subset Y$ with dimensions of fibers over W being $\geq d$, then the $\mathcal{H}^{2d-\dim X}(f|_{f^{-1}W})_*\underline{\mathbb{C}}[\dim X] \neq 0$, i.e. $(f|_{f^{-1}W})_*\underline{\mathbb{C}}[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$ only if $\dim W \geq 2d - \dim X$. So we restrict to proper morphisms with a nice property on the dimension of fibers:

Definition 1.2. Let X be an irreducible variety, $f : X \to Y$ is called **semismall** if there exists a stratification $\{Y_t\}_{t\in\mathscr{T}}$ of Y, such that for each stratum Y_t and each point $y \in Y_t \cap f(X)$

$$2 \dim f^{-1}(y) + \dim Y_t \leq \dim X$$

f is called **small** with respect to a dense open subset $W \subset Y$ if $\forall y \in W$, $f^{-1}(y)$ is a finite set, and there exists a stratification $\{Y_t\}_{t\in\mathscr{T}}$, such that W is a union of strata and such that $\forall y \in Y_t \cap f(X) \subset Y - W$

$$2 \dim f^{-1}(y) + \dim Y_t < \dim X$$

Proposition 1.3 (Equivalent definition of semismallness)**.** *Let X be an irreducible variety. Then $f : X \to Y$ is semismall if and only if*

$$\dim X \times_Y X \leq \dim X$$

Proof. If f is semismall, then there is a stratification $\{Y_t\}_{t\in\mathscr{T}}$, such that for each stratum Y_t and each point $y \in Y_t \cap f(X)$

$$\dim(f \times f)^{-1}(y) + \dim Y_t = 2 \dim f^{-1}(y) + \dim Y_t \leq \dim X$$

which implies that $\dim(f \times f)^{-1}Y_t \leq \dim X$. Since $X \times_Y X = \amalg_t(f \times f)^{-1}Y_t$, we conclude that $\dim X \times_Y X \leq \dim X$.

Let's prove the "if" part by constructing a stratification. Let's prove a lemma first.

Lemma 1.4. *If $f : X \to Y$ is a morphism, then there exists a stratification $\{Y_t\}_{t\in\mathscr{T}}$ such that $\forall t \in \mathscr{T}$, the fibers of $f|_{f^{-1}Y_t}$ are either empty or of dimensions $d(t)$ which only depend on t, and*

$$\dim f^{-1}Y_t = d(t) + \dim Y_t \quad ; \quad \dim(f\times f)^{-1}Y_t = 2d(t) + \dim Y_t$$

Proof. We prove it by Noetherian induction. First of all, we replace Y with the closure of $f(X)$, endowed with reduced structure sheaf. Suppose that the lemma is true for any proper closed subvariety of Y.

According to the generic flatness theorem [GD65] (EGA IV_2, Théorème 6.9.1) there is an irreducible open subvariety $U \subset Y$ such that $f|_{f^{-1}U}$ is flat. Shrink U if necessary, we assume that each irreducible component of $f^{-1}U$ is mapped surjectively to U. Then we apply Theorem 15.1 of [Mat89] to $f|_{f^{-1}U}$ and we see that $\forall y \in U$ and $\forall x \in f^{-1}(y)$, $\dim \mathcal{O}_{X,x} \le \dim f^{-1}(y) + \dim \mathcal{O}_{Y,y}$, and there is at least one $x \in f^{-1}(y)$ such that the equality is obtained. In other words

$$\sup_{x\in f^{-1}(y)} \dim \mathcal{O}_{X,x} = \dim f^{-1}(y) + \dim \mathcal{O}_{Y,y} = \dim f^{-1}(y) + \dim U$$

Since each irreducible components of $f^{-1}U$ is mapped surjectively to U, there exists $x \in f^{-1}(y)$ such that x lies in the component with maximal dimension, i.e. $\dim f^{-1}U$, so

$$\sup_{x\in f^{-1}(y)} \dim \mathcal{O}_{X,x} = \dim f^{-1}U$$

As a consequence, $\dim f^{-1}(y) = \dim f^{-1}U - \dim U$ is constant on U.

Apply Theorem 15.1 of [Mat89] to the flat morphism $(f\times f)|_{f^{-1}U\times_Y f^{-1}U}$, we see that

$$\sup_{x\in(f\times f)^{-1}(y)} \dim \mathcal{O}_{X\times_Y X,x} = \dim f^{-1}(y)\times f^{-1}(y) + \dim \mathcal{O}_{Y,y} = 2\dim f^{-1}(y) + \dim U$$

The RHS is constant on U, so

$$\dim(f\times f)^{-1}U = \sup_{y\in U} \sup_{x\in(f\times f)^{-1}(y)} \dim \mathcal{O}_{X\times_Y X,x} = 2\dim f^{-1}(y) + \dim U$$

∎

According to the lemma, there is a stratification $\{Y_t\}_{t\in\mathscr{T}}$ such that $\forall t \in \mathscr{T}$, fibers of $f|_{f^{-1}Y_t}$ are either empty or of dimensions $d(t)$ which only depend on t and

$$\dim(f\times f)^{-1}Y_t = 2d(t) + \dim Y_t$$

If $\dim X \times_Y X \le \dim X$, then $(f\times f)^{-1}Y_t \le \dim X\times_Y X \le \dim X$, which implies that $2d(t) + \dim Y_t \le \dim X$, so f is semismall. ∎

Exercise 1.5. For any $f : X \to Y$, inequality $\dim X \times_Y X \ge \dim X$ always hold.

Exercise 1.6. Consider morphisms $f : X \to Y$ and $g : Y \to Z$, if $g\circ f$ is semismall, then f is semismall.

Exercise 1.7 (**Composition of semismall maps can fail to be semismall**). Blowing up subvariety $\{x = w = 0\}$ inside the affine cone $\{xy = zw\} \subset \mathbb{C}^4$, followed by blowing up the preimage of $\{0\}$, each blow up is semismall, but the composition is not. Show that the composition is directly blowing up $\{0\}$.

2. Semismall and small pushforward

Theorem 2.1. *Assume that X is smooth, $f : X \to Y$ is a* ***semismall*** *morphism, then*

(1) *If $F \in {}^p\mathrm{D}^{\leq 0}_{\mathrm{loc}}(X)$, then $f_!F \in {}^p\mathrm{D}^{\leq 0}_{c}(Y)$;*
(2) *If $F \in {}^p\mathrm{D}^{\geq 0}_{\mathrm{loc}}(X)$, then $f_*F \in {}^p\mathrm{D}^{\geq 0}_{c}(Y)$.*

Note that (2) follows from (1) by Verdier duality.

Proof. It suffices to prove that if $F \in \mathrm{Loc}(X)[\dim X]$ then $f_!F \in {}^p\mathrm{D}^{\leq 0}_{c}(Y)$, i.e.

$$\dim \mathrm{Supp}(\mathcal{H}^i f_! F) \leq -i$$

From the definition of semismallness, we know that there is a stratification $\{Y_t\}_{t \in \mathscr{T}}$, such that for each stratum $i_t : Y_t \hookrightarrow Y$ and each point $y \in Y_t \cap f(X)$, $2 \dim f^{-1}(y) + \dim Y_t \leq \dim X$, or equivalently $-\dim X + 2 \dim f^{-1}(y) \leq -\dim Y_t$. We see that $F|_{f^{-1}Y_t}$ is of cohomological degree $-\dim X$, hence the homological bound of $f_!$ [Bai18], i.e. $\mathcal{H}^i f_! K = 0$ for $i > 2d$ and $K \in \mathrm{Shv}_c(X)$, implies that

$$i_t^* f_! F \cong (f|_{f^{-1}Y_t})_! F|_{f^{-1}Y_t} \in \mathrm{D}_c^{\leq -\dim Y_t}(Y_t)$$

Since $i_t^* \mathcal{H}^i f_! F = \mathcal{H}^i i_t^* f_! F$, we have following inequality on strata

$$\dim \mathrm{Supp}(i_t^* \mathcal{H}^i f_! F) \leq -i$$

The LHS is $\dim(\mathrm{Supp}(\mathcal{H}^i f_! F) \cap Y_t)$. Let t runs through $\mathscr{T}$, then we have the desired inequality

$$\dim \mathrm{Supp}(\mathcal{H}^i f_! F) \leq -i$$

■

Corollary 2.2. *Assume that X is smooth, $f : X \to Y$ is a* ***proper*** *morphism, TFAE:*

(a) *f is semismall;*
(b) *$f_* : \mathrm{Loc}(X)[\dim X] \to \mathrm{Perv}(Y)$;*
(c) *$f_* \underline{\mathbb{C}}[\dim X] \in \mathrm{Perv}(Y)$.*

Proof. By the previous theorem, (a) implies (b). (b) implies (c) tautologically. It remains to prove that (c) implies (a). Assume that (a) is false.

By Lemma 1.4, there is a stratification $\{Y_t\}_{t \in \mathscr{T}}$ such that $\forall t \in \mathscr{T}$, fibers of $f|_{f^{-1}Y_t}$ are either empty or of dimensions $d(t)$ which only depends on t and

$$\dim(f \times f)^{-1} Y_t = 2d(t) + \dim Y_t$$

Since f is not semismall, there must be at least one stratum Y_t such that $2d(t) + \dim Y_t > \dim X$, or equivalently $-\dim X + 2 \dim f^{-1}(y) > -\dim Y_t$. We need the following lemma

Lemma 2.3. *If X is a proper variety of dimension d, then $\mathrm{H}^{2d}(X, \mathbb{C}) \neq 0$.*

Proof. By Verdier duality, the statement $\mathrm{H}^{2d}(X, \mathbb{C}) \neq 0$ is equivalent to $\mathrm{H}^{-2d}(X, \omega_X) \neq 0$, where ω_X is the dualizing complex on X. Take a smooth open subvariety $j : U \hookrightarrow X$ with complement $i : Z \hookrightarrow X$ such that $\dim Z < d$, consider the distinguished triangle

$$i_* \omega_Z \to \omega_X \to j_* \omega_U \to$$

Apply the functor $R\Gamma(X,-)$ and we get exact sequence

$$\mathrm{H}^{-2d}(Z,\omega_Z)\to\mathrm{H}^{-2d}(X,\omega_X)\to\mathrm{H}^{-2d}(U,\omega_U)\to\mathrm{H}^{-2d+1}(Z,\omega_Z)$$

Note that $R\Gamma(Z,\omega_Z)=R\Gamma_c(Z,\underline{\mathbb{C}})^\vee R\Gamma(Z,\underline{\mathbb{C}})^\vee$ because Z is proper, and since $\dim Z<d$, $\mathrm{H}^i(Z,\underline{\mathbb{C}})=0$ for $i>2d-2$, hence $\mathrm{H}^{-2d+1}(Z,\omega_Z)=\mathrm{H}^{-2d}(Z,\omega_Z)=0$. We see that $\mathrm{H}^{-2d}(X,\omega_X)\cong\mathrm{H}^{-2d}(U,\omega_U)$. Since there is at least one connected component of U with dimension d, denote it by U_0, then $\omega_{U_0}=\underline{\mathbb{C}}[2d]$, so $\mathrm{H}^{-2d}(U_0,\omega_{U_0})=\mathrm{H}^0(U_0,\mathbb{C})=\mathbb{C}\neq 0$, hence we conclude that $\mathrm{H}^{-2d}(X,\omega_X)\neq 0$. ■

Apply the lemma and we get $\mathcal{H}^{-\dim X+2\dim f^{-1}(y)}(f|_{f^{-1}Y_t})_*\underline{\mathbb{C}}\neq 0$, so that $i_t^*f_*\underline{\mathbb{C}}[\dim X]\notin\mathrm{D}_c^{\leq-\dim Y_t}(Y_t)$, hence $f_*\underline{\mathbb{C}}[\dim X]\notin\mathrm{Perv}(Y)$, i.e. (c) is false. ■

It turns out that we can say more for proper small morphisms.

Theorem 2.4. *Assume that X is smooth. $f:X\to Y$ is a* ***proper small*** *morphism with respect to dense open $j:W\hookrightarrow Y$, $\mathcal{L}\in\mathrm{Loc}(X)$, denote $f|_{f^{-1}W}$ by f' (f' is finite since it's proper and quasi-finite) then*

$$f_*\mathcal{L}[\dim X]\cong j_{!*}(f'_*\mathcal{L}[\dim X])$$

Proof. Since small morphism is obviously semismall, we have $f_*\mathcal{L}[\dim X]\in\mathrm{Perv}(Y)$ from Corорllary 2.2. By a property of intermediate extension ([Ach18] Lemma 4.2.8), it suffices to prove that

(1) $i_t^*f_*\mathcal{L}[\dim X]\in\mathrm{D}_c^{\leq-\dim Y_t-1}(Y_t)$ for every stratum $Y_t\subset Y-W$;
(2) $i_t^!f_*\mathcal{L}[\dim X]\in\mathrm{D}_c^{\geq-\dim Y_t+1}(Y_t)$ for every stratum $Y_t\subset Y-W$.

Notice that (2) follows from (1) by the Verdier duality, so it suffices to prove (1). This follows from replacing $\leq$ by $<$ in dimension inequalities of the proof of Theorem 2.1, details left as an exercise. ■

3. Generalization

In many situations, we don't have a semismall or small morphisms, instead, morphisms are semismall or small when they are restricted to subvarieties of the source. They are called stratified semismall or stratified small morphisms, and they share similar properties to semismall or small morphisms. We are going to give precise definitions and explain their properties.

Definition 3.1. Let X be a variety equiped with a good stratification $\{X_s\}_{s\in\mathscr{S}}$, $f:X\to Y$ is called **stratified semismall** if $\forall s\in\mathscr{S}$, $f|_{X_s}$ is semismall.

f is called **stratified small** with respect to a dense open subset $W\subset Y$ if $\forall s\in\mathscr{S}$, $f|_{X_s}$ is stratified small with respect to W.

Exercise 3.2. f is stratified semismall if and only if there exists a stratification $\{Y_t\}_{t\in\mathscr{T}}$, such that for any $s\in\mathscr{S}$ and $t\in\mathscr{T}$ and each point $y\in Y_t\cap f(X_s)$

$$2\dim(f^{-1}(y)\cap X_s)+\dim Y_t\leq\dim X_s$$

Similarly, f is stratified small with respect to a dense open subset $W \subset Y$ if and only if $\forall y \in W$, $f^{-1}(y)$ is a finite set, and there exists a stratification $\{Y_t\}_{t\in\mathscr{T}}$, such that W is a union of strata and such that $\forall s$ and $\forall y \in Y_t \cap f(X) \subset Y - W$

$$2\dim(f^{-1}(y) \cap X_s) + \dim Y_t < \dim X_s$$

Hint. Sufficiency is obvious, necessity can be proven in the following way: Suppose that we have a family of stratification $\{\mathscr{T}_s\}_{s\in\mathscr{S}}$ of Y such that for any $s \in \mathscr{S}$ and $r \in \mathscr{T}_s$ and each point $y \in Y_r \cap f(X_s)$

$$2\dim(f^{-1}(y) \cap X_s) + \dim Y_r \le \dim X_s$$

then find a common refinement of $\{\mathscr{T}_s\}_{s\in\mathscr{S}}$, denote it by $\mathscr{T}$, we claim that this is a stratification of Y which satisfies the condition in the definition of stratified semismall. In fact, $\forall t \in \mathscr{T}$ and $\forall y \in Y_t \cap f(X_s)$, there exists a $r \in \mathscr{T}_s$ such that $Y_t \subset Y_r$, since $\mathscr{T}$ is a refinement of $\mathscr{T}_s$, in particular $\dim Y_t \le \dim Y_r$, hence

$$2\dim(f^{-1}(y) \cap X_s) + \dim Y_r \le \dim X_s$$

The argument is similar for stratified small morphism. ■

Theorem 3.3. *Assume that X has a good stratification $\{X_s\}_{s\in\mathscr{S}}$, $f : X \to Y$ is a **stratified semismall** morphism, then*

1. *If $F \in {}^p\mathrm{D}^{\le 0}_{\mathscr{S}}(X)$, then $f_! F \in {}^p\mathrm{D}^{\le 0}_c(Y)$;*
2. *If $F \in {}^p\mathrm{D}^{\ge 0}_{\mathscr{S}}(X)$, then $f_* F \in {}^p\mathrm{D}^{\ge 0}_c(Y)$.*

Moreover, if f is also proper then $f_* : D_{\mathscr{S}}(X) \to D_c(Y)$ is t-exact.

Proof. (2) follows from (1) by the Verdier duality. The the case when f is also proper follows from combining (1) with (2). Let's prove (1) by proving that $\forall s \in \mathscr{S}$, and for any $\mathcal{L} \in \mathrm{Loc}(X_s)$, we have $f_!\,\mathrm{IC}(X_s, \mathcal{L}) \in {}^p\mathrm{D}^{\le 0}_c(Y)$. We proceed by Noetherian induction: Suppose that for any $G \in {}^p\mathrm{D}^{\le 0}_{\mathscr{S}}(X)$ supported on $\bar{X}_s - X_s$, we have $f_! G \in {}^p\mathrm{D}^{\le 0}_c(Y)$, we want to prove that $f_!\,\mathrm{IC}(X_s, \mathcal{L}) \in {}^p\mathrm{D}^{\le 0}_c(Y)$.

Notice that there is a short exact sequence of perverse sheaves

$$0 \to K \to i_{s!}\mathcal{L}[\dim X_s] \to \mathrm{IC}(X_s, \mathcal{L}) \to 0$$

such that $\mathrm{Supp}\, K \subset \bar{X}_s - X_s$, so there is a distinguished triangle

$$f_! K \to f_! i_{s!}\mathcal{L}[\dim X_s] \to f_!\,\mathrm{IC}(X_s, \mathcal{L}) \to$$

Since we have $f_! K \in {}^p\mathrm{D}^{\le 0}_c(Y)$, it suffices to prove that $f_! i_{s!}\mathcal{L}[\dim X_s] = (f|_{X_s})_! \in {}^p\mathrm{D}^{\le 0}_c(Y)$. This follows from Theorem 2.1 because $f|_{X_s}$ is semismall. ■

Corollary 3.4. *Assume that X has a good stratification $\{X_s\}_{s\in\mathscr{S}}$, $f : X \to Y$ is a **proper** morphism, TFAE:*

(a) *f is stratified semismall;*
(b) *$f_* : \mathrm{Perv}_{\mathscr{S}}(X) \to \mathrm{Perv}(Y)$;*
(c) *$\forall s \in \mathscr{S}$, $f_*\,\mathrm{IC}(X_s, \underline{\mathbb{C}}) \in \mathrm{Perv}(Y)$;*
(d) *$\forall s \in \mathscr{S}$, $f_* \underline{\mathbb{C}}_{\bar{X}_s}[\dim X_s] \in {}^p\mathrm{D}^{\le 0}_c(Y)$.*

Proof. By the previous theorem, (a) implies (b). (b) implies (c) tautologically. (b) also implies (d) since f_* is t-exact and $\underline{\mathbb{C}}_{\bar{X}_s}[\dim X_s] \in {}^p\mathrm{D}^{\leq 0}_{\mathscr{S}}(X)$.

(d) implies (a): By Lemma 1.4, there is a stratification $\{Y_t\}_{t\in\mathscr{T}}$ such that $\forall t \in \mathscr{T}$, fibers of $f|_{f^{-1}Y_t\cap X_s}$ are either empty or of dimensions $d(s,t)$ which only depends on s and t and

$$\dim(f\times f)^{-1}Y_t \cap (X_s \times_Y X_s) = 2d(s,t) + \dim Y_t$$

Assume that there is a stratum $j_\sigma : X_\sigma \hookrightarrow X$, such that $f|_{X_\sigma}$ is not semismall, then there must be at least one stratum Y_t such that $2d(\sigma,t) + \dim Y_t > \dim X_\sigma$, or equivalently $-\dim X_\sigma + 2\dim f^{-1}(y)\cap X_\sigma > -\dim Y_t$. Note that $d_0 := \dim \mathrm{Supp}\,\underline{\mathbb{C}}_{f^{-1}(y)\cap \bar{X}_\sigma} \geq d(\sigma,t)$. Apply the Lemma 2.3, and we get $\mathcal{H}^{-\dim X+2d_0}(f|_{f^{-1}Y_t})_*\underline{\mathbb{C}}_{f^{-1}Y_t\cap\bar{X}_\sigma} \neq 0$, so that $i_t^* f_*\underline{\mathbb{C}}_{\bar{X}_\sigma}[\dim X_\sigma] \notin \mathrm{D}_c^{\leq -\dim Y_t}(Y_t)$, hence $f_*\underline{\mathbb{C}}_{\bar{X}_\sigma}[\dim X_\sigma] \notin {}^p\mathrm{D}_c^{\leq 0}(Y)$, a contradiction.

(c) implies (d): We proceed by Noetherian induction on startification $\mathscr{S}$. By induction hypothesis, there is an open stratum $j_\sigma : X_\sigma \hookrightarrow X$ of dimension $\dim X$ with complement Z in $\bar{X}_\sigma$, $i : Z \hookrightarrow \bar{X}_\sigma$, such that $\forall s \in \mathscr{S} - \{\sigma\}$, $f_*\,\underline{\mathbb{C}}_{\bar{X}_s}[\dim X_s] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$, we want to show that $f_*\,\underline{\mathbb{C}}_{\bar{X}_\sigma}[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$. First of all, there is a distinguished triangle

$$j_{\sigma!}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \to \underline{\mathbb{C}}_{\bar{X}_\sigma}[\dim X] \to i_*\,\underline{\mathbb{C}}_Z[\dim X] \to$$

Since we have proven that (d) implies (a), so $(f|_{X-X_\sigma})_*$ sends ${}^p\mathrm{D}^{\leq 0}_{\mathscr{S}-\{\sigma\}}(X - X_\sigma)$ to ${}^p\mathrm{D}_c^{\leq 0}(Y)$. We see that $\underline{\mathbb{C}}_Z[\dim X] \in {}^p\mathrm{D}^{\leq 0}_{\mathscr{S}-\{\sigma\}}(X - X_\sigma)$ so $f_* i_*\,\underline{\mathbb{C}}_Z[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$. As a result, it suffices to prove that $f_* j_{\sigma!}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$. Note that there is another distinguished triangle

$$K \to j_{\sigma!}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \to j_{\sigma!*}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \to$$

such that $K \in {}^p\mathrm{D}^{\leq 0}_{\mathscr{S}}(X)$ and supported on Z, hence $f_*K \in {}^p\mathrm{D}_c^{\leq 0}(Y)$. The third term is $\mathrm{IC}(X_s, \underline{\mathbb{C}})$ by definition, so $f_* j_{\sigma!*}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$ by assumption, hence

$$f_* j_{\sigma!}\,\underline{\mathbb{C}}_{X_\sigma}[\dim X] \in {}^p\mathrm{D}_c^{\leq 0}(Y)$$

■

Similar to Theorem 2.4, we have following

Theorem 3.5. *Assume that X has a good stratification $\{X_s\}_{s\in\mathscr{S}}$, $f : X \to Y$ is a* ***proper stratified small*** *morphism with respect to dense open $j : W \hookrightarrow Y$, denote $f|_{f^{-1}W}$ by f', then $\forall F \in \mathrm{Perv}_{\mathscr{S}}(X)$*

$$f_*F \cong j_{!*}(f'_*F|_{f^{-1}W})$$

Proof. Exercise. ■

4. More examples

Kaledin's Theorem. Recall that a smooth variety X is called symplectic if $\dim X$ is even and there exists a 2-form $\Omega \in \Gamma(X, \Omega_X^2)$ such that $\Omega^{\dim X/2}$ is nonzero everywhere, i.e. Ω is nondegenerate. We mention the following theorem ([Kal06] Lemma 2.11) without proof:

Theorem 4.1. *A projective birational morphism from a smooth symplectic variety is semismall.*

This theorem tells us that there is a large class of semismall morphisms. We pick one of them: **Springer resolution**. Consider a reductive group G over $\mathbb{C}$ and the nilpotent cone $\mathcal{N}$ of its Lie algebra $\mathfrak{g}$. Choose a Borel subgroup of G, then there is a projective birational morphism $f : \widetilde{\mathcal{N}} := T^*(G/B) \to \mathcal{N}$ constructed from

$$T^*(G/B) = G \underset{B}{\times} \mathfrak{u} \to \mathfrak{g}$$

where $\mathfrak{u}$ is the Lie algebra of unipotent radical U of B. As a cotangent bundle of a smooth variety, $T^*(G/B)$ is automatically symplectic: the symplectic form $\Omega = d\alpha$ where α is the tautological one form coming from regarding the identity map $T^*(G/B) \to T^*(G/B)$ as a section of $\pi^*\Omega^1_{G/B}$ for $\pi : T^*(G/B) \to G/B$ the natural projection. By Kaledin's Theorem , f is semismall.

In fact, there is another way of proving that f is semismall, using following result ([Hum11] 6.7)

Theorem 4.2 (Dimension Formula)**.** *If G is a reductive algebraic group of rank r over an algebraically closed field, u is a nilpotent element in the Lie algebra $\mathfrak{g}$, then*

$$\dim Z_G(u) = r + 2 \dim f^{-1}(u)$$

From this formula, we see that nilpotent orbit $G \cdot u$ has dimension

$$\dim G - \dim Z_G(u) = \dim G - r - 2 \dim f^{-1}(u) = \dim \widetilde{\mathcal{N}} - 2 \dim f^{-1}(u)$$

i.e. $\dim G \cdot u + 2 \dim f^{-1}(u) = \dim \widetilde{\mathcal{N}}$, since thers are only finitely many nilpotent orbits ([Hum11] 3.9 for groups over complex numbers or fields with good charateristics, [Lus76] for general case) so f is semismall.

Grothendieck's alteration. Grothendieck proposed the following strengthening of Springer resolution:

$$f : \widetilde{\mathfrak{g}} := G \underset{B}{\times} \mathfrak{b} \to \mathfrak{g}$$

sending a pair $(g, x) \in G \times \mathfrak{g}$ to $\mathrm{Ad}_g(x) \in \mathfrak{g}$. It can be shown that f is surjective and projective. Moreover, f can be embedded into a commutative diagram

$$\begin{tikzcd} \widetilde{\mathfrak{g}} \arrow[r, "p"] \arrow[d, "f"] & \mathfrak{t} \arrow[d, "q"] \\ \mathfrak{g} \arrow[r, "\pi"] & \mathfrak{t}/\mathcal{W} \end{tikzcd}$$

Note that $\mathfrak{t}/\mathcal{W} \cong \mathfrak{g}/G$. It has following properties:
(1) Restrict to the regular locus $\mathfrak{g}_{\mathrm{reg}}$, the commutative diagram is Cartesian:

$$\begin{tikzcd} G \underset{B}{\times} \mathfrak{b}_{\mathrm{reg}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{t} \arrow[d] \\ \mathfrak{g}_{\mathrm{reg}} \arrow[r] & \mathfrak{t}/\mathcal{W} \end{tikzcd}$$

(2) Taking fibers over $\{0\}$ of projections $\widetilde{\mathfrak{g}} \to \mathfrak{t}$ and $\mathfrak{g} \to \mathfrak{t}/\mathcal{W}$, we recover Springer's resolution

$$\widetilde{\mathcal{N}} = G \underset{B}{\times} \mathfrak{u} \to \mathcal{N}$$

(3) Restrict to the regular semisimple locus $G \underset{B}{\times} \mathfrak{b}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ is a $\mathcal{W}$-torsor.

Proposition 4.3. *Grothendieck's alteration $f : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is **small** with respect to $\mathfrak{g}_{rs}$.*

Sketch of proof. We learn from the property (3) that f is finite on $f^{-1}\mathfrak{g}_{\mathrm{rs}}$. Consider a stratification of $\mathfrak{t}$ by by $\{\mathfrak{t}_{\underline{\alpha}}\}_{\underline{\alpha}\subset\Phi}$, where Φ is the set of positive roots, and $\mathfrak{t}_{\underline{\alpha}}$ is the hyperplane $\{\underline{\alpha} = 0\}$ removing the union of sub-hyperplanes $\{\underline{\beta} = 0\}$ for $\underline{\beta}$ running through all subsets of Φ such that contain $\underline{\alpha}$ properly. Note that the open stratum $\mathfrak{t}_{\underline{\emptyset}}$ is the regular locus of $\mathfrak{t}$. $\mathfrak{t}_{\underline{\alpha}}$ has a property that for $t \in \mathfrak{t}_{\underline{\alpha}}$, $Z_G(t)$ is connected reductive of rank r and does not depend on t, denote it by $Z_G(\mathfrak{t}_{\underline{\alpha}})$.

Notice that every elements of $\mathfrak{g}$ has a unique Jordan decomposition $x = x_s + x_n$ where $x_s \in \mathfrak{g}_{\mathrm{reg}}$ and $x_n \in \mathcal{N}$, and $[x_s, x_n] = 0$. We can conjugate x_s to an element $t \in \mathfrak{t}$, so x_s is send to an element in the nilpotent cone of $Z_G(t)$, hence x is in the G-orbit of some subvariety $\mathfrak{t}_{\underline{\alpha}} \times S_{\underline{\alpha}}$, where $S_{\underline{\alpha}}$ is a $Z_G(\mathfrak{t}_{\underline{\alpha}})$-orbit in the nilpotent cone of $Z_G(\mathfrak{t}_{\underline{\alpha}})$.

On the one hand, it is not hard to see that $f^{-1}(x)$ is a finite disjoint union of Springer fibers associated to the nilpotent orbit $S_{\underline{\alpha}}$ of reductive group $Z_G(\mathfrak{t}_{\underline{\alpha}})$, hence we can apply the dimension formula:

$$2 \dim f^{-1}(x) + r = \dim Z_{Z_G(\mathfrak{t}_{\underline{\alpha}})}(S_{\underline{\alpha}})$$

On the other hand, $G \cdot (\mathfrak{t}_{\underline{\alpha}} \times S_{\underline{\alpha}})$ is of dimension

$$\dim \mathfrak{t}_{\underline{\alpha}} + \dim G - \dim Z_G(x) = \dim \mathfrak{t}_{\underline{\alpha}} + \dim G - \dim Z_{Z_G(\mathfrak{t}_{\underline{\alpha}})}(S_{\underline{\alpha}})$$

hence we have

$$\dim G \cdot (\mathfrak{t}_{\underline{\alpha}} \times S_{\underline{\alpha}}) + 2 \dim f^{-1}(x) = \dim \mathfrak{t}_{\underline{\alpha}} + \dim G - r$$

since $\mathfrak{t}_{\underline{\alpha}}$ is a proper subvariety of $\mathfrak{t}$ if and only if $\underline{\alpha} \neq \emptyset$, so

$$\dim G \cdot (\mathfrak{t}_{\underline{\alpha}} \times S_{\underline{\alpha}}) + 2 \dim f^{-1}(x) < \dim G = \dim \widetilde{\mathfrak{g}}$$

if and only if $\underline{\alpha} \neq \emptyset$. Hence we conclude that $f : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small with respect to $G \cdot \mathfrak{t}_{\underline{\emptyset}} = \mathfrak{g}_{\mathrm{rs}}$. ■

Hilbert-Chow maps of smooth surfaces. Recall that for a quasi-projective variety X, we have the Hilbert scheme $\underline{\mathrm{Hilb}}_X$ parametrizing closed subvarieties which are proper. There is a connected component of $\underline{\mathrm{Hilb}}_X$ denoted by $\underline{\mathrm{Hilb}}^n_X$, parametrizing closed subvariety of length n, i.e. those $Z \hookrightarrow X$ consisting of finite many points with $\dim \mathcal{O}_Z(Z) = n$. Note that $\underline{\mathrm{Hilb}}^n_X$ is also quasi-projective.

There is a projective and surjective morphism called Hilbert-Chow, sending $\underline{\mathrm{Hilb}}^n_{X,\mathrm{red}}$ (i.e. with reduced structure sheaf) to the nth symmetric power of X, denoted by $X^{(n)}$, which sends an element $Z \in \underline{\mathrm{Hilb}}^n_{X,\mathrm{red}}$ to the finite collection of points

$$\{\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z,x}) \cdot [x]\}_{x\in \mathrm{Supp}\,\mathcal{O}_Z}$$

since $\sum_{x\in\mathrm{Supp}\,\mathcal{O}_Z} \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z,x}) = n$ by definition, the image does live in $X^{(n)}$.

Also recall that for a smooth quasi-projective surface S, $\underline{\mathrm{Hilb}}_S^n$ is smooth of dimension $2n$. All of the basic facts about Hilbert schemes and Hilbert-Chow maps mentioned above can be found in [Fan05] Part 2 and Part 3.

Theorem 4.4. *Suppose that S is a smooth quasi-projective surface over a field k, then the Hilbert-Chow map*

$$f : \underline{\mathrm{Hilb}}_S^n \to S^{(n)}$$

is semismall.

Sketch of proof. First of all, we notice that there is a natural stratification on $S^{(n)}$ given by

$$S^{(n)} = \coprod_{\lambda \in \mathrm{Part}(n)} S_\lambda^{(n)}$$

where λ runs through partitions of n, and $S_\lambda^{(n)}$ is the locally closed subvariety of $S^{(n)}$ parametrizing l points with multiplicity $\{\lambda_1, \lambda_2, \cdots, \lambda_l\}$ such that $\sum_{i=1}^{l} \lambda_i = n$. Note that $\overline{S_\lambda^{(n)}}$ is a diagonal embedding of $S^{(l)}$ into $S^{(n)}$, in particular $\dim S_\lambda^{(n)} = 2l$. It's easy to see that fiber of f over a point $\{x_1, x_2, \cdots, x_l\}$ is a product

$$\prod_{i=1}^{l} \underline{\mathrm{Hilb}}^{\lambda_i}(k[\![x, y]\!])$$

where $\underline{\mathrm{Hilb}}^m(k[\![x, y]\!])$ is the scheme parametrizing quotients of $k[\![x, y]\!]$ of length m. More precisely, it's an *fppf* sheaf sending a k-algebra R to the set of quotients $R[x, y]/I$ which are locally free R-modules of rank m and supported on the zero section of $\mathrm{Spec}\, R[x, y]$. We claim that

$$\dim \underline{\mathrm{Hilb}}^m(k[\![x, y]\!]) \leq m - 1$$

It follows from the claim that

$$\dim S_\lambda^{(n)} + 2 \sum_{i=1}^{l} \dim \underline{\mathrm{Hilb}}^{\lambda_i}(k[\![x, y]\!]) \leq 2l + 2 \cdot \sum_{i=1}^{l} (\lambda_i - 1) = 2n$$

and we finished the proof. It remains to prove the claim, we put it into a lemma below. ∎

Lemma 4.5. *Suppose that $\underline{\mathrm{Hilb}}^m(k[\![x, y]\!])$ is the scheme parametrizing quotiens of $k[\![x, y]\!]$ of length m, then*

$$\dim \underline{\mathrm{Hilb}}^m(k[\![x, y]\!]) \leq m - 1$$

Sketch of proof. Define the following moduli space $\mathcal{M}$ of triples

$$(R \in \mathrm{Alg}_{/k}) \mapsto \{(X, Y, v) | (X, Y) \in \mathcal{N}_{\mathfrak{sl}_m}(R) \times \mathcal{N}_{\mathfrak{sl}_m}(R), v \in R^m, [X, Y] = 0, R[X, Y]v = R^m\}$$

i.e. X and Y are two commuting nilpotent $\mathfrak{sl}_m(R)$-matrices, $v \in R^m$ and $R[X, Y]v$ generates R^m. It's an exercise to show that $\mathcal{M}$ is representable. We also define a morphism $\pi : \mathcal{M} \to \underline{\mathrm{Hilb}}^m(k[\![x, y]\!])$ by sending a triple (X, Y, v) to the module of $k[\![x, y]\!]$ generated by v. π is obviously invariant under the GL_m action, we claim that this is a GL_m-torsor:

(1) π is a surjective morphism between *fppf* sheaves because for any quotient $R[x,y]/I$ which is locally free R-module of rank m and supported on the zero section of $\operatorname{Spec} R[x,y]$, we can localize R to make $R[x,y]/I$ a free module, and associate a triple by sending 1 to v, and x, y to matrices representing them on some basis.

(2) For a k-algebra R, suppose that there are two R-points of $\mathcal{M}$ represented by triples (X,Y,v) and (X',Y',v'), such that $\pi(X,Y,v)\cong\pi(X',Y',v')$ as R-algebra. Since they algebra they generate are isomprphisc, if $\sum_{i,j} a_{ij}X^iY^jv=0$ then $\sum_{i,j} a_{ij}X'^iY'^jv'=0$. It follows that we can define a map from R^m to R^m:

$$\phi:\sum_{i,j} a_{ij}X^iY^jv\mapsto\sum_{i,j} a_{ij}X'^iY'^jv'$$

This is a surjective morphism between free modules hence an ismorphism, i.e. represented by a $\mathrm{GL}_m(R)$-matrix. We conclude that (X,Y,v) and (X',Y',v') are related by a $\mathrm{GL}_m(R)$ transform.

(3) $\mathrm{GL}_m(R)$-action on $\mathcal{M}(R)$ is free, bacause if $g\in\mathrm{GL}_m(R)$ fixes X, Y and v, then it fixes every X^iY^jv, but those elements generates R^m, hence g fixes R^m, i.e. $g=\mathrm{Id}$.

So we see that π is a GL_m-torsor hence $\dim\underline{\mathrm{Hilb}}^m(k[\![x,y]\!])=\dim\mathcal{M}-m^2$.

Next, we observe that $\mathcal{M}$ is a subvariety of the following moduli space

$$\mathcal{M}_0:=\{(X,Y,v)|(X,Y)\in\mathcal{N}_{\mathfrak{sl}_m}\times\mathcal{N}_{\mathfrak{sl}_m},v\in k^m,[X,Y]=0\}$$

so $\dim\mathcal{M}\le\dim\mathcal{M}_0$. Now $\mathcal{M}_0=\mathcal{M}_1\times\mathbb{A}^m$ where

$$\mathcal{M}_1:=\{(X,Y)\in\mathcal{N}_{\mathfrak{sl}_m}\times\mathcal{N}_{\mathfrak{sl}_m}|[X,Y]=0\}$$

hence $\dim\mathcal{M}_0=\dim\mathcal{M}_1+m$. $\mathcal{M}_1$ is a subvariety of

$$\mathcal{M}_2:=\{(X,Y)\in\mathcal{N}_{\mathfrak{sl}_m}\times\mathfrak{sl}_m|[X,Y]=0\}$$

so we have $\dim\mathcal{M}_1\le\dim\mathcal{M}_2$. It remains to estimate the dimension of $\mathcal{M}_2$.

Consider the projection $p:\mathcal{M}_2\to\mathcal{N}_{\mathfrak{sl}_m}$, on every nilpotent orbit $\mathrm{SL}_m\cdot X$, fibers of p are conjugates of $Z_{\mathfrak{sl}_m}(X)$, so

$$\dim p^{-1}(\mathrm{SL}_m\cdot X)\le\dim\mathrm{SL}_m\cdot X+\dim Z_{\mathfrak{sl}_m}(X)=\dim\mathrm{SL}_m\cdot X+\dim Z_{\mathrm{SL}_m}(X)$$

The last equation comes from the smoothness of $Z_{\mathrm{SL}_m}(X)$, which is clear when $\operatorname{char}k=0$ (Cartier's Theorem [Car62]), and is also true in general by an easy computation (leave as an exercise). As a result, $\dim\mathcal{M}_2\le\dim\mathrm{SL}_m=m^2-1$

To sum up, we have

$$\begin{aligned}\dim\underline{\mathrm{Hilb}}^m(k[\![x,y]\!])=\dim\mathcal{M}-m^2&\le\dim\mathcal{M}_0-m^2\\&=\dim\mathcal{M}_1+m-m^2\\&\le\dim\mathcal{M}_2+m-m^2\\&\le m^2-1+m-m^2=m-1\end{aligned}$$

■

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