

# Bohr-Sommerfeld-Heisenberg Quantization of the Mathematical Pendulum\*

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## Abstract

In this paper we give the Bohr-Sommerfeld-Heisenberg quantization of the mathematical pendulum.

## 1 Introduction

The Dirac's formulation of quantum mechanics [7] can be described as a precursor of the theory of  $\mathbb{C}^*$  algebras. Quantum observables are self adjoint operators in a complex vector space of quantum states. In chapter 3 of [8] Dirac represents quantum states as functions on the spectrum of the maximal abelian subalgebra (complete set of commuting observables). Classical Hamiltonian mechanics may be regarded as the limit of quantum mechanics when  $\hbar$  tends to zero. There are quantum systems without classical analogues.

Quantization is an attempt to find a quantum system corresponding to a given classical system. Since there may be several different approaches, quantization may give inequivalent results. Because quantum observables may be represented as operators on the space of functions on the spectrum of the maximal abelian subalgebra, the usual approach to quantization is to identify a complete set of commuting observables and to study operators on its spectrum.

For a completely integrable Hamiltonian system, Bohr-Sommerfeld quantization [1, 12] of the action variables gives rise to a space of quantum states and a complete set of commuting observables acting of this space of states.

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Bohr-Sommerfeld theory does not provide operators of transition between the eigenstates of operators corresponding to the actions. These transitions are accounted for by shifting operators. Because the general theory of these operators requires an extension of geometric quantization to locally Hamiltonian vector fields, which is far a field from the topic of this paper, we refer the reader to [6]. However, we do treat a special case relevant to this paper in the appendix. The commutation relations satisfied by the shifting operators are the same as the commutation relations satisfied by formal quantization of the functions  $e^{\pm i\vartheta}$ , where  $\vartheta$  is an angle in the action angle coordinates for the integrable system. Moreover, if  $\vartheta$  were a single-valued function, then its Hamiltonian vector field  $X_\vartheta$  would generate a local group  $e^{tX_\vartheta}$  of local symplectomorphisms of the phase space preserving the Bohr-Sommerfeld polarization, which would lift to a local group  $e^{tZ_\vartheta}$  of local quantomorphisms. Since the angle  $\vartheta$  is a multi-valued function,  $e^{tZ_\vartheta}$  is not well defined for  $t \neq nh$ , where  $h$  is Planck's constant and  $n \in \mathbb{Z}$ . However, the shifting operators, given by  $e^{\pm hZ_\vartheta}$  are well defined and correspond to the operators of multiplication by  $e^{\pm i\vartheta}$ . The existence of shifting operators answers Heisenberg's criticism [10] of the Bohr-Sommerfeld theory.

In geometric quantization, a complete set of commuting observables corresponds to a polarization. For a completely integrable Hamiltonian system with a regular foliation by Lagrangian tori, we get Bohr-Sommerfeld theory by choosing a polarization tangent to the tori of the foliation [4]. Taking into account the existence of shifting operators, we obtain a *full* geometrically based quantum theory. We do not try to compare the results of our quantization scheme with observations. For readers who would like to compare the energy spectra of the Schrodinger and the Bohr-Sommerfeld quantizations of the mathematical pendulum, we provide implicit equations for the energy spectrum in Bohr-Sommerfeld theory. In interesting completely integrable systems [5], the foliation by tori is not regular and we have to take into account the singularities of the polarization to obtain the Bohr-Sommerfeld quantum spectrum.

In this paper we discuss how to treat the singularities in the mathematical pendulum.

## 2 The classical mathematical pendulum

### 2.1 The basic setup

We consider the classical mathematical pendulum, which is a Hamiltonian system on  $T^*S^1 = \mathbb{R} \times S^1 = \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ , the cotangent bundle of the

circle  $S^1$ , with coordinates  $(p, \alpha)$ , symplectic form  $\omega = dp \wedge d\alpha$ , and 1-form  $\theta = p d\alpha$ . The Hamiltonian of the system is

$$H : T^*S^1 \rightarrow \mathbb{R} : (p, \alpha) \mapsto \frac{1}{2} p^2 - \cos \alpha + 1. \quad (1)$$

The Hamiltonian vector field  $X_H$  of  $H$  satisfies  $X_H \lrcorner (dp \wedge d\alpha) = -p dp - \sin \alpha d\alpha$  so that

$$X_H(p, \alpha) = -\sin \alpha \frac{\partial}{\partial p} + p \frac{\partial}{\partial \alpha}. \quad (2)$$

Its integral curves are solutions of Hamilton's equations

$$\frac{dp}{dt} = -\sin \alpha \quad \text{and} \quad \frac{d\alpha}{dt} = p. \quad (3)$$

The Hamiltonian  $H$  has two critical points: one at  $(0, 0)$  with  $H(0, 0) = 0$  and the other at  $(0, \pi)$  with  $H(0, \pi) = 2$ . These correspond to a stable elliptic and an unstable hyperbolic equilibrium point of  $X_H$ , respectively.

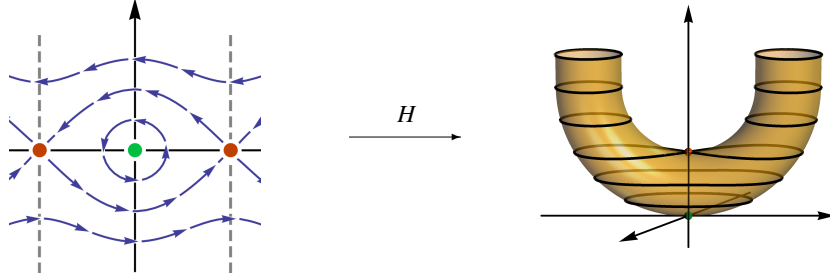


Figure 1. The graph of  $H(x, y) = \frac{1}{2}y^2 - \cos x + 1$  with  $(x, y) \in [-\pi, \pi] \times \mathbb{R}$ .

## 2.2 Action-angle coordinates

In this subsection we find action-angle coordinates  $(I, \vartheta)$  for the mathematical pendulum.

First, we introduce the action function  $I$  on  $T^*S^1$  such that for every connected component  $C(e)$  of the energy level  $H^{-1}(e)$ , the restriction of  $I$  to  $C(e)$  is

$$I(e) = I|_{C(e)} = \frac{1}{2\pi} \int_{C(e)} \theta = \frac{1}{2\pi} \int_{C(e)} p d\alpha. \quad (4)$$

Before giving explicit expressions for  $I$  and  $\vartheta$  we compute the Poisson bracket  $\{I, \vartheta\}$  as follows:

$$\{I, \vartheta\} = L_{X_\vartheta} I = \frac{1}{2\pi} \int_{C(e)} L_{X_\vartheta} \theta = \frac{1}{2\pi} \int_{C(e)} [X_\vartheta \lrcorner d\theta + d(X_\vartheta \lrcorner \theta)]$$

$$= -\frac{1}{2\pi} \int_{\gamma} d\vartheta = -1, \quad (5)$$

since  $\omega = d\theta$  and  $C(e)$  is parametrized by a periodic integral curve  $\gamma$  of  $X_H$  of period  $T = T(e)$ . We reparametrize  $C(e)$  using  $\vartheta = \frac{2\pi}{T} t$ , which is the angle function. Because the matrix of the symplectic form  $\omega$  in action angle coordinates is

$$\left( \begin{pmatrix} 0 & \{I, \vartheta\} \\ \{\vartheta, I\} & 0 \end{pmatrix}^{-1} \right)^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

it follows that  $\omega = dI \wedge d\vartheta$ . Similarly, the Poisson bracket  $\{I, H\}$  is computed as follows:

$$\begin{aligned} \{I, H\} &= L_{X_H} I = \frac{1}{2\pi} \int_{C(e)} L_{X_H} \theta = \frac{1}{2\pi} \int_{C(e)} [X_H \lrcorner d\theta + d(X_H \lrcorner \theta)] \\ &= \frac{1}{2\pi} \int_{\gamma} d(-H + X_H \lrcorner \theta) = 0, \end{aligned}$$

since the curve  $\gamma$  is closed. Thus  $I$  is constant on the integral curves of  $X_H$ . So  $I$  is constant on  $C(e)$ . Consequently,

$$\frac{1}{2\pi} \int_{C(e)} I d\vartheta = \frac{1}{2\pi} I|_{C(e)} \int_{\gamma} d\vartheta = I(e). \quad (6)$$

We now give explicit expressions for the action  $I$  and the angle  $\vartheta$  of the mathematical pendulum. There are two cases.

CASE 1.  $0 < e < 2$ .

We denote by  $I_0$  the restriction of  $I$  to the region  $P_0 = \{(p, \alpha) \in T^*S^1 \mid H(p, \alpha) < 2\}$ . Because  $(0, 0)$  is a nondegenerate minimum of the Hamiltonian  $H$  with minimum value 0, for  $e$  near 0 the level set  $H^{-1}(e)$  is diffeomorphic to a circle  $S^1$  and hence is connected. From the Morse isotopy lemma it follows that for every  $e$  with  $0 < e < 2$  the level set  $H^{-1}(e)$  is diffeomorphic to a circle and hence is connected. By definition

$$I_0(e) = \frac{1}{2\pi} \int_{H^{-1}(e)} p d\alpha = \frac{1}{\pi} \int_{\alpha^-}^{\alpha^+} \sqrt{2(e - (1 - \cos \alpha))} d\alpha, \quad (7)$$

where  $e = 1 - \cos \alpha^{\pm}$ , which implies that  $\alpha^- = -\alpha^+$ , since  $\cos$  is an even function. Therefore

$$I_0(e) = \frac{4}{\pi} e \int_0^{\pi/2} \frac{\cos^2 \varphi}{\sqrt{1 - \frac{e}{2} \sin^2 \varphi}} d\varphi, \quad (8)$$

using the identity  $\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}$  and the change of variables  $\sin \frac{\alpha}{2} = \sin \frac{\alpha^+}{2} \sin \varphi$ . We check some limiting cases. First when  $e \nearrow 2$  we obtain  $\lim_{e \nearrow 2} I_0(e) = \frac{8}{\pi}$ . When  $e \searrow 0$  we find that  $I_0(e) \sim \frac{4e}{\pi} \int_0^{\pi/2} \cos^2 \varphi \, d\varphi = e$ , which is what is given by the harmonic oscillator.

We now find the corresponding angle  $\vartheta_0$ . By definition

$$\vartheta_0 = \frac{2\pi}{T} t = \frac{2\pi}{T} \int_{-\alpha^+}^{\alpha} \frac{d\alpha}{\sqrt{2(e - (1 - \cos \alpha))}} = \frac{4\pi}{T} \int_0^{\varphi} \frac{1}{\sqrt{1 - \frac{e}{2} \sin^2 \varphi}} \, d\varphi,$$

where  $T = T(e)$  is the period of the motion of the mathematical pendulum on  $H^{-1}(e)$ . From Hamilton's equations it follows that

$$T = 4 \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{e}{2} \sin^2 \varphi}} \, d\varphi. \quad (9)$$

Again we check some limiting cases. First, when  $e \nearrow 2$  we find that  $T \nearrow \infty$ . So  $\vartheta_0 \searrow 0$ . Second, when  $e \searrow 0$  we get  $T \searrow 4 \int_0^{\pi/2} d\varphi = 2\pi$ . So  $\vartheta_0 \searrow 2\varphi = \alpha$ , which checks with the angle given by the harmonic oscillator.

CASE 2.  $e > 2$ .

First we find the restrictions  $I_{\pm}$  of  $I$  to the regions  $P_{\pm} = \{(p, \alpha) \in T^*S^1 \mid H(p, \alpha) > 2, \pm p > 0\}$ . Because  $(0, \pi)$  is a nondegenerate critical point of Morse index 1 of the Hamiltonian  $H$  with critical value 2, for  $e > 2$  but near to 2 the level set  $H^{-1}(e)$  is diffeomorphic to the disjoint union of two circles  $C_{\pm}(e)$ . By the Morse isotopy lemma it follows that for all  $e > 2$  the level set  $H^{-1}(e)$  is diffeomorphic to  $C_-(e) \amalg C_+(e)$ . By definition

$$\begin{aligned} I_{\pm}(e) &= \frac{1}{2\pi} \int_{C_{\pm}(e)} p \, d\alpha = \frac{1}{\pi} \int_{-\pi}^{\pi} \sqrt{2(e - (1 - \cos \alpha))} \, d\alpha \\ &= \frac{4\sqrt{2e}}{\pi} \int_0^{\pi/2} \sqrt{1 - \frac{2}{e} \sin^2 \varphi} \, d\varphi. \end{aligned} \quad (10)$$

We check two limiting cases. When  $e \searrow 2$ ,  $\lim_{e \searrow 2} I_{\pm}(e) = \frac{4}{\pi} \int_0^{\pi/2} \cos \varphi \, d\varphi = \frac{4}{\pi}$ , which is one half of the action  $I(e)$  at  $e = 2$ . This is correct because as  $e \searrow 2$  the component  $C_{\pm}(e)$  of  $H^{-1}(e)$  converges to  $H^{-1}(2) \cap \{\pm p \geq 0\}$ . When  $e \nearrow \infty$ , we get  $I_{\pm}(e) \sim \sqrt{2e}$ .

We now find the corresponding angle  $\vartheta_{\pm}$ . By definition

$$\vartheta_{\pm} = \frac{2\pi}{T_{\pm}} t = \frac{2\pi}{T_{\pm}} \int_{-\pi}^{\alpha} \frac{d\alpha}{\sqrt{2(e - (1 - \cos \alpha))}} = \frac{2\pi}{T_{\pm}} \sqrt{\frac{2}{e}} \int_0^{\varphi} \frac{1}{\sqrt{1 - \frac{2}{e} \sin^2 \varphi}} \, d\varphi.$$

where  $T_{\pm} = T_{\pm}(e)$  is the period of the motion of the mathematical pendulum on  $H^{-1}(e)$ . From Hamilton's equations it follows that

$$T_{\pm} = \int_{-\pi}^{\pi} \frac{d\alpha}{\sqrt{2(e - (1 - \cos \alpha))}} = \sqrt{\frac{2}{e}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{2}{e} \sin^2 \varphi}} d\varphi. \quad (11)$$

Again we check some limiting cases. First, when  $e \searrow 2$  we find that  $T_{\pm} \nearrow \infty$ . So  $\vartheta_{\pm} \searrow 0$ . Second, when  $e \nearrow \infty$  we get  $T_{\pm} \sim \frac{\pi}{\sqrt{2e}}$ . So  $\vartheta \sim 4\varphi = 2\alpha$ .

It follows from the above discussion that the action function  $I$ , defined by equation (4) is continuous on  $[0, \infty)$ . However,  $I(e)$  is not smooth at  $e = 2$ , see Dullin [9].

### 3 Elements of geometric quantization

In this section, we review the elements of geometric quantization applicable to the mathematical pendulum following [11].

Consider a trivial complex line bundle  $L = \mathbb{C} \times T^*S^1$  with projection map  $\rho : L \rightarrow T^*S^1 : (z, (p, \alpha)) \mapsto (p, \alpha)$  and trivializing section  $\lambda_0 : T^*S^1 \rightarrow L : (p, \alpha) \mapsto (1, (p, \alpha))$ . Define a connection  $\nabla$  on  $L$  by setting

$$\nabla \lambda_0 = -i\hbar^{-1}\theta \otimes \lambda_0, \quad (12)$$

where  $\hbar$  is Planck's constant divided by  $2\pi$  and  $\theta = p d\alpha$  is the canonical 1-form on  $T^*S^1$ . Since  $\omega = d\theta$ , it follows that the curvature of the connection  $\nabla$  is  $\frac{-1}{2\pi\hbar} \omega$ .

We consider the geometric quantization of the mathematical pendulum with respect to the singular polarization  $D$  of  $T^*S^1$  consisting of all integral curves of the Hamiltonian vector field  $X_H$  (2)<sup>1</sup> associated to the Hamiltonian function  $H$  (1). This means that quantum states of the mathematical pendulum are represented by sections of the prequantization line bundle  $L$  that are covariantly constant along  $D$ . It should be noted that this representation is not unique. Multiplication of every section by a constant phase factor leads to an equivalent representation. For  $e \notin \{0, 2\}$ , the leaves of  $D$  are smooth and are topological circles due to the conclusions of subsection 2.2. Moreover, this polarization has singularities consisting of the equilibrium points

<sup>1</sup>Throughout this paper we will use the shorthand *mathsymbol(Number)* to mean *mathsymbol* given in equation (*Number*). For example,  $X_H(2)$  means  $X_H$  given in equation (2).

$(0, 0)$  and  $(0, \pi)$  of  $X_H$  and two homoclinic orbits of  $X_H$ , which have  $(0, \pi)$  as a common boundary. Therefore, we are extending geometric quantization to a *singular* polarization, which leads to the difficulties encountered here.

## 4 Bohr-Sommerfeld conditions

Consider an integral curve  $\gamma : \mathbb{R} \rightarrow T^*S^1 : t \mapsto \gamma(t) = (p(t), \alpha(t))$  of the Hamiltonian vector field  $X_H$ . Suppose that  $e = H(\gamma(t))$  is not 0 or 2. Then,  $\gamma$  is periodic with period  $T \neq 0$ . The cases when  $e = 0$  and  $e = 2$  will be discussed separately.

Let  $\sigma : T^*S^1 \rightarrow L$  be a section of the prequantization line bundle that is covariantly constant along  $D$ . Then  $\gamma^*\sigma : \mathbb{R} \rightarrow L$  is a horizontal lift of  $\gamma$  to  $L$ .  $\gamma^*\sigma$  is periodic with period  $T$  if either the restriction of the connection  $\nabla$  to the image of  $\gamma$  has trivial holonomy group, or  $\sigma$  restricted to the image of  $\gamma$  is identically zero.

**Theorem 4.1** *Let  $\gamma : [0, T] \rightarrow T^*S^1$  be a periodic integral curve of  $X_H$  with period  $T \neq 0$ . The holonomy group of  $\nabla$ , restricted to the image  $\text{im } \gamma$  of  $\gamma$ , is trivial if and only if the action integral*

$$I_\gamma = I|_{\text{im } \gamma} = \frac{1}{2\pi} \int_0^T \gamma^*\theta \, dt = n\hbar, \quad (13)$$

for some  $n \in \mathbb{Z}$ .

**Proof.** Consider an integral curve  $\gamma : \mathbb{R} \rightarrow T^*S^1 : t \mapsto \gamma(t) = (p(t), \alpha(t))$  of the Hamiltonian vector field  $X_H$ . It satisfies Hamilton's equations (3). Suppose that  $e = H(\gamma(t)) \in (0, 2)$ . Then, the curve  $\gamma$  is periodic with period  $T$ , see (9). Let

$$\tilde{\gamma} : [0, T] \rightarrow L : t \mapsto (z(\gamma(t)), \gamma(t)) = z(\gamma(t)) \lambda_0(\gamma(t))$$

be a horizontal lift of  $\gamma$ . Then the covariant derivative  $\frac{D}{dt}\tilde{\gamma}(t)$  of  $\tilde{\gamma}$  must vanish. Equation (12) implies that

$$\begin{aligned} \frac{D}{dt}\tilde{\gamma}(t) &= \frac{D}{dt}(z(\gamma(t))\lambda_0(\gamma(t))) \\ &= \frac{dz}{dt}\lambda_0(\gamma(t)) - i\hbar^{-1}z\langle\theta | X_H\rangle(\gamma(t))\lambda_0(\gamma(t)) \\ &= \left(\frac{dz}{dt} - i\hbar^{-1}z\langle p \, d\alpha | -\sin \alpha \frac{\partial}{\partial p} + p \frac{\partial}{\partial \alpha}\rangle\right)\lambda_0(\gamma(t)) \end{aligned}$$

$$= \left( \frac{dz}{dt} - i\hbar^{-1}p(t)^2 z \right) \lambda_0(\gamma(t)),$$

where  $\frac{1}{2}p(t)^2 - \cos \alpha(t) + 1 = e$ . Hence,  $p(t) = \pm\sqrt{2(e - (1 - \cos \alpha(t)))}$ . Because  $\frac{dz}{dt} = \frac{dz}{d\alpha} \frac{d\alpha}{dt} = \frac{dz}{d\alpha} p$ , the curve  $\tilde{\gamma}$  is horizontal (= covariantly constant) if  $\frac{dz}{d\alpha} p(t) - i\hbar^{-1}p(t)^2 z = p(t) \left( \frac{dz}{d\alpha} - i\hbar^{-1}p(t)z \right) = 0$ , that is,

$$-i\hbar \frac{1}{z} \frac{dz}{d\alpha} = \pm\sqrt{2(e - (1 - \cos \alpha))}. \quad (14)$$

Here the + sign corresponds to  $\alpha \in [0, \alpha^+]$  and the - sign corresponds to  $\alpha \in [\alpha^-, 0] = [-\alpha^+, 0]$ . Integrating (14) from  $\alpha^-$  to  $\alpha^+$  and using the fact that  $\cos$  is an even function, we get

$$-i\hbar \ln \left( \frac{z(\alpha^+)}{z(\alpha^-)} \right) = 2 \int_{\alpha^-}^{\alpha^+} \sqrt{2(e - (1 - \cos \alpha))} d\alpha = 2\pi I(e)$$

by equation (6). The horizontal lift  $\tilde{\gamma}$  of the closed curve  $\gamma$  is a closed curve in the line bundle  $L$  if and only if  $z(\alpha^+) = z(\alpha^-)$ . Since  $\ln$  is a multivalued function and  $\ln 1 = 2\pi ni$ , it follows that  $\tilde{\gamma}$  is a closed curve in  $L$  if and only if we have  $I|_{C(e)} = n\hbar$ .

For  $e > 2$ , we have  $H^{-1}(e) = C_-(e) \amalg C_+(e)$ , and there are integral curves  $\gamma^-$  and  $\gamma^+$  of  $X_H$  such that where  $C_-(e)$  is the image of  $\gamma^-$  and  $C_+(e)$  are image of  $\gamma^+$ . The same argument as in the preceding paragraph shows that the horizontal lift  $\tilde{\gamma}^-$  of  $\gamma^-$  is a closed curve in  $L$  if and only if  $I_-(e) = m_- \hbar$ , where  $m_-$  is an integer. Similarly, the horizontal lift  $\tilde{\gamma}^+$  of  $\gamma^+$  is a closed curve in  $L$  if and only if  $I_+(e) = m_+ \hbar$ , where  $m_+$  is an integer. Since  $I_-(e) = I_+(e)$ , it follows that  $m_- = m_+$ . Moreover,  $I_-(e) = I|_{C_-(e)}$  and  $I_+(e) = I|_{C_+(e)}$ . ■

Equation (13) gives the Bohr-Sommerfeld conditions discussed in the introduction. The action integral is independent of the parametrization of  $\gamma$  within its orientation class. However, the change of orientation of  $\gamma$  would lead to the change from  $n$  to  $-n$ . Therefore, the Bohr-Sommerfeld condition (13) depends only on the image of  $\gamma$ . In the following, we shall refer to the image an integral curve  $\gamma$  of  $X_H$  that satisfies equation (13) as a *Bohr-Sommerfeld torus*. The integer  $n$  on the right hand side of equation (13) is called the *quantum number* of the corresponding Bohr-Sommerfeld torus. Since integral curves of  $X_H$  preserve the Hamiltonian  $H$ , we may rewrite equation (13) in the form

$$I|_{C(e)} = \frac{1}{2\pi} \int_{C(e)} p d\alpha = n\hbar, \quad (15)$$



where  $C(e)$  is a connected component of the energy level  $H^{-1}(e)$ . Thus, Bohr-Sommerfeld conditions (13) impose conditions on the energy. The set of values of the energy allowed by Bohr-Sommerfeld conditions is interpreted as the *quantum energy spectrum* of the system.

From the discussion preceding theorem 4 it follows that a section  $\sigma$  of the prequantum line bundle, which is covariantly constant along  $D$ , has support contained in the union of Bohr-Sommerfeld tori and the energy levels  $H^{-1}(0)$  and  $H^{-1}(2)$ . Since  $H^{-1}(0)$  is a critical point, the restriction of  $\sigma$  to  $H^{-1}(0)$  is the value of  $\sigma$  at  $H^{-1}(0)$  which is not restricted by the condition that  $\sigma$  is covariantly constant along  $D$ . So we may allow the value  $e = 0$  in equation (15). On the one hand, we consider  $H^{-1}(0)$  as a (singular) Bohr-Sommerfeld torus corresponding to the quantum number  $n = 0$ . On the other hand, we assume that the singular level set  $H^{-1}(2)$  is *not* a (singular) Bohr-Sommerfeld torus.

Since a section of the prequantization line bundle that is covariantly constant along  $D$  has its support in the union of Bohr-Sommerfeld tori, which has empty interior, such sections can be smooth only in the sense of distributions. Therefore, we adopt the following definition.

**Definition 4.2** *A quantum state of the mathematical pendulum is a section  $\sigma$  of the prequantization line bundle  $\rho$ , whose support lies in the union of Bohr-Sommerfeld tori such that for each Bohr-Sommerfeld torus  $C$  the restriction  $\sigma|_C$  of  $\sigma$  to  $C$  is a smooth covariantly constant section of  $\rho|_C$ .*

Let  $\mathfrak{H}$  be the space of quantum states of the mathematical pendulum. For each Bohr-Sommerfeld torus  $C$ , we choose a non-vanishing smooth covariantly constant section  $\sigma$  of  $L|_C$ . The family  $\{\sigma|_C\}$  is a basis of  $\mathfrak{H}$ , which we shall refer to as a Bohr-Sommerfeld basis. Give  $\mathfrak{H}$  a hermitian scalar product  $(\cdot | \cdot)$  so that the Bohr-Sommerfeld basis  $\{\sigma|_C\}$  is orthonormal. Thus, we have obtained a vector space structure on the space of states of the mathematical pendulum. Note that this structure is not uniquely determined by the geometry of the classical phase space. We have the freedom of multiplying each basis vector  $\sigma|_C$  by a nonzero complex number.

**Definition 4.3** *A function  $f \in C^\infty(T^*S^1)$  is Bohr-Sommerfeld quantizable if it is constant on Bohr-Sommerfeld tori. Bohr-Sommerfeld quantization assigns to a quantizable function  $f$  a linear operator  $\mathbf{Q}_f$  on  $\mathfrak{H}$  such that, for each Bohr-Sommerfeld torus  $C$*

$$\mathbf{Q}_f \sigma|_C = f|_C \sigma|_C. \tag{16}$$

Observe that the operators  $\mathbf{Q}_f$  corresponding to Bohr-Sommerfeld quantizable functions  $f$  are diagonal in the Bohr-Sommerfeld basis. Since the Bohr-Sommerfeld tori are closed and mutually disjoint, for any function  $C \mapsto \lambda_C$  on the collection of Bohr-Sommerfeld tori, there exists a function  $f \in C^\infty(T^*S^1)$  such that  $f|_C = \lambda_C$ . Thus, each basis vector  $\sigma|_C$  is an eigenvector of the operator  $\mathbf{Q}_f$  corresponding to an eigenvalue  $\lambda_C$ .

## 5 Quantization away from the singularity

### 5.1 Structure of the Bohr-Sommerfeld basis

We now study of the structure of the Bohr-Sommerfeld basis  $\{\sigma|_C\}$ .

The energy level  $H^{-1}(2)$  divides  $T^*S^1$  into three open subsets:  $P_0 = \{(p, \alpha) \mid H(p, \alpha) < 2\}$  and  $P_\mp = \{(p, \alpha) \mid H(p, \alpha) > 2 \text{ and } \mp p > 0\}$ . Let  $\mathfrak{H}_0$  be the subspace of  $\mathfrak{H}$  consisting of sections with support in  $P_0$ . Similarly, let  $\mathfrak{H}_\mp$  be the subspaces of  $\mathfrak{H}$  consisting of sections with support in  $P_\mp$ . Then  $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_- \oplus \mathfrak{H}_+$  and  $\{\sigma|_C\}_{0,\pm} = \{\sigma|_C \mid C \subseteq P_{0,\pm}\}$  are bases of  $\mathfrak{H}_0$ ,  $\mathfrak{H}_+$ , and  $\mathfrak{H}_-$ , respectively.

A Bohr-Sommerfeld torus in  $P_0$  can be labelled by its quantum number  $n = 0, \dots, N$ , where  $N$  is the largest nonnegative integer such that  $N\hbar < I(2)$ . Thus  $\{\sigma|_C\}_0 = \{\sigma_0^0, \dots, \sigma_N^0\}$ , where the subscript  $n = 0, \dots, N$  is the quantum number of the state  $\sigma_n^0$  and the superscript 0 reminds us that  $\sigma_n^0$  lies in  $\mathfrak{H}_0$ . Similarly a Bohr-Sommerfeld torus in  $P_\pm$  can be labeled by its quantum number  $m \geq M$ , where  $2M$  is the smallest even nonnegative integer greater than or equal to  $N + 1$ . In other words,

$$M = \min\{m \in \mathbb{Z}_{>0} \mid 2m \geq N + 1\} = \begin{cases} \frac{1}{2}(N + 2), & \text{if } N \text{ is even} \\ \frac{1}{2}(N + 1), & \text{if } N \text{ is odd.} \end{cases} \quad (17)$$

Hence the Bohr-Sommerfeld basis of  $\mathfrak{H}_\pm$  is  $\{\sigma|_C\}_\pm = \{\sigma_m^\pm, m \geq M\}$ , where the subscript  $m$  is the quantum number of the state  $\sigma_m^\pm$  in  $\mathfrak{H}_\pm$ . Since  $N < 2M$  the basis  $\{\sigma|_C\}$  of  $\mathfrak{H}$  has the lattice structure

$$\begin{array}{c} \sigma_0^0 \Leftrightarrow \sigma_1^0 \Leftrightarrow \dots \Leftrightarrow \sigma_N^0 \begin{array}{l} \nearrow \sigma_M^+ \Leftrightarrow \sigma_{M+1}^+ \Leftrightarrow \dots \\ \searrow \sigma_M^- \Leftrightarrow \sigma_{M+1}^- \Leftrightarrow \dots \end{array} \end{array} \quad (18)$$

The structure of the Bohr-Sommerfeld set can be used to study the energy spectrum of the mathematical pendulum, which consists of values of  $e_n$  such

that a connected component  $C(e_n)$  of the energy level  $H^{-1}(e_n)$  satisfies Bohr-Sommerfeld conditions

$$\frac{1}{2\pi} \int_{C(e_n)} p d\alpha = n\hbar$$

for some integer  $n$ . See the discussion following equation (12). The part of energy spectrum contained in the interval  $[0, 2]$  is simple, and it can be obtained by solving for  $e_n$  equation

$$n\hbar = \frac{4}{\pi} e_n \int_0^{\pi/2} \frac{\cos^2 \varphi}{\sqrt{1 - \frac{e_n}{2} \sin^2 \varphi}} d\varphi,$$

where  $0 \leq n \leq N$ , and  $N$  is the largest positive integer such that  $e_N < 2$ . We have assumed that  $e = 2$  is not in the energy spectrum of the mathematical pendulum. For  $e > 2$ , The part of the energy spectrum contained in the half line  $[2, \infty)$  can be obtained by solving for  $e_m$  the equation

$$m\hbar = \frac{1}{\pi} \int_{-\pi}^{\pi} \sqrt{2(e_m - (1 - \cos \alpha))} d\alpha,$$

where  $2m \geq N + 1$  ensures that  $e_m > 2$ . In this range, each eigenspace is 2-dimensional.

## 5.2 Transitions between quantum states

In this subsection we discuss the transitions between quantum states given by the horizontal arrows in diagram (18). The transitions from  $\sigma_N^0$  to  $\sigma_M^\pm$  given by slanted arrows in diagram (18) involve crossing the energy level 2, where the action  $I$  is continuous but not differentiable. This requires understanding the  $\mathbb{Z}_2$  symmetry of the mathematical pendulum, which will be treated in the next section.

In diagram (18) transitions involving the right pointing horizontal arrows correspond to the action of an operator  $\mathbf{b}$  on  $\mathfrak{H}$  such that

$$\mathbf{b} \sigma_n^0 = \sigma_{n+1}^0, \text{ for } n = 0, 1, \dots, N - 1 \quad (19)$$

and

$$\mathbf{b} \sigma_m^\pm = \sigma_{m+1}^\pm, \text{ for } m = M, M + 1, \dots \quad (20)$$

We refer to  $\mathbf{b}$  as the *raising* operator on  $\mathfrak{H}$ . Transitions involving the left pointing horizontal arrows give rise to the *lowering* operator  $\mathbf{a}$  such that

$$\mathbf{a} \sigma_n^0 = \sigma_{n-1}^0, \text{ for } n = 1, \dots, N \quad (21)$$

and

$$\mathbf{a}\sigma_m^\pm = \sigma_{m-1}^\pm, \text{ for } m = M + 1, \dots \quad (22)$$

Since  $\sigma_0^0$  is the lowest point in the lattice, we require that

$$\mathbf{a}\sigma_0^0 = 0. \quad (23)$$

Thus in diagram (18) the lowering operator  $\mathbf{a}$  corresponds to left pointing horizontal arrows, while the raising operator  $\mathbf{b}$  corresponds to right pointing horizontal arrows.<sup>2</sup> In order to interpret the slanted arrows in the diagram we need to discuss the  $\mathbb{Z}_2$  symmetry of the mathematical pendulum. The shifting operators  $\mathbf{a}$  and  $\mathbf{b}$  will be constructed by lifting the shifting operator in the  $\mathbb{Z}_2$ -reduced quantum system. In the appendix we construct this shifting operator using geometric quantization.

In order to identify the function whose quantization might lead to the operator  $\mathbf{b}$  we extend Dirac's quantization rule

$$[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = i\hbar \mathbf{Q}_{\{f_1, f_2\}} \quad (24)$$

to complex valued functions. Action angle coordinates  $(I, \vartheta)$  on  $P_0 \cup P_+ \cup P_-$  restrict to action angle coordinates  $(I_0, \vartheta_0)$  on  $(P_0, \omega|_{P_0})$  and  $(I_\pm, \vartheta_\pm)$  on  $(P_\pm, \omega|_{P_\pm})$ , respectively. The latter action angle coordinates have been computed in section 2.2. They satisfy the Poisson bracket relations  $\{I_0, \vartheta_0\} = -1$  and  $\{I_\pm, \vartheta_\pm\} = -1$  on  $P_0$  and  $P_\pm$ , respectively. Therefore

$$\{I_0, e^{i\vartheta_0}\} = -ie^{i\vartheta_0} \text{ and } \{I_\pm, e^{i\vartheta_\pm}\} = -ie^{i\vartheta_\pm}. \quad (25)$$

If we introduce the quantum operator  $\mathbf{Q}_{e^{i\vartheta}}$  on  $\mathfrak{H}$ , equations (24) and (25) imply

$$[\mathbf{Q}_{I_0}, \mathbf{Q}_{e^{i\vartheta_0}}] = -i\hbar \mathbf{Q}_{e^{i\vartheta_0}} \text{ and } [\mathbf{Q}_{I_\pm}, \mathbf{Q}_{e^{i\vartheta_\pm}}] = -i\hbar \mathbf{Q}_{e^{i\vartheta_\pm}}. \quad (26)$$

On the other hand, equation (16) yields

$$\mathbf{Q}_{I_0}\sigma_n^0 = n\hbar\sigma_n^0, \text{ for } n = 0, 1, \dots, N \quad (27)$$

and

$$\mathbf{Q}_{I_\pm}\sigma_m^\pm = m\hbar\sigma_m^\pm, \text{ for } m = M, M + 1, \dots \quad (28)$$

---

<sup>2</sup>For small positive quantum numbers, when the value of  $e$  is slightly above zero, the mathematical pendulum is well approximated by the 1-dimensional harmonic oscillator. In this situation the shifting operators  $\mathbf{a}$  and  $\mathbf{b}$  are well approximated by the classical raising and lowering operators of the harmonic oscillator. We do not discuss the complex analytic nature of this approximation.

Equations (19) and (20) imply that the operator  $\mathbf{b}$  and the quantized actions satisfy the commutation relations

$$[\mathbf{Q}_{I_0}, \mathbf{b}] \sigma_n^0 = -i\hbar \mathbf{b} \sigma_n^0 \quad \text{and} \quad [\mathbf{Q}_{I_\pm}, \mathbf{b}] \sigma_m^\pm = -i\hbar \mathbf{b} \sigma_m^\pm \quad (29)$$

for  $n = 0, 1, \dots, N-1$  and  $m = M, M+1, \dots$ , respectively. Comparing equations (26) and (29) shows that the raising operator  $\mathbf{b}$  defined in equations (19) and (20) satisfy the same commutation relations as the operator  $\mathbf{Q}_{e^{i\vartheta}}$ .

It is of interest to see to what degree the raising and the lowering operators on  $\mathfrak{H}$  correspond to quantization of classical functions on  $P = P_0 \cup P_- \cup P_+$ . Their restrictions to the subspaces  $\mathfrak{H}_0$ ,  $\mathfrak{H}_-$  and  $\mathfrak{H}_+$  of  $\mathfrak{H}$  with supports in  $\mathfrak{H}_0$ ,  $\mathfrak{H}_-$  and  $\mathfrak{H}_+$ , respectively, can be related to quantization of the functions  $e^{\pm i\vartheta_0}$ ,  $e^{\pm i\vartheta_-}$ , and  $e^{\pm i\vartheta_+}$  on  $P_0$ ,  $P_+$  and  $P_-$ , respectively.

Choose the basic sections of basic sections  $\{\sigma_n^0\}_{n=1}^N$  in  $\mathfrak{H}_0$  so that  $e^{i\vartheta_0} \sigma_n^0 = \mathbf{a} \sigma_{n+1}^0$  for all  $1 \leq n \leq N$ . Because  $\mathbf{b}_0 \circ \mathbf{a}_0 \sigma_n^0 = \sigma_n^0$  for all  $1 \leq n \leq N$ , it follows that the restriction  $\mathbf{b}_0$  of the raising operator  $\mathbf{b}$  to  $\mathfrak{H}_0$  may be interpreted as the operator  $\mathbf{Q}_{e^{i\vartheta_0}}$ , which sends  $\sigma_n^0$  into  $e^{i\vartheta_0} \sigma_n^0$  for  $1 \leq n \leq N$ . Similarly, the restriction  $\mathbf{b}_\mp$  of the raising operator  $\mathbf{b}$  to  $\mathfrak{H}_\mp$  may be interpreted as the operator  $\mathbf{Q}_{e^{i\vartheta_\mp}}$  of multiplication by  $e^{i\vartheta_{0,\mp}}$ . In other words,  $\mathbf{b}_- = \mathbf{Q}_{e^{i\vartheta_-}}$ . Similarly,  $\mathbf{b}_+ = \mathbf{Q}_{e^{i\vartheta_+}}$ , is the restriction of the raising operator  $\mathbf{b}$  to  $\mathfrak{H}_+$ . On the other hand, the restriction  $\mathbf{b}_0$  of the raising operator  $\mathbf{b}$  to  $\mathfrak{H}_0$  agrees with  $e^{i\vartheta_0}$  except on  $\sigma_n^0$  because  $\mathbf{b} \sigma_n^0$  is not in  $\mathfrak{H}_0$ . In the same way, the lowering operators  $\mathbf{a}_0$ ,  $\mathbf{a}_-$ , and  $\mathbf{a}_+$  may be interpreted in terms of the multiplication operators  $\mathbf{Q}_{e^{i\vartheta_0}}$ ,  $\mathbf{Q}_{e^{i\vartheta_-}}$ , and  $\mathbf{Q}_{e^{i\vartheta_+}}$ , respectively.

## 6 The $\mathbb{Z}_2$ -symmetry

The mathematical pendulum  $(H, T^*S^1, \omega)$  has a  $\mathbb{Z}_2$ -symmetry generated by

$$\zeta : T^*S^1 \rightarrow T^*S^1 : (p, \alpha) \mapsto (-p, -\alpha), \quad (30)$$

because the Hamiltonian  $H$  (1) and the symplectic form  $\omega = dp \wedge d\alpha$  are invariant. In more detail, for every  $(p, \alpha) \in T^*S^1$  we have  $(\zeta^*H)(p, \alpha) = H(-p, -\alpha) = H(p, \alpha)$ . So  $H$  is invariant. Also the 1-form  $\theta = p d\alpha$  is invariant, because  $(\zeta^*\theta)(p, \alpha) = (-p)d(-\alpha) = \theta(p, \alpha)$ . This implies that the 2-form  $\omega$  is invariant, because  $\zeta^*\omega = \zeta^*(d\theta) = d(\zeta^*\theta) = \omega$ .

The quantized mathematical pendulum  $(H, T^*S^1, \omega)$  has quantum line bundle

$$\rho : L = \mathbb{C} \times T^*S^1 \rightarrow T^*S^1 : (z, (p, \alpha)) \mapsto (p, \alpha) \quad (31)$$

with covariant derivative  $\nabla = -i\hbar\theta \otimes \lambda_0$ , where  $\lambda_0 : T^*S^1 \rightarrow L : (p, \alpha) \mapsto (1, (p, \alpha))$  is a section, which trivializes the bundle  $\rho$ . The bundle space  $L$  has a  $\mathbb{Z}_2$ -symmetry generated by the mapping

$$\mu : L \rightarrow L : (z, (p, \alpha)) \mapsto (z, (-p, -\alpha)). \quad (32)$$

The mapping  $\mu$  covers the  $\mathbb{Z}_2$ -symmetry of the mathematical pendulum generated by  $\zeta$  (30), since  $\rho \circ \mu = \zeta \circ \rho$ .

### 6.1 The $\mathbb{Z}_2$ -symmetric quantum system

Consider the  $\mathbb{Z}_2$ -symmetry on  $L$  generated by the mapping  $\mu$  (32). The  $\mathbb{Z}_2$ -symmetric quantized system is the mathematical pendulum with  $\mathbb{Z}_2$ -symmetry generated by  $\zeta$  (30) and quantum line bundle  $\rho$  (31) having the  $\mathbb{Z}_2$ -symmetry generated by  $\mu$  (32).

Let  $\Gamma(\rho)$  be the vector space of smooth sections of the bundle  $\rho$ .

**Lemma 6.1.1** *The  $\mathbb{Z}_2$  action on  $L$  generated by the mapping  $\mu$  (32) induces a  $\mathbb{Z}_2$ -action on  $\Gamma(\rho)$  generated by the linear map*

$$\mu^* : \Gamma(\rho) \rightarrow \Gamma(\rho) : \sigma \mapsto \mu^*(\sigma) = \mu^{-1} \circ \sigma \circ \zeta. \quad (33)$$

**Proof.** Let  $\sigma : T^*S^1 \rightarrow L : (p, \alpha) \mapsto (z(p, \alpha), (p, \alpha))$  be a smooth section of the bundle  $\rho$ . Then

$$(\mu^* \sigma)(p, \alpha) = \mu^{-1}(z(\zeta(p, \alpha)), \zeta(p, \alpha)) = (z(\zeta(p, \alpha)), (p, \alpha)) \quad (34)$$

is a section of the bundle  $\rho$ . So

$$\mu^*(\mu^*(\sigma))(p, \alpha) = (z(\zeta^2(p, \alpha)), (p, \alpha)) = \sigma(p, \alpha),$$

that is,  $(\mu^*)^2 \sigma = \sigma$ . ■

The mapping  $\zeta : T^*S^1 \rightarrow T^*S^1$  (30) acts on the set of all Bohr-Sommerfeld tori. Hence it induces an operator

$$\mathbf{P} : \mathfrak{H} \rightarrow \mathfrak{H} : \sigma|_C \mapsto \mu^*(\sigma|_C), \quad (35)$$

which we call the *parity* operator. Since the mapping  $\mu$  (32) generates a  $\mathbb{Z}_2$ -action on  $L$ , it follows that  $\mu^*$  (33) generates a representation of  $\mathbb{Z}_2$  on  $\mathfrak{H}$ . From the fact that the Hamiltonian  $H$  (1) is invariant under the  $\mathbb{Z}_2$ -action on  $T^*S^1$ , we get  $[\mathbf{Q}_H, \mathbf{P}] = 0$ . Moreover,

**Lemma 6.1.2** The maps

$$\mathbf{P}_{0,\pm} = \mathbf{P}|_{\mathfrak{H}_{0,\pm}} : \mathfrak{H}_{0,\pm} \rightarrow \mathfrak{H}_{0,\mp}$$

are bijective involutions, which implies  $\mathbf{P}_0^{-1} = \mathbf{P}_0$  and  $\mathbf{P}_{\pm}^{-1} = \mathbf{P}_{\mp}$ .

**Proof.** Recall that if  $\sigma \in \Gamma(\rho)$ , then  $\sigma = f\lambda_0$  for some smooth function  $f$  on  $T^*S^1$ . The support  $\text{supp } \sigma$  of  $\sigma$  is the support  $\text{supp } f$  of the function  $f$ , which is  $\{(p, \alpha) \in T^*S^1 | f(p, \alpha) \neq 0\}$ . Suppose that  $\sigma|_{C_{0,\pm}(e)} = z_{0,\pm}(p, \alpha)\lambda_0(p, \alpha) \in \mathfrak{H}_{0,\pm}$ . Then  $\text{supp } \sigma|_{C_{0,\pm}(e)} = C_{0,\pm}(e)$ . From (34) we get  $(\mathbf{P}_{0,\pm}(\sigma|_{C_{0,\pm}(e)}))(p, \alpha) = z_{0,\pm}(\zeta(p, \alpha))\lambda_0(p, \alpha)$ . So we obtain

$$(p, \alpha) \in \text{supp } \mathbf{P}_{0,\pm}(\sigma|_{C_{0,\pm}(e)}) \text{ if and only if } \zeta(p, \alpha) \in \text{supp } z_{0,\pm} = C_{0,\pm}(e)$$

if and only if  $(p, \alpha) \in \zeta(C_{0,\pm}(e)) = C_{0,\mp}(e)$ . So  $\mathbf{P}_{0,\pm}(\sigma|_{C_{0,\pm}(e)}) \in \mathfrak{H}_{0,\mp}$ . Thus  $\mathbf{P}_{0,\pm}$  maps  $\mathfrak{H}_{0,\pm}$  onto  $\mathfrak{H}_{0,\mp}$ . From

$$\begin{aligned} (\mathbf{P}_{0,\pm}^2 \sigma|_{C_{0,\pm}(e)})(p, \alpha) &= z_{0,\pm}(\zeta^2(p, \alpha))\lambda_0(p, \alpha) \\ &= z_{0,\pm}(p, \alpha)\lambda_0(p, \alpha) = \sigma|_{C_{0,\pm}(e)}(p, \alpha) \end{aligned}$$

we get  $\mathbf{P}_{0,\pm}^2 = \text{id}|_{\mathfrak{H}_{0,\pm}}$ . The operator  $\mathbf{P}_{0,\pm}$  is injective, for if  $\mathbf{P}_{0,\pm}(\sigma|_{C_{0,\pm}(e)}) = \mathbf{P}_{0,\pm}(\sigma'|_{C_{0,\pm}(e)})$ , then

$$\sigma|_{C_{0,\pm}(e)} = \mathbf{P}_{0,\pm}^2(\sigma|_{C_{0,\pm}(e)}) = \mathbf{P}_{0,\pm}^2(\sigma'|_{C_{0,\pm}(e)}) = \sigma'|_{C_{0,\pm}(e)}.$$

Thus the operator  $\mathbf{P}_{0,\pm}$  is bijective. Since  $\mathbf{P}_{0,\pm}^2 = \text{id}|_{\mathfrak{H}_{\pm}}$ , it follows that  $\mathbf{P}_0^{-1} = \mathbf{P}_0$  and  $\mathbf{P}_{\pm}^{-1} = \mathbf{P}_{\mp}$ . ■

We say that a quantum state  $\sigma|_C$  of the  $\mathbb{Z}_2$ -quantized mathematical pendulum  $(H, T^*S^1, \omega)$  with quantum line bundle  $\rho$  is *even* if it is covariantly constant even section  $\sigma|_C$  of  $\rho$ , that is,  $\mathbf{P}(\sigma|_C) = \sigma|_C$ . Let  $\mathfrak{H}^{\text{even}}$  be the vector space spanned by the even quantum states.  $\sigma|_C$  is an *odd* quantum state if it is covariantly constant odd section  $\sigma|_C$  of  $\rho$ , that is,  $\mathbf{P}(\sigma|_C) = -\sigma|_C$ . Let  $\mathfrak{H}^{\text{odd}}$  be the vector space spanned by the odd quantum states.

**Claim 6.1.3** We have

$$\mathfrak{H} = \mathfrak{H}^{\text{even}} \oplus \mathfrak{H}^{\text{odd}}. \quad (36)$$

**Proof.** The proof of this claim is standard, but we include it for completeness. Suppose that the section  $\sigma|_C$  is in  $\mathfrak{H}$ . Then  $\mathbf{P}(\sigma|_C) \in \mathfrak{H}$ . So the sections  $\sigma_{\text{even}}|_C = \frac{1}{2}(\sigma|_C + \mathbf{P}(\sigma|_C))$  and  $\sigma_{\text{odd}}|_C = \frac{1}{2}(\sigma|_C - \mathbf{P}(\sigma|_C))$  lie in  $\mathfrak{H}$  and  $\sigma|_C = \sigma_{\text{even}}|_C + \sigma_{\text{odd}}|_C$ . Now  $\sigma_{\text{even}}|_C \in \mathfrak{H}^{\text{even}}$ , because it is covariantly constant, has support  $C$ , and is an even section, since

$$\mathbf{P}(\sigma_{\text{even}}|_C) = \frac{1}{2}(\mathbf{P}(\sigma|_C) + \mathbf{P}^2(\sigma|_C)) = \frac{1}{2}(\sigma|_C + \mathbf{P}(\sigma|_C)) = \sigma_{\text{even}}|_C.$$

Similarly,  $\sigma_{\text{odd}}|_C \in \mathfrak{H}^{\text{odd}}$ . Thus  $\mathfrak{H} = \mathfrak{H}^{\text{even}} + \mathfrak{H}^{\text{odd}}$ . The preceding sum is direct since  $\mathfrak{H}^{\text{even}} \cap \mathfrak{H}^{\text{odd}} = \{0\}$ . For if  $\sigma|_C \in \mathfrak{H}^{\text{even}} \cap \mathfrak{H}^{\text{odd}}$ , then  $\sigma|_C = \mathbf{P}(\sigma|_C) = -\sigma|_C$ , which implies  $\sigma|_C = 0$ . ■

**Theorem 6.1.4** *Let  $\sigma|_C$  be a nonzero even or odd quantum state in  $\mathfrak{H}^{\text{even}} \cap \mathfrak{H}_0$  or  $\mathfrak{H}^{\text{odd}} \cap \mathfrak{H}_0$ , respectively. Then its quantum number is even or odd, respectively.*

**Proof.** Write  $\sigma(p, \alpha) = z(p, \alpha)\lambda_0$  for  $(p, \alpha) \in T^\times S^1 = T^*S^1 \setminus \{(0, 0), (0, \pi)\}$ , where  $z(p, \alpha) = \varepsilon z(-p, -\alpha)$  with  $\varepsilon = 1$  if  $\sigma$  is even and  $\varepsilon = -1$  if  $\sigma$  is odd. Note that  $z|_C$  is nowhere vanishing, for if it vanished at some point in  $C$  then it would be identically zero on  $C$ , since  $\sigma|_C$  is covariantly constant. But this contradicts our hypothesis. Because  $\sigma|_C \in \mathfrak{H}_0$  by hypothesis, it follows that  $C = C(e) = (H^\times)^{-1}(e)$ , where  $H^\times = H|T^\times S^1$  and  $0 < e < 2$ .

Let  $\gamma$  be a closed curve in  $T^\times S^1$ , which parametrizes  $C$ . The image of  $\gamma$  is  $\{(p, \alpha) \in T^\times S^1 \mid \frac{1}{2}p^2 - \cos \alpha + 1 = e\}$ , which is clearly invariant under the  $\mathbb{Z}_2$ -action generated by  $\zeta$  (30). Setting  $p = 0$  we get  $1 - \cos \alpha = e$ , which has a solution  $\alpha^+ \in [0, \pi]$  and another solution  $\alpha^- = -\alpha^+$ . Thus we can parametrize  $\gamma$  by  $\alpha$ , say  $\gamma(\alpha) = (p(\alpha), \alpha)$  for  $\alpha \in [-\alpha^+, \alpha^+]$ . Let

$$\rho^\times : L^\times = \mathbb{C} \times T^\times S^1 \rightarrow T^\times S^1 : (z, (p, \alpha)) \mapsto (p, \alpha)$$

be the quantum line bundle with covariant derivative  $\nabla^\times = \nabla|\Gamma(\rho^\times)$ . Let  $\tilde{\gamma}$  be the horizontal lift of  $\gamma$ . Parametrize the image of  $\tilde{\gamma}$  by  $\alpha$ , namely,  $\tilde{\gamma}(\alpha) = (z(p(\alpha), \alpha), (p(\alpha), \alpha))$  for  $\alpha \in [-\alpha^+, \alpha^+]$ . So  $\tilde{\gamma}$  is a section of  $L^\times$ .

Suppose that for some  $\alpha_0 \in [-\alpha^+, \alpha^+]$  we have  $z(\gamma(-\alpha_0)) = \varepsilon z(\gamma(\alpha_0))$ , where  $\varepsilon = \pm 1$ . Then  $z(\gamma(-\alpha_0)) \neq 0$ , since  $\sigma|_C$  is nowhere zero. Because  $\tilde{\gamma}$  is the horizontal lift of  $\gamma$ , using the covariant derivative  $\nabla^\times$ , integrating equation (14) we get

$$\begin{aligned} -i \frac{\hbar}{2\pi} \ln \frac{z(\gamma(\alpha_0))}{z(\gamma(-\alpha_0))} &= \frac{2}{2\pi} \int_{-\alpha_0}^{\alpha_0} \sqrt{2(e - (1 - \cos \alpha))} d\alpha \\ &= \begin{cases} 2m\hbar, & \text{if } \varepsilon = 1 \\ (2m - 1)\hbar, & \text{if } \varepsilon = -1 \end{cases} \end{aligned} \quad (37)$$

for some positive integer  $m$ . The last equality in (37) follows because the hypothesis  $z(\gamma(-\alpha_0)) = \varepsilon z(\gamma(\alpha_0))$  implies that

$$\ln \frac{z(\gamma(\alpha_0))}{z(\gamma(-\alpha_0))} = \ln \varepsilon = \begin{cases} 2m\pi i, & \text{if } \varepsilon = 1 \\ (2m - 1)\pi i, & \text{if } \varepsilon = -1 \end{cases}$$



for some integer  $m$ . Since  $\sqrt{2(e - (1 - \cos \alpha))} > 0$  on  $(-\alpha_0, \alpha_0)$ , we see that  $m > 0$ . From the fact that the Bohr-Sommerfeld torus  $C = C(e) = (H^\times)^{-1}(e)$  has quantum number  $n$ , that is,

$$\frac{2}{2\pi} \int_{-\alpha^+}^{\alpha^+} \sqrt{2(e - (1 - \cos \alpha))} d\alpha = n\hbar$$

and  $\sqrt{2(e - (1 - \cos \alpha))} > 0$  on  $(-\alpha^+, \alpha^+)$ , we obtain  $\begin{cases} 0 \leq 2m \leq n, & \text{if } \varepsilon = 1 \\ 0 \leq 2m - 1 \leq n, & \text{if } \varepsilon = -1. \end{cases}$

We now show that

$$\begin{cases} n = 2m, & \text{if } \varepsilon = 1 \\ n = 2m - 1, & \text{if } \varepsilon = -1. \end{cases} \quad (38)$$

Consider the function

$$F(\alpha_0) = \frac{1}{2\pi} \int_{-\alpha_0}^{\alpha_0} \sqrt{2(e - (1 - \cos \alpha))} d\alpha.$$

For  $\alpha_0 = 0$  we get  $F(0) = 0$ ; while for  $\alpha_0 = \alpha^+$ , we get  $F(\alpha^+) = n\hbar$ . By continuity, for every  $0 \leq k \leq n$  there is an angle  $\alpha_k \in [0, \alpha^+]$  such that  $F(\alpha_k) = k\hbar$ . Hence (38) holds. ■

**Corollary 6.1.5** *Let  $\sigma|_C$  be a nonzero even or odd quantum state in  $\mathfrak{H}^{\text{even}} \cap \mathfrak{H}_0$  or  $\mathfrak{H}^{\text{odd}} \cap \mathfrak{H}_0$ , respectively. Then the quantum number of  $\sigma|_C$  is the number of  $\mathbb{Z}_2$ -orbits of  $\varepsilon \mu$  (32) on the image of the horizontal lift of the curve  $\gamma$ , which parametrizes  $C$ .*

**Proof.** Suppose that for some  $\alpha_0 \in [-\alpha^+, \alpha^+]$  we have  $\tilde{\gamma}(-\alpha_0) = \varepsilon \mu(\tilde{\gamma}(\alpha_0))$ . Then  $z(\gamma(-\alpha_0)) = \varepsilon z(\gamma(\alpha_0))$ . Repeating the argument of theorem 8 which proves equations (37) and (38) shows that the quantum number  $n$  of the Bohr-Sommerfeld torus  $C = (H^\times)^{-1}(e)$  is equal to the number of  $\alpha_k \in [0, \alpha^+]$  such that  $z(\gamma(\alpha_k)) = \varepsilon z(\gamma(-\alpha_k))$ . In other words,  $n$  is the number of  $\mathbb{Z}_2$  orbits of  $\varepsilon \mu$  on the image of  $\tilde{\gamma}$ . ■

It follows from lemma 6.1.2 that the operators  $\mathbf{P}_\pm$  enable us to go from  $\mathfrak{H}_+$  to  $\mathfrak{H}_-$  and back. Thus they play the role of shifting operators. In particular  $\{\mathbf{P}_+ \sigma_m^+\}_{m \geq M}$  is a basis of  $\mathfrak{H}_-$ . Because the parity operator  $\mathbf{P}$  induces a  $\mathbb{Z}_2$ -symmetry on the Hilbert space  $\mathfrak{H}$ , by averaging the given inner product, we may assume that the parity operator  $\mathbf{P}$  preserves the new inner product on  $\mathfrak{H}$ . In order to simplify the presentations we choose the orthonormal bases  $\{\sigma_m^\pm\}$  of  $\mathfrak{H}_\pm$  so that  $\mathbf{P}_\pm \sigma_m^\pm = \sigma_m^\mp$ . In order to construct operators relating  $\mathfrak{H}_0$  to  $\mathfrak{H}_\pm$  we need to show that reduction of the  $\mathbb{Z}_2$ -symmetry of the mathematical pendulum gives rise to the quantized  $\mathbb{Z}_2$ -reduced mathematical pendulum.

## 6.2 $\mathbb{Z}_2$ -quantization and reduction

In this subsection discuss quantization of the  $\mathbb{Z}_2$ -reduced mathematical pendulum.

### 6.2.1 Reduction of the $\mathbb{Z}_2$ -symmetry

Here we reduce the  $\mathbb{Z}_2$ -symmetry of the mathematical pendulum  $(H, T^*S^1, \omega)$  generated by  $\zeta$  (30).

First we determine the reduced phase space  $\tilde{P}$ , which is the space  $T^*S^1/\mathbb{Z}_2$  of orbits of the  $\mathbb{Z}_2$ -symmetry on  $T^*S^1$ . To start with we use cut and paste geometric methods to construct the  $\mathbb{Z}_2$ -orbit space. Recall that a connected subset  $\Delta$  of  $T^*S^1$  is a *fundamental domain* for the  $\mathbb{Z}_2$ -symmetry generated by  $\zeta$  (30), if it contains exactly one point of each  $\mathbb{Z}_2$ -orbit in  $T^*S^1$ .

**Claim 6.2.1** *The set  $\Delta = \{(p, \alpha) \in T^*S^1 \mid p > 0 \text{ or } p = 0 \ \& \ \alpha \in [0, \pi]\}$  is a fundamental domain for the  $\mathbb{Z}_2$ -action generated by  $\zeta$ .*

**Proof.** Clearly  $\Delta$  is connected. Let  $(p, \alpha) \in T^*S^1 \setminus \Delta$ . If  $p \neq 0$ , then  $p < 0$ . So  $\zeta(p, \alpha) = (-p, -\alpha) \in \Delta$ . Suppose that  $p = 0$  and  $\alpha \in (\pi, 2\pi)$ . Then  $-\alpha \in (0, \pi)$ . So  $\zeta(0, \alpha) = (0, -\alpha) \in \Delta$ . Hence  $\zeta(T^*S^1 \setminus \Delta) \subseteq \Delta$ , which implies  $T^*S^1 \setminus \Delta = \zeta(\zeta(T^*S^1 \setminus \Delta)) \subseteq \zeta(\Delta)$ . Consequently,

$$T^*S^1 = (T^*S^1 \setminus \Delta) \cup \Delta \subseteq \zeta(\Delta) \cup \Delta \subseteq T^*S^1,$$

that is,  $\zeta(\Delta) \cup \Delta = T^*S^1$ . Note that  $\Delta \cap \zeta(\Delta) = \{(0, 0), (0, \pi)\}$ , which are the fixed points of the  $\mathbb{Z}_2$ -action. ■

Look at the closure  $\overline{\Delta}$  of  $\Delta$  in  $T^*S^1$  and identify the points on the boundary of  $\overline{\Delta}$ , which lie on the same  $\mathbb{Z}_2$ -orbit. The resulting space is a model for the orbit space  $T^*S^1/\mathbb{Z}_2$ .

We now give another construction for the reduced phase space using invariant theory and the concept of a differential space, see [2]. The algebra of real analytic functions on  $T^*S^1$ , which are invariant under the symmetry group  $\mathbb{Z}_2$ , is generated by

$$\tau_1 = \cos \alpha, \quad \tau_2 = p \sin \alpha, \quad \tau_3 = \frac{1}{2} p^2 - \cos \alpha + 1. \quad (39)$$

These invariant functions are subject to the relation

$$C(\tau) = \frac{1}{2} \tau_2^2 - (\tau_3 + \tau_1 - 1)(1 - \tau_1^2) = 0, \quad |\tau_1| \leq 1 \ \& \ \tau_3 \geq 0, \quad (40)$$

which defines the  $\mathbb{Z}_2$ -orbit space  $\tilde{P} = T^*S^1/\mathbb{Z}_2$  as a semialgebraic variety in  $\mathbb{R}^3$  with coordinates  $\tau = (\tau_1, \tau_2, \tau_3)$ . We say that a function  $f$  on  $\tilde{P}$

is *smooth* if there is a smooth function  $F$  on  $\mathbb{R}^3$  such that  $f = F|_{\tilde{P}}$ . Let

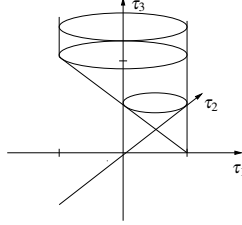


Figure 2. The  $\mathbb{Z}_2$ -reduced space in  $\mathbb{R}^3$ .

$C^\infty(\tilde{P})$  be the space of smooth functions on  $\tilde{P}$ . Then  $(\tilde{P}, C^\infty(\tilde{P}))$  is a locally compact subcartesian differential space, because  $\tilde{P}$  is a semialgebraic.

Next we construct the reduced Hamiltonian. Since the Hamiltonian  $H$  of the mathematical pendulum is invariant under the  $\mathbb{Z}_2$ -symmetry, it induces a smooth function  $\tilde{H}$  on  $\tilde{P}$  given by restricting the smooth function  $\tau_3 : \mathbb{R}^3 \rightarrow \mathbb{R} : \tau \mapsto \tau_3$  to  $\tilde{P}$ .

In order to have dynamics on  $\tilde{P}$ , we first need a Poisson bracket  $\{ , \}_{\mathbb{R}^3}$  on  $C^\infty(\mathbb{R}^3)$ . A calculation using the Poisson bracket  $\{ , \}$  on  $P$  shows that

$$\begin{aligned} \{\tau_1, \tau_2\} &= \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_3} \\ \{\tau_2, \tau_3\} &= 2\tau_1(\tau_3 + \tau_1 - 1) + \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_1} \\ \{\tau_3, \tau_1\} &= \tau_2 = \frac{\partial C}{\partial \tau_2} \end{aligned}$$

For every  $F, G \in C^\infty(\mathbb{R}^3)$  let

$$\{F, G\}_{\mathbb{R}^3} = \sum_{i,j} \frac{\partial F}{\partial \tau_i} \frac{\partial G}{\partial \tau_j} \{\tau_i, \tau_j\} = \langle \text{grad } F \times \text{grad } G, \text{grad } C \rangle, \quad (41)$$

Here  $\langle , \rangle$  is the Euclidean inner product on  $\mathbb{R}^3$  and  $\times$  is the vector product. Then  $\{ , \}_{\mathbb{R}^3}$  is a Poisson bracket on  $C^\infty(\mathbb{R}^3)$ . On  $C^\infty(\tilde{P})$  define a Poisson bracket  $\{ , \}_{\tilde{P}}$  as follows. Suppose that  $f, g \in C^\infty(\tilde{P})$ . Then there are  $F, G \in C^\infty(\mathbb{R}^3)$  such that  $f = F|_{\tilde{P}}$  and  $g = G|_{\tilde{P}}$ . Let  $\{f, g\}_{\tilde{P}} = \{F, G\}_{\mathbb{R}^3}|_{\tilde{P}}$ . Because of (41), the defining function  $C$  (40) of  $\tilde{P}$  is a Casimir in the Poisson algebra  $\mathcal{A} = (C^\infty(\mathbb{R}^3), \{ , \}_{\mathbb{R}^3}, \cdot)$ . Hence, the collection  $\mathcal{I}$  of all smooth functions on  $\mathbb{R}^3$ , which vanish identically on  $\tilde{P}$ , is a Poisson ideal in  $\mathcal{A}$ .

Consequently, the Poisson bracket  $\{, \}_{\tilde{P}}$  is well defined and  $\mathcal{B} = \mathcal{A}/\mathcal{I} = (C^\infty(\tilde{P}) = C^\infty(\mathbb{R}^3)|_{\mathcal{I}}, \{, \}_{\tilde{P}}, \cdot)$  is a Poisson algebra.

Consider the derivation  $-\text{ad}_{\tau_3}$  on the Poisson algebra  $\mathcal{A}$ . This derivation gives rise to the  $\mathbb{Z}_2$ -reduced Hamiltonian vector field  $-\text{ad}_{\tilde{H}}$  on the locally compact subcartesian differential space  $(\tilde{P}, C^\infty(\tilde{P}))$  associated to the  $\mathbb{Z}_2$ -reduced Hamiltonian  $\tilde{H}$ . To see this note that on  $\mathbb{R}^3$  the integral curves of  $-\text{ad}_{\tau_3}$  satisfy

$$\begin{aligned}\dot{\tau}_1 &= \{\tau_1, \tau_3\}_{\mathbb{R}^3} = -\tau_2 \\ \dot{\tau}_2 &= \{\tau_2, \tau_3\}_{\mathbb{R}^3} = 2\tau_1(\tau_3 + \tau_1 - 1) + \tau_1^2 - 1 \\ \dot{\tau}_3 &= \{\tau_3, \tau_3\}_{\mathbb{R}^3} = 0.\end{aligned}$$

Because  $C$  is a Casimir of the Poisson algebra  $\mathcal{A}$ , we obtain  $0 = \{C, \tau_3\}_{\mathbb{R}^3}$ . In other words,  $C$  is an integral of  $-\text{ad}_{\tau_3}$ . A calculation shows that  $-\text{ad}_{\tau_3}$  leaves the sets  $C^{-1}(0)$ ,  $\{\tau_3 + \tau_1 - 1 = 0\}$ , and  $\{\tau_1 = \pm 1\}$  invariant. Thus the  $\mathbb{Z}_2$ -reduced space  $\tilde{P}$  is invariant under the flow of  $-\text{ad}_{\tau_3}$ . Consequently, the  $\mathbb{Z}_2$ -reduced Hamiltonian vector field  $-\text{ad}_{\tilde{H}}$ , where  $\tilde{H} = \tau_3|_{\tilde{P}}$ , is defined on  $\tilde{P}$ . Because the Hamiltonian vector field  $X_H$  of the mathematical pendulum is complete, the reduced vector field  $-\text{ad}_{\tilde{H}}$  is complete. Its flow  $\varphi_t^{\tilde{H}}$  is a 1-parameter group of diffeomorphisms of  $\tilde{P}$ . In fact, for  $\tilde{p} \in \tilde{H}^{-1}(e)$  the closure of the integral curve  $t \mapsto \varphi_t^{\tilde{H}}(\tilde{p})$  is a connected component of the level set  $\tilde{H}^{-1}(e)$ , since a level set of the reduced Hamiltonian  $\tilde{H}$  is compact.

We now take a closer look at the  $\mathbb{Z}_2$ -reduction mapping

$$\tilde{\pi} : T^*S^1 \rightarrow \tilde{P} \subseteq \mathbb{R}^3 : (p, \alpha) \mapsto \tau(p, \alpha). \quad (42)$$

The  $\mathbb{Z}_2$ -action on  $T^*S^1$  has two fixed points:  $p_0 = (0, 0)$  and  $p_2 = (0, \pi)$ . So the reduced space  $\tilde{P}$  has two singular points  $\tilde{p}_0 = (1, 0, 0)$  and  $\tilde{p}_2 = (-1, 0, 2)$ , which are conical. The set  $\tilde{P}^\times$  of nonsingular points of  $\tilde{P}$  is  $\tilde{P} \setminus \{\tilde{p}_0, \tilde{p}_2\}$ , which is a smooth manifold that is diffeomorphic to  $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ . The  $\mathbb{Z}_2$ -orbit map  $\tilde{\pi}$  (42) restricted to  $T^\times S^1 = T^*S^1 \setminus \{p_0, p_2\}$  is the proper submersion

$$\tilde{\pi}^\times : T^\times S^1 \rightarrow \tilde{P}^\times : (p, \alpha) \mapsto \tau(p, \alpha), \quad (43)$$

whose fiber  $(\tilde{\pi}^\times)^{-1}(\tau)$  at  $\tau \in \tilde{P}^\times$  is two distinct points. Thus we have proved

**Lemma 6.2.2** *The  $\mathbb{Z}_2$ -orbit map  $\tilde{\pi}^\times$  (43) is a 2 to 1 covering map.*

The 1-form  $\theta^\times = \theta|_{T^\times S^1} = (p\alpha)|_{T^\times S^1}$  on  $T^\times S^1$  is invariant under the  $\mathbb{Z}_2$ -action, since  $T^\times S^1$  is a smooth  $\mathbb{Z}_2$ -invariant manifold. Hence  $\theta^\times$  pushes

down under the  $\mathbb{Z}_2$ -orbit map  $\tilde{\pi}^\times$  (43) to a 1-form  $\tilde{\theta}^\times$  on the smooth manifold  $\tilde{P}^\times$ . So  $(\tilde{\pi}^\times)^*\tilde{\theta}^\times = \theta^\times$ . Here are explicit expressions for the 1-form  $\tilde{\theta}^\times$ .

**Claim 6.2.3** *On  $\tilde{U}_1 = \tilde{P}^\times \setminus \{\tau_1 = \pm 1\}$  we have  $\tilde{\theta}^\times|_{\tilde{U}_1} = -\tau_2(1 - \tau_1^2)^{-1} d\tau_1$ ; while on  $\tilde{U}_2 = \tilde{P}^\times \setminus (\{\tau_1 = 0\} \cup \{\tau_3 + \tau_1 - 1 = 0\})$  we have  $\tilde{\theta}^\times|_{\tilde{U}_2} = (2(\tau_3 + \tau_1 - 1)d\tau_2 - \tau_2 d\tau_1 - \tau_2 d\tau_3)(2\tau_1(\tau_3 + \tau_1 - 1))^{-1}$ . Note  $\tilde{P}^\times = \tilde{U}_1 \cup \tilde{U}_2$ .*

**Proof.** On  $U_1 = T^\times S^1 \setminus \{(p, \pm\pi) \in T^\times S^1 \mid p \in \mathbb{R}^\times\}$  we have

$$(\tilde{\pi}^\times)^*(\tilde{\theta}^\times|_{\tilde{U}_1}) = -\frac{p \sin \alpha}{1 - \cos^2 \alpha} d(\cos \alpha) = (p d\alpha)|_{U_1} = \theta^\times|_{U_1};$$

while on  $U_2 = T^\times S^1 \setminus \{(p, \pm\pi/2) \in T^\times S^1 \mid p \in \mathbb{R}^\times\} \cup \{(0, \alpha) \in T^\times S^1\}$  we have

$$\begin{aligned} (\tilde{\pi}^\times)^*(\tilde{\theta}^\times|_{\tilde{U}_2}) &= \frac{(p^2 d(p \sin \alpha) - p \sin \alpha d(\cos \alpha) - p \sin \alpha (p dp - d(\cos \alpha)))}{p^2 \cos \alpha} \\ &= (p d\alpha)|_{U_2} = \theta^\times|_{U_2}. \end{aligned}$$

Note that for  $i = 1, 2$  we have  $(\tilde{\pi}^\times)^{-1}(\tilde{U}_i) = U_i$  and  $T^\times S^1 = U_1 \cup U_2$ . ■

Because the 2-form  $\omega = d\theta$  on  $T^*S^1$  is invariant under the  $\mathbb{Z}_2$ -symmetry generated by  $\zeta$  (30), the 2-form  $\omega^\times = \omega|_{T^\times S^1} = d(\theta|_{T^\times S^1}) = d\theta^\times$  on  $T^\times S^1$  is  $\mathbb{Z}_2$ -invariant. Hence  $\omega^\times$  pushes down to a 2-form  $\tilde{\omega}^\times$  on the  $\mathbb{Z}_2$ -reduced space  $\tilde{P}^\times$ . Now

$$(\tilde{\pi}^\times)^*(d\theta^\times) = d((\tilde{\pi}^\times)^*\tilde{\theta}^\times) = d\theta^\times = \omega^\times = (\tilde{\pi}^\times)^*\tilde{\omega}^\times.$$

Since  $\tilde{\pi}^*$  is surjective, we obtain  $\tilde{\omega}^\times = d\tilde{\theta}^\times$ . Thus the punctured  $\mathbb{Z}_2$ -reduced space  $\tilde{P}^\times$  is a symplectic manifold with symplectic form  $\tilde{\omega}^\times = d\tilde{\theta}^\times$ .

We now compute the  $\mathbb{Z}_2$ -reduced actions. On  $\tilde{P}^\times \setminus \{\tau_1 = \pm 1\}$  the 1-form  $\tilde{\theta}^\times$  is  $-\tau_2(1 - \tau_1^2)^{-1} d\tau_1$ , where  $\tau_2 = \mp\sqrt{2(\tau_3 + \tau_1 - 1)(1 - \tau_1^2)}$ . So  $\tilde{\theta}^\times = \pm\sqrt{\frac{2(\tau_3 + \tau_1 - 1)}{1 - \tau_1^2}} d\tau_1$ . The  $\mathbb{Z}_2$ -reduced Hamiltonian on  $\tilde{P}^\times$  is  $\tilde{H}^\times = \tilde{H}|_{\tilde{P}^\times}$ .

Consequently, the reduced action  $\tilde{I}^\times : (0, 2) \cup (2, \infty) \rightarrow \mathbb{R}$  on a connected component  $\tilde{C}(e)$  of the level set  $(\tilde{H}^\times)^{-1}(e)$  is

$$\tilde{I}^\times(e) = \frac{1}{2\pi} \int_{\tilde{C}(e)} \tilde{\theta}^\times = \frac{1}{\pi} \int_{\max(1-e, -1)}^1 \frac{\sqrt{2(e + \tau_1 - 1)}}{\sqrt{1 - \tau_1^2}} d\tau_1, \quad (44)$$

when  $0 < e < 2$  or  $e > 2$ . We now calculate the integral in (44). First we consider the case when  $0 < e < 2$ . Letting  $u^2 = \tau_1 - (1 - e)$ ,  $u = \sqrt{e}v$ , and then  $v = \cos \varphi$ , we get successively

$$\tilde{I}^\times(e) = \frac{\sqrt{2}}{\pi} \int_0^{\sqrt{e}} \frac{2u^2}{\sqrt{(e - u^2)(2 - e + u^2)}} du$$

$$\begin{aligned}
&= \frac{2\sqrt{2}}{\pi} e \int_0^1 \frac{v^2}{\sqrt{(1-v^2)(2-e+ev^2)}} dv \\
&= \frac{2}{\pi} e \int_0^{\pi/2} \frac{\cos^2 \varphi}{\sqrt{1-\frac{e}{2}\sin^2 \varphi}} d\varphi \geq 0.
\end{aligned} \tag{45}$$

Next we treat the case when  $e > 2$ . Letting  $\tau_1 = \cos \vartheta$  and  $\vartheta = 2\varphi$  successively, we get

$$\begin{aligned}
\tilde{I}^\times(e) &= \frac{\sqrt{2}}{\pi} \int_0^{\pi/2} \sqrt{e-1+\cos \vartheta} d\vartheta = \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} \sqrt{e-2+2\cos^2 \varphi} d\varphi \\
&= \frac{2\sqrt{2e}}{\pi} \int_0^{\pi/2} \sqrt{1-\frac{2}{e}\sin^2 \varphi} d\varphi \geq 0.
\end{aligned} \tag{46}$$

Note that the  $\mathbb{Z}_2$ -reduced action  $\tilde{I}^\times$  (45) and (46) is one half the original action  $I^\times = I|_{T \times S^1}$  (8) and (10). Moreover, the function  $e \mapsto \tilde{I}^\times(e)$  is continuous at  $e = 2$ . Dullin [9] shows,  $\tilde{I}^\times$  has a logarithm term in its series expansion in  $e - 2$ , which shows  $\tilde{I}^\times$  is not differentiable at  $e = 2$ .

The corresponding  $\mathbb{Z}_2$ -reduced angle  $\tilde{\vartheta}^\times$  is

$$\tilde{\vartheta}^\times = \frac{2\pi}{\tilde{T}} t = \frac{2\pi}{\tilde{T}} \int_t^1 \frac{2 d\tau_1}{\sqrt{2(1-\tau_1^2)(e-1+\tau_1)}}, \tag{47}$$

where  $t \in [\max(1-e, -1), 1]$  and

$$\tilde{T} = \tilde{T}(e) = \int_{-1}^1 \frac{2 d\tau_1}{\sqrt{2(1-\tau_1^2)(e-1+\tau_1)}}. \tag{48}$$

In order to simplify the notation, in the following we will use  $\tilde{I}$  for the reduced action on the reduced space  $\tilde{P}$  and  $\tilde{\vartheta}$  for the reduced angle, which is not defined at the singular points  $\tilde{p}_0$  and  $\tilde{p}_2$  of  $\tilde{P}$ . Note that  $\tilde{I}^\times = \tilde{I}|_{\tilde{P}^\times}$  and  $\tilde{\vartheta}^\times = \tilde{\vartheta}|_{\tilde{P}^\times}$ .

### 6.2.2 Reduction of the $\mathbb{Z}_2$ -quantum symmetry

In this subsection we reduce the  $\mathbb{Z}_2$ -symmetry of the quantized mathematical pendulum. In other words, we reduce the  $\mathbb{Z}_2$ -action on  $L \subseteq \mathbb{C} \times \mathbb{R}^2$  generated by the mapping  $\mu$  (32). We use invariant theory.

The algebra of invariant real analytic functions is generated by

$$\sigma_1 = z, \quad \tau_1 = \cos \alpha, \quad \tau_2 = p \sin \alpha, \quad \tau_3 = \frac{1}{2} p^2 - \cos \alpha + 1$$

subject to the relation

$$\begin{aligned} \frac{1}{2} \tau_2^2 &= \frac{1}{2} p^2 \sin^2 \alpha = \frac{1}{2} p^2 (1 - \cos^2 \alpha) \\ &= (\tau_1 + \tau_3 - 1)(1 - \tau_1^2), \quad |\tau_1| \leq 1 \ \& \ \tau_3 \geq 0. \end{aligned} \quad (49)$$

Equation (49) defines the  $\mathbb{Z}_2$ -orbit space  $\tilde{\mathcal{P}} = L/\mathbb{Z}_2$ . The Hilbert mapping

$$\tilde{\zeta} : L \rightarrow \tilde{\mathcal{P}} : (z, (p, \alpha)) \mapsto (\sigma_1(z), \tau_1(p, \alpha), \tau_2(p, \alpha), \tau_3(p, \alpha)) = (\sigma_1(z), \tau(p, \alpha))$$

is the orbit map of the  $\mathbb{Z}_2$ -action. The  $\mathbb{Z}_2$ -orbit space  $\tilde{\mathcal{P}}$  is  $\mathbb{C} \times (T^*S^1/\mathbb{Z}_2)$ , which is a semialgebraic variety with two singular planes  $\mathbb{C} \times \{(\pm 1, 0, 1 \mp 1)\}$ . We view  $\tilde{\mathcal{P}}$  as a complex ‘‘line bundle’’ over the  $\mathbb{Z}_2$ -orbit space  $\tilde{P}$  with bundle projection

$$\tilde{\omega} : \tilde{\mathcal{P}} = \mathbb{C} \times \tilde{P} \rightarrow \tilde{P} : (\sigma_1, \tau = (\tau_1, \tau_2, \tau_3)) \mapsto \tau.$$

Here (49) is the defining relation of the  $\mathbb{Z}_2$ -orbit space  $\tilde{P}$  with orbit mapping  $\tilde{\pi} : T^*S^1 \rightarrow \tilde{P} : (p, \alpha) \mapsto \tau(p, \alpha)$ . Consider the smooth manifold  $\tilde{\mathcal{P}}^\times = \tilde{\mathcal{P}} \setminus \{(\pm 1, 0, 1 \mp 1)\} = \mathbb{C} \times \tilde{P}^\times$  of nonsingular points of  $\tilde{\mathcal{P}}$ , where  $\tilde{P}^\times = T^\times S^1/\mathbb{Z}_2$  and  $T^\times S^1 = T^*S^1 \setminus \{(0, 0), (0, \pi)\}$ . Restricting  $\tilde{\omega}$  to  $\mathbb{C} \times \tilde{P}^\times$  gives a trivial smooth bundle  $\tilde{\rho}^\times : \tilde{L}^\times = \mathbb{C} \times \tilde{P}^\times \rightarrow \tilde{P}^\times$ . The bundle  $\tilde{\rho}^\times$  serves as the quantum bundle of the  $\mathbb{Z}_2$ -reduced mathematical pendulum  $(\tilde{H}^\times, \tilde{P}^\times, \tilde{\omega}^\times)$ .

### 6.2.3 The $\mathbb{Z}_2$ -reduced quantum system

In this subsection we quantize the  $\mathbb{Z}_2$ -reduced quantum mathematical pendulum, namely, the  $\mathbb{Z}_2$ -reduced mathematical pendulum  $(\tilde{H}^\times, \tilde{P}^\times, \tilde{\omega}^\times)$  with the  $\mathbb{Z}_2$ -reduced quantum bundle

$$\tilde{\rho}^\times : \tilde{L}^\times = \mathbb{C} \times \tilde{P}^\times \rightarrow \tilde{P}^\times : (z, \tau) \mapsto \tau$$

and trivializing section  $\tilde{\lambda}_0 : \tilde{P}^\times \rightarrow \tilde{L}^\times : \tau \mapsto (1, \tau)$ .

We need to find a connection  $\tilde{\nabla}^\times$  on the smooth sections of the bundle  $\tilde{\rho}^\times$ , which is related to the original connection  $\nabla^\times$  on  $P^\times$ . On smooth sections of the bundle  $\rho^\times$  we have a connection, whose covariant derivative  $\nabla_X^\times$  in the direction of the smooth vector field  $X$  on  $T^\times S^1$  acts on the section  $\lambda_0^\times$  by  $\nabla_X^\times \lambda_0^\times = -i\hbar^{-1}(X \lrcorner \theta^\times) \lambda_0^\times$ . Suppose that  $\tilde{X}$  is a smooth vector field on  $\tilde{P}^\times$ , which is  $\tilde{\pi}^\times$ -related to the vector field  $X$ , that is,  $T\tilde{\pi}^\times X = \tilde{X} \circ \tilde{\pi}^\times$ . On the line bundle  $\tilde{\rho}^\times$  with trivializing section  $\tilde{\lambda}_0^\times$  define a connection  $\tilde{\nabla}^\times$  by  $(\tilde{\pi}^\times)^*(\tilde{\nabla}_{\tilde{X}}^\times \tilde{\lambda}_0^\times) = \nabla_X^\times \lambda_0^\times$ . In other words,

**Fact 6.2.3.1**

$$\tilde{\nabla}_{\tilde{X}}^{\times} \tilde{\lambda}_0^{\times} = -i\hbar^{-1}(\tilde{X} \lrcorner \tilde{\theta}^{\times}) \tilde{\lambda}_0^{\times}. \quad (50)$$

**Proof.** Equation (50) follows because by definition

$$(\tilde{\pi}^{\times})^*(\tilde{\nabla}_{\tilde{X}}^{\times} \tilde{\lambda}_0^{\times}) = \nabla_X^{\times} \lambda_0^{\times} = -i\hbar^{-1}(X \lrcorner \theta^{\times}) \lambda_0^{\times};$$

whereas

$$\begin{aligned} (\tilde{\pi}^{\times})^*(-i\hbar^{-1}(\tilde{X} \lrcorner \tilde{\theta}^{\times}) \tilde{\lambda}_0^{\times}) &= -i\hbar^{-1}((\tilde{X} \lrcorner \tilde{\theta}^{\times}) \circ \tilde{\pi}^{\times}) \tilde{\lambda}_0^{\times} \circ \tilde{\pi}^{\times} \\ &= -i\hbar^{-1}(\tilde{X} \circ \tilde{\pi}^{\times} \lrcorner \tilde{\theta}^{\times} \circ \tilde{\pi}^{\times}) \tilde{\lambda}_0^{\times} \circ \tilde{\pi}^{\times} \\ &= -i\hbar^{-1}(T\tilde{\pi}^{\times} X \lrcorner T\tilde{\pi}^{\times} \theta^{\times}) \tilde{\lambda}_0^{\times} \circ \tilde{\pi}^{\times} \\ &= -i\hbar^{-1}(X \lrcorner \theta^{\times}) ((T\tilde{\pi}^{\times})^t \tilde{\lambda}_0^{\times} \circ \tilde{\pi}^{\times}) \\ &= -i\hbar^{-1}(X \lrcorner \theta^{\times}) \lambda_0^{\times}, \quad \text{since } (\tilde{\pi}^{\times})^* \tilde{\lambda}_0^{\times} = \lambda_0^{\times}. \end{aligned}$$

Thus  $(\tilde{\pi}^{\times})^*(\tilde{\nabla}_{\tilde{X}}^{\times} \tilde{\lambda}_0^{\times}) = (\tilde{\pi}^{\times})^*(-i\hbar^{-1}(\tilde{X} \lrcorner \tilde{\theta}^{\times}) \tilde{\lambda}_0^{\times})$ , which implies (50) since  $\tilde{\pi}^{\times}$  is surjective. ■

Next we determine the quantization rules for the  $\mathbb{Z}_2$ -reduced Hamiltonian system  $(\tilde{H}^{\times} = \tilde{H}|_{\tilde{P}^{\times}}, \tilde{P}^{\times}, \tilde{\omega}^{\times})$  with quantum line bundle  $\tilde{\rho}^{\times}$  and trivializing section  $\tilde{\lambda}_0$ . The mapping  $\tilde{P}^{\times} \rightarrow T\tilde{P}^{\times} : p \mapsto \text{span}\{X_{\tilde{H}^{\times}}(p)\}$  defines a smooth Lagrangian distribution  $\tilde{D}$  on the symplectic manifold  $(\tilde{P}^{\times}, \tilde{\omega}^{\times})$ , which is a polarization of  $(\tilde{P}^{\times}, \tilde{\omega}^{\times})$ . A leaf of  $\tilde{D}$  is a connected component of a level set  $(\tilde{H}^{\times})^{-1}(e)$  of the  $\mathbb{Z}_2$ -reduced Hamiltonian  $\tilde{H}^{\times}$  on  $\tilde{P}^{\times}$ , which is a smooth  $S^1$  when  $e \in (0, 2) \cup (2, \infty)$ .

Let  $\gamma : \mathbb{R} \rightarrow \tilde{P}^{\times}$  be an integral curve of  $X_{\tilde{H}^{\times}}$  of energy  $e \in (0, 2) \cup (2, \infty)$ . Then  $\gamma$  is periodic of primitive period  $\tilde{T} = \tilde{T}(e) > 0$ . Also  $\gamma$  parametrizes a connected component  $\tilde{C}^{\times}(e)$  of the smooth level set  $(\tilde{H}^{\times})^{-1}(e)$ . Parallel transport the section  $\tilde{\lambda}$  of the  $C^{\times}$ -bundle  $\tilde{\rho}^{\times}$  along  $\gamma$  using the connection  $\tilde{\nabla}^{\times}$ . Then at every point  $\gamma(t)$  in  $(\tilde{H}^{\times})^{-1}(e)$  we have

$$0 = (\tilde{\nabla}_{X_{\tilde{H}^{\times}}}^{\times} \tilde{\lambda})(\gamma(t)) = (L_{X_{\tilde{H}^{\times}}} f)(\gamma(t)) \tilde{\lambda}_0 - \frac{i}{\hbar}(X_{\tilde{H}^{\times}} \lrcorner \tilde{\theta}^{\times})(\gamma(t)) f(\gamma(t)) \tilde{\lambda}_0,$$

that is,

$$\frac{dF(t)}{dt} - \frac{i}{\hbar}(X_{\tilde{H}^{\times}} \lrcorner \tilde{\theta}^{\times})(\gamma(t)) F(t) = 0, \quad (51)$$

where  $F(t) = f(\gamma(t))$ . For equation (51) to have a nonvanishing solution

$$F(\tilde{T}) = F(0) \exp\left(\frac{i}{\hbar} \int_0^{\tilde{T}} (X_{\tilde{H}^{\times}} \lrcorner \tilde{\theta}^{\times})(\gamma(t)) dt\right) = A(\tilde{T})F(0),$$



the holonomy  $A(\tilde{T})$  of the connection  $\tilde{\nabla}^\times$  along  $\gamma$  must equal 1, because  $F(\tilde{T}) = f(\gamma(\tilde{T})) = f(\gamma(0)) = F(0) \neq 0$ . Consequently, for some  $k \in \mathbb{Z}$  we have

$$\begin{aligned} k &= \frac{1}{\hbar} \int_0^{\tilde{T}} (X_{\tilde{H}^\times} \lrcorner \tilde{\theta}^\times)(\gamma(t)) dt = \frac{1}{\hbar} \int_0^{\tilde{T}} \tilde{\theta}^\times(\gamma(t)) X_{\tilde{H}^\times}(\gamma(t)) dt \\ &= \frac{1}{\hbar} \int_0^{\tilde{T}} \tilde{\theta}^\times(\gamma(t)) \frac{d\gamma(t)}{dt} dt = \frac{1}{\hbar} \int_0^{\tilde{T}} \gamma^* \tilde{\theta}^\times = \frac{1}{\hbar} \int_{(\tilde{H}^\times)^{-1}(e)} \tilde{\theta}^\times \end{aligned}$$

In other words, when  $e \in (0, 2) \cup (2, \infty)$  the quantization rule for the  $(\mathbb{Z}_2, \cdot)$ -reduced quantized Hamiltonian system  $(\tilde{H}^\times, \tilde{P}^\times, \tilde{\omega}^\times)$  with quantum bundle  $\tilde{\rho}^\times : \tilde{L}^\times \rightarrow \tilde{P}^\times$  is

$$0 \leq \tilde{I}^\times(e) = \frac{1}{2\pi} \int_{\tilde{C}^\times(e)} \tilde{\theta}^\times = k\hbar, \quad \text{for some } k \in \mathbb{Z}_{\geq 0}, \quad (52)$$

where  $\tilde{I}^\times$  is the action (44) of the  $\mathbb{Z}_2$ -reduced mathematical pendulum.

**Lemma 6.2.3.2** *The reduction mapping  $\tilde{\pi}^\times : T^\times S^1 \rightarrow \tilde{P}^\times : (p, \alpha) \mapsto \tau(p, \alpha)$  (43) maps a Bohr-Sommerfeld torus  $C(e)$  of the mathematical pendulum  $(H^\times, T^\times S^1, \omega^\times)$  onto a Bohr-Sommerfeld torus  $\tilde{C}(e)$  of the  $\mathbb{Z}_2$ -reduced mathematical pendulum  $(\tilde{H}^\times, \tilde{P}^\times, \tilde{\omega}^\times)$ .*

**Proof.** By lemma 6.2.3.1 the  $\mathbb{Z}_2$ -reduction mapping  $\tilde{\pi}^\times$  is a 2 to 1 covering map. Its preimage of the  $e$ -level set  $(\tilde{H}^\times)^{-1}(e)$  of the  $\mathbb{Z}_2$ -reduced Hamiltonian on the  $\mathbb{Z}_2$ -reduced phase space  $\tilde{P}^\times$  is  $(H^\times)^{-1}(e)$  if  $0 < e < 1$  or  $e > 2$ . Thus the image of a connected component  $C(e)$  of  $(H^\times)^{-1}(e)$  under  $\tilde{\pi}^\times$  is  $(\tilde{H}^\times)^{-1}(e)$ . Since

$$\begin{aligned} (\tilde{\pi}^\times)^* \left( \tilde{I}^\times|_{\tilde{C}(e)} \right) &= (\tilde{\pi}^\times)^* \left( \frac{1}{2\pi} \int_{(\tilde{H}^\times)^{-1}(e)} \tilde{\vartheta}^\times \right) \\ &= \begin{cases} \frac{2}{2\pi} \int_{(H^\times)^{-1}(e)} \vartheta^\times, & \text{if } 0 < e < 2 \\ \frac{2}{2\pi} \int_{C_\pm(e)} \vartheta^\times, & \text{if } 2 < e \end{cases} = \frac{1}{2\pi} \int_{C(e)} \vartheta^\times = (I^\times)|_{C(e)}, \end{aligned}$$

the image under  $\tilde{\pi}^\times$  of a Bohr-Sommerfeld torus of the mathematical pendulum is a Bohr-Sommerfeld torus of the  $\mathbb{Z}_2$ -reduced mathematical pendulum. ■

For every positive integer  $k$ , let  $\tilde{\sigma}_k$  be a section of the line bundle  $\tilde{\rho}^\times$ , which is supported and covariantly constant on the level set  $(\tilde{I}^\times)^{-1}(k\hbar)$ . As before we add the quantum number 0, which corresponds to a section supported on the singular Bohr-Sommerfeld torus corresponding to the singular

point  $(1, 0, 0)$  of  $\tilde{P}$ . The collection  $\{\tilde{\sigma}_k\}_{k \in \mathbb{Z}_{\geq 0}}$  is an orthonormal basis of the space  $\tilde{\mathfrak{H}}$  of quantum states of the  $\mathbb{Z}_2$ -reduced mathematical pendulum.

Since the quantum states  $\{\tilde{\sigma}_k\}_{k \in \mathbb{Z}_{\geq 0}}$  are ordered by increasing  $k$ , there exist shifting operators  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}$  such that

$$\begin{aligned}\tilde{\mathbf{b}}\tilde{\sigma}_k &= \tilde{\sigma}_{k+1}, \quad \text{for } k \geq 0 \\ \tilde{\mathbf{a}}\tilde{\sigma}_k &= \tilde{\sigma}_{k-1}, \quad \text{for } k > 0 \text{ and } \tilde{\mathbf{a}}\tilde{\sigma}_0 = 0.\end{aligned}\tag{53}$$

Because the local lattice structure of the set of Bohr-Sommerfeld tori on  $\tilde{P}$  is linear, the shifting operators  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}$  are also well defined across the singularity at the reduced energy value  $e = 2$ . As before, the operators  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}$  satisfy the same commutation relations as the quantum operators  $\mathbf{Q}_{e-i\tilde{\vartheta}}$  and  $\mathbf{Q}_{e+i\tilde{\vartheta}}$ , respectively.

#### 6.2.4 Lifting the shifting operators

In this subsection we use the isomorphism  $\mathbf{R} : \mathfrak{H}^{\text{even}} \rightarrow \tilde{\mathfrak{H}}$  to lift the shifting operators on  $\tilde{\mathfrak{H}}$  to shifting operators on  $\mathfrak{H}^{\text{even}}$ .

We define  $\mathbf{R}$  as the operator which sends the basis  $\{\sigma_{2k}^0, k = 1, \dots, K; \sigma_m^+ + \sigma_m^-, m = M, M+1, \dots\}$  of  $\mathfrak{H}^{\text{even}}$  to the basis  $\{\tilde{\sigma}_k\}$  of  $\tilde{\mathfrak{H}}$  as follows

$$\begin{aligned}\mathbf{R}(\sigma_{2k}^0) &= \tilde{\sigma}_k, \quad \text{for } k = 1, \dots, K \\ \mathbf{R}(\sigma_m^+ + \sigma_m^-) &= \tilde{\sigma}_m, \quad \text{for } m \geq M.\end{aligned}\tag{54}$$

Recall that  $2M = \begin{cases} N+2, & \text{if } N \text{ is even} \\ N+1, & \text{if } N \text{ is odd} \end{cases}$  and  $2K = \begin{cases} N, & \text{if } N \text{ is even} \\ N-1, & \text{if } N \text{ is odd} \end{cases}$ . To define shifting operators on  $\mathfrak{H}^{\text{even}}$  recall that equation (53) defines the shifting operators  $\tilde{\mathbf{b}}$  and  $\tilde{\mathbf{a}}$  on  $\tilde{\mathfrak{H}}$ . We may lift the shifting operator  $\tilde{\mathbf{a}}$  to the shifting operator  $\mathbf{a}^{\text{even}}$  on  $\mathfrak{H}^{\text{even}}$  by setting

$$\begin{aligned}\mathbf{R}\mathbf{a}^{\text{even}}\sigma_{2k}^0 &= \tilde{\mathbf{a}}\mathbf{R}\sigma_{2k}^0 \quad \text{for } k \leq K \\ \mathbf{R}\mathbf{a}^{\text{even}}(\sigma_m^+ + \sigma_m^-) &= \tilde{\mathbf{a}}\mathbf{R}(\sigma_m^+ + \sigma_m^-) \quad \text{for } m \geq M\end{aligned}\tag{55a}$$

and lift the shifting operator  $\tilde{\mathbf{b}}$  to the shifting operator  $\mathbf{b}^{\text{even}}$  on  $\mathfrak{H}^{\text{even}}$  by setting

$$\begin{aligned}\mathbf{R}\mathbf{b}^{\text{even}}\sigma_{2k}^0 &= \tilde{\mathbf{b}}\mathbf{R}\sigma_{2k}^0 \quad \text{for } k \leq K \\ \mathbf{R}\mathbf{b}^{\text{even}}(\sigma_m^+ + \sigma_m^-) &= \tilde{\mathbf{b}}\mathbf{R}(\sigma_m^+ + \sigma_m^-) \quad \text{for } m \geq M.\end{aligned}\tag{55b}$$

If  $k+1 \leq K$ , then

$$\mathbf{R}\mathbf{b}^{\text{even}}\sigma_{2k}^0 = \tilde{\mathbf{b}}\mathbf{R}\sigma_{2k}^0 = \tilde{\mathbf{b}}\tilde{\sigma}_k = \tilde{\sigma}_{k+1} = \mathbf{R}\sigma_{2k+2}^0.$$

So  $\mathbf{b}^{\text{even}}\sigma_{2k}^0 = \sigma_{2k+2}^0$ , because  $\mathbf{R}$  is injective. Since  $n = 2k$ , it follows that  $\mathbf{b}^{\text{even}}$  raises the quantum number  $n$  by 2, provided that  $n + 1 < N$ . Hence  $\sigma_{n+2}^0 = \mathbf{b}\sigma_{n+1}^0 = \mathbf{b}\mathbf{b}\sigma_n^0 = \mathbf{b}^{\text{even}}\sigma_n^0$ . A similar argument shows that  $\mathbf{b}^{\text{even}}$  raises the quantum number  $m \geq M$  by 1 and that  $\mathbf{b}^{\text{even}}(\sigma_m^+ + \sigma_m^-) = \mathbf{Q}_{e^{i\vartheta}}(\sigma_m^+ + \sigma_m^-)$ . Analogous results can be obtained for the lowering operator  $\mathbf{a}^{\text{even}}$ . In particular, if  $k \leq K$ , then

$$\mathbf{R}\mathbf{a}^{\text{even}}\sigma_{2k}^0 = \tilde{\mathbf{a}}\mathbf{R}\sigma_{2k}^0 = \tilde{\mathbf{a}}\tilde{\sigma}_k = \tilde{\sigma}_{k-1} = \mathbf{R}\sigma_{2k-2}^0.$$

This implies that  $\mathbf{a}^{\text{even}}\sigma_n^0 = \sigma_{n-2}^0 = \mathbf{Q}_{e^{-2i\vartheta}}\sigma_n^0$  for  $0 < n \leq N$ . Similarly, for  $m > M$  we get  $\mathbf{a}^{\text{even}}(\sigma_m^+ + \sigma_m^-) = (m-1)(\sigma_m^+ + \sigma_m^-)$ .

## 7 Crossing the singularity

The operators  $\mathbf{a}^{\text{even}}$  and  $\mathbf{b}^{\text{even}}$ , defined in equation (55a) and (55b), respectively allow for shifting quantum states which cross the singular level set  $H^{-1}(2)$ . In order to write this out explicitly, we need to consider the cases when  $N$  is even or odd separately.

We look at the operators  $\mathbf{a}^{\text{even}}$  and  $\mathbf{b}^{\text{even}}$  when  $N = 2K$ ,  $M = \frac{1}{2}(N+2) = K+1$ , and  $\sigma_N^0 = \sigma_{2K}^0 \in \mathfrak{H}^{\text{even}} \cap \mathfrak{H}_0$  together with

$$\sigma_M^+ + \sigma_M^- = \sigma_{K+1}^+ + \sigma_{K+1}^- \in \mathfrak{H}^{\text{even}} \cap (\mathfrak{H}_+ \oplus \mathfrak{H}_-).$$

In this case equation (55b) yields

$$\begin{aligned} \mathbf{R}\mathbf{b}^{\text{even}}\sigma_N^0 &= \mathbf{R}\mathbf{b}^{\text{even}}\sigma_{2K}^0 = \tilde{\mathbf{b}}\mathbf{R}\sigma_{2K}^0 = \tilde{\mathbf{b}}\tilde{\sigma}_K \\ &= \tilde{\sigma}_{K+1} = \tilde{\sigma}_M = \mathbf{R}(\sigma_M^+ + \sigma_M^-). \end{aligned} \quad (56)$$

Since  $\mathbf{R} : \mathfrak{H}^{\text{even}} \rightarrow \tilde{\mathfrak{H}}$  is injective, we get

$$\mathbf{b}^{\text{even}}\sigma_N^0 = \sigma_M^+ + \sigma_M^-. \quad (57)$$

Let  $\mathbf{pr}^\pm : \mathfrak{H}_+ \oplus \mathfrak{H}_- \rightarrow \mathfrak{H}_\pm : \sigma_m^+ + \sigma_m^- \mapsto \sigma_m^\pm$ . From (57) we get

$$\mathbf{pr}^+\mathbf{b}^{\text{even}}\sigma_N^0 = \sigma_M^+ \quad \text{and} \quad \mathbf{pr}^-\mathbf{b}^{\text{even}}\sigma_N^0 = \sigma_M^-, \quad (58)$$

which represents the transition given by the right pointing top and bottom slanted arrows in diagram (18) when  $N$  is even. Similarly,

$$\begin{aligned} \mathbf{R}\mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) &= \tilde{\mathbf{a}}\mathbf{R}(\sigma_M^+ + \sigma_M^-) = \tilde{\mathbf{a}}\mathbf{R}(\sigma_{K+1}^+ + \sigma_{K+1}^-) \\ &= \tilde{\mathbf{a}}\tilde{\sigma}_{K+1} = \tilde{\sigma}_K = \mathbf{R}\sigma_{2K}^0 = \mathbf{R}\sigma_N^0, \end{aligned}$$

which implies

$$\mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) = \sigma_N^0. \quad (59)$$

Next we look at the operators  $\mathbf{a}^{\text{even}}$  and  $\mathbf{b}^{\text{even}}$  when  $N = 2K + 1$  and  $M = K + 1$  and  $\sigma_N^0 = \sigma_{2K+1}^0 \in \mathfrak{H}^{\text{odd}} \cap \mathfrak{H}_0$ . Then by theorem 8 we have  $\sigma_{N-1}^0 = \sigma_{2K}^0 \in \mathfrak{H}^{\text{even}} \cap \mathfrak{H}_0$ . Moreover,  $\sigma_M^+ + \sigma_M^- = \sigma_{K+1}^+ + \sigma_{K+1}^- \in \mathfrak{H}^{\text{even}} \cap (\mathfrak{H}_+ \oplus \mathfrak{H}_-)$ . We can cross directly from  $\sigma_{N-1}^0 = \sigma_{2K}^0$  to  $\sigma_M^+ + \sigma_M^- = \sigma_{K+1}^+ + \sigma_{K+1}^-$  using the operator  $\mathbf{b}^{\text{even}}$ . In other words,

$$\mathbf{b}^{\text{even}}\sigma_{N-1}^0 = \sigma_M^+ + \sigma_M^-. \quad (60)$$

Therefore, in order to cross from  $\sigma_N^0$  to  $\sigma_M^+ + \sigma_M^-$ , we first go to  $\sigma_{N-1}^0$  and then to  $\sigma_M^+ + \sigma_M^-$ . So  $\mathbf{b}^{\text{even}}\mathbf{a}\sigma_N^0 = \sigma_M^+ + \sigma_M^-$ . Hence for odd  $N$ , we have

$$\mathbf{pr}^+\mathbf{b}^{\text{even}}\mathbf{a}\sigma_N^0 = \sigma_M^+ \quad \text{and} \quad \mathbf{pr}^-\mathbf{b}^{\text{even}}\mathbf{a}\sigma_N^0 = \sigma_M^-, \quad (61)$$

which represents the right pointing top and bottom slanted arrows in diagram (18) when  $N$  is odd. Similarly,

$$\begin{aligned} \mathbf{R}\mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) &= \tilde{\mathbf{a}}\mathbf{R}(\sigma_M^+ + \sigma_M^-) = \tilde{\mathbf{a}}\mathbf{R}(\sigma_{K+1}^+ + \sigma_{K+1}^-) \\ &= \tilde{\mathbf{a}}\tilde{\sigma}_{K+1} = \tilde{\sigma}_K = \mathbf{R}\sigma_{2K}^0 = \mathbf{R}\sigma_{N-1}^0, \end{aligned}$$

which implies

$$\mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) = \sigma_{N-1}^0. \quad (62)$$

Let

$$\iota_{\pm} : \mathfrak{H}_{\pm} \rightarrow \mathfrak{H}_+ \oplus \mathfrak{H}_- : \sigma_m^{\pm} \mapsto \frac{1}{2}(\sigma_m^{\pm} + \mathbf{P}\sigma_m^{\pm}) = \sigma_m^+ + \sigma_m^-. \quad (63)$$

Using the injection mapping  $\iota_{\pm}$  and equations (59) and (62) when  $N = 2M - 2$  we have

$$\hat{\mathbf{a}}_{\pm}^{\text{even}}(\sigma_M^{\pm}) = \mathbf{a}^{\text{even}}\iota_{\pm}(\sigma_M^{\pm}) = \mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) = \sigma_N^0; \quad (64a)$$

while when  $N = 2M - 1$  we have

$$\hat{\mathbf{a}}_{\pm}^{\text{even}}(\sigma_M^{\pm}) = \mathbf{b}\mathbf{a}^{\text{even}}\iota_{\pm}(\sigma_M^{\pm}) = \mathbf{b}\mathbf{a}^{\text{even}}(\sigma_M^+ + \sigma_M^-) = \mathbf{b}\sigma_{N-1}^0 = \sigma_N^0. \quad (64b)$$

The operator  $\hat{\mathbf{a}}_+^{\text{even}}$  represents the transition given by the left pointing top slanting arrow in diagram (18); while the operator  $\hat{\mathbf{a}}_-^{\text{even}}$  represents the transition given by the left pointing bottom slanting arrow in the diagram.

## 8 Appendix: construction of the lowering operator

In this appendix we construct the lowering operator  $\mathbf{a}$  for the quantized  $\mathbb{Z}_2$ -reduced system on  $T^\times S^1$ .

On phase space  $T^*\tilde{S}^1 = \mathbb{R} \times \tilde{S}^1$ , where  $\tilde{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$  with coordinates  $(p, \vartheta)$  and symplectic form  $\omega = dp \wedge d\vartheta$  consider the trivial (right) principal bundle

$$\pi^\times : L^\times = \mathbb{C}^\times \times T^*\tilde{S}^1 \rightarrow T^*\tilde{S}^1 : (b, (p, \vartheta)) \mapsto (p, \vartheta)$$

with connection 1-form  $\beta = \frac{1}{2\pi} \frac{db}{b} - \frac{1}{h} p d\vartheta$ . The curvature  $d\beta$  of  $\beta$  is  $-\frac{1}{h}\omega$ , which we suppose has integer de Rham cohomology.

The action integral of Bohr-Sommerfeld quantization is  $I = \int_0^{2\pi} p d\vartheta = 2\pi p$ . The variable conjugate to  $I$  is  $\theta = \vartheta/2\pi$ , since  $\omega = dI \wedge d\theta$ . Let  $(I, \theta)$  be coordinates on  $T^*S^1 = \mathbb{R} \times S^1$  with  $S^1 = \mathbb{R}/\mathbb{Z}$  and let  $\omega = dI \wedge d\theta$  be the symplectic form on  $T^*S^1$ . The vector field  $X = -\frac{\partial}{\partial I} = -\frac{1}{2\pi} \frac{\partial}{\partial p}$  on  $T^*S^1$  is locally Hamiltonian, since

$$L_X \omega = d(X \lrcorner \omega) + X \lrcorner d\omega = d(-d\theta) = 0,$$

and has *local* Hamiltonian  $\theta$ , since  $-d\theta = X \lrcorner \omega$  *locally*. The flow of  $X$  is

$$e^{tX} : T^*S^1 \rightarrow T^*S^1 : (I, \theta) \mapsto (I - t, \theta) = (2\pi(p - \frac{1}{2\pi}t), \vartheta/2\pi). \quad (65)$$

Note that the diffeomorphism  $e^{hX}$  sends the Bohr-Sommerfeld torus  $T_n$ , defined by  $\{I = nh\}$ , onto the Bohr-Sommerfeld torus  $T_{n-1}$ , defined by  $\{I = (n-1)h\}$ .

In what follows we find a quantomorphism  $\Phi_h$  of  $(L, \lambda)$ , which covers  $e^{hX}$ . In other words,  $\Phi_h$  is a diffeomorphism of  $L$  into itself such that  $\Phi_h^* \lambda = \lambda$  and  $\pi \circ \Phi_h = e^{hX} \circ \pi$ . Here

$$\pi : L = \mathbb{C} \times T^*S^1 \rightarrow T^*S^1 : (z, (I, \theta)) \mapsto (I, \theta)$$

is the line bundle, associated to the  $\mathbb{C}^\times$  principal bundle  $\pi^\times : L^\times \rightarrow T^*S^1$ , with connection 1-form  $\lambda = \frac{1}{2\pi i} dz - \frac{1}{h} I d\theta$ .

The vector field  $X$  is *integral*, that is, 1) there is an *good covering*  $\mathcal{U} = \{U_i\}_{i \in I}$  of  $T^*S^1$  by open sets  $U_i$ ,  $i \in I$ , where every finite intersection of elements of  $\mathcal{U}$  is either empty or contractible; 2) for every  $U_i, U_j \in \mathcal{U}$  such that  $U_i \cap U_j \neq \emptyset$  we have  $\theta|_{U_i} - \theta|_{U_j}$  is an integer on  $U_i \cap U_j$ . The local Hamiltonian functions  $\theta|_{U_i}$  for  $i \in I$  piece together to give a smooth mapping  $[\theta] : T^*S^1 \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ , which is the ‘‘coordinate’’  $\theta$ , that is,  $[\theta] = \vartheta/2\pi$ .

Consider the vector field  $Z$  on  $L^\times$ , whose flow is

$$e^{tZ} : L^\times \rightarrow L^\times : (b, (I, \theta)) \mapsto (b e^{-2\pi i t \theta / h}, (I - t, \theta)). \quad (66)$$

The flow of  $Z$  preserves the connection 1-form  $\beta$ , since for every  $(b, I, \theta) \in L^\times$  we have

$$\begin{aligned} ((e^{tZ})^* \beta)(b, I, \theta) &= \frac{1}{2\pi i} d \ln [b e^{-2\pi i t \theta / h}] - \frac{1}{h} (I - t) d\theta \\ &= \left( \frac{1}{2\pi i} \frac{db}{b} - \frac{1}{h} I d\theta \right) - \frac{1}{h} t d\theta + \frac{1}{h} t dd\theta = \beta(b, I, \theta). \end{aligned}$$

We have

$$e^{tZ} = e^{-tY_{\theta/h}} \circ e^{t \text{lift} X}, \quad (67)$$

where  $Y_{\theta/h}(b, I, \theta) = \frac{d}{dt} \Big|_{t=0} (b e^{2\pi i t \theta / h}, I, \theta)$  is a vector field on  $(L^\times, \beta)$ , whose flow is  $e^{tY_{\theta/h}}(b, I, \theta) = (b e^{2\pi i t \theta / h}, I, \theta)$ , and  $\text{lift} X$  is a vector field on  $(L^\times, \beta)$ , which is the horizontal lift of the vector field  $X$ , that is,  $\text{lift} X(b, I, \theta) \in \ker \beta(b, I, \theta)$  for every  $(b, I, \theta) \in L^\times$ . The vector fields  $\text{lift} X$  and  $X$  are  $\pi^\times$ -related, that is,  $T_{(b, I, \theta)} \pi^\times (\text{lift} X(b, I, \theta)) = X(\pi^\times(b, I, \theta))$  for every  $(b, I, \theta) \in L^\times$ . The flow of  $\text{lift} X$  is

$$e^{t \text{lift} X}(b, I, \theta) = (b, e^{tX}(I, \theta)) = (b, I - t, \theta).$$

Note that the flows of the vector fields  $Y_{\theta/h}$  and  $\text{lift} X$  commute.

We now look at the universal covering space  $(T^*\mathbb{R}, \tilde{\omega})$  of  $(T^*S^1, \omega)$  with coordinates  $(p, q)$  and symplectic form  $\tilde{\omega} = dp \wedge dq$ . The universal covering map is given by

$$\kappa : T^*\mathbb{R} \rightarrow T^*S^1 : (p, q) \mapsto \left( \frac{1}{2\pi} I, \theta \right) = (p, q \bmod \mathbb{Z}),$$

since  $\mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z} : q \mapsto q \bmod \mathbb{Z}$  is the smooth universal covering map of  $S^1$ . Pull the local Hamiltonian vector field  $X$  on  $T^*S^1$  back by the covering map  $\kappa$  to a vector field  $\tilde{X}$  on  $T^*\mathbb{R}$ , which is  $\kappa$ -related to  $X$ , that is,  $T_{(p, q)} \kappa (\tilde{X}(p, q)) = X(\kappa(p, q))$  for every  $(p, q) \in T^*\mathbb{R}$ . The integral vector field  $\tilde{X}$  is the Hamiltonian vector field  $X_{q-c} = -\frac{\partial}{\partial p}$  associated to the Hamiltonian function  $q : T^*\mathbb{R} \rightarrow \mathbb{R} : (p, q) \mapsto q$ . The constant  $c$  can be chosen to be 0, because the smooth mappings  $[q] : T^*\mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z} : (p, q) \mapsto q \bmod \mathbb{Z}$  and  $\kappa^*[\theta] : T^*\mathbb{R} \rightarrow S^1 : (q, p) \mapsto [\theta](\kappa(p, q))$  are equal, namely,

$$[q] = \kappa^*[\theta]. \quad (68)$$

A calculation shows that

$$\kappa \circ e^{tXq} = e^{tX} \circ \kappa. \quad (69a)$$

The  $\mathbb{C}^\times$  bundle  $\pi^\times : L^\times \rightarrow T^*S^1$  pulls back under the covering map  $\kappa$  to the  $C^\times$  bundle  $\tilde{\pi}^\times : \tilde{L}^\times \rightarrow T^*\mathbb{R} : (b, p, q) \mapsto (p, q)$  with connection 1-form  $\tilde{\beta} = \kappa^*\beta = \frac{1}{2\pi i} \frac{db}{b} - \frac{1}{h} p dq$ . The flow  $e^{tZ}$  on  $L^\times$  lifts to the 1-parameter group of diffeomorphisms

$$e^{t\tilde{Z}} : \tilde{L}^\times \rightarrow \tilde{L}^\times : (b, p, q) \mapsto (b e^{-2\pi i t q/h}, p - t, q), \quad (70)$$

which preserves  $\tilde{\beta}$ . We have  $e^{t\tilde{Z}} = e^{-t\tilde{Y}_{q/h}} \circ e^{t \text{lift} X_q}$ , where  $\tilde{Y}_{q/h}$  is the vector field on  $\tilde{L}^\times$ , whose value at  $(p, q)$  is  $\left. \frac{d}{dt} \right|_{t=0} (b e^{-2\pi i t q/h}, p, q)$ . Its flow is given by  $e^{tY_{q/h}}(b, p, q) = (b e^{2\pi i t q/h}, p, q)$ . Also  $\text{lift} X_q$  is a vector field on  $\tilde{L}^\times$ , which is the horizontal lift of  $X_q$  using the connection 1-form  $\tilde{\beta}$  on  $\tilde{L}^\times$ . The flow of  $\text{lift} X_q$  is  $e^{t \text{lift} X_q}(b, p, q) = (b, p - t, q)$ . Note that the flows  $e^{tY_{q/h}}$  and  $e^{t \text{lift} X_q}$  commute. Since  $\tilde{L}^\times = \kappa^*L^\times$ , there is a smooth mapping

$$\kappa^\times : \tilde{L}^\times \rightarrow L^\times : (b, p, q) \mapsto (b, p, q \bmod \mathbb{Z}) = (b, \frac{1}{2\pi} I, \theta),$$

which covers  $\kappa$ , that is,  $\pi^\times \circ \kappa^\times = \kappa \circ \tilde{\pi}^\times$ . The flows  $e^{t \text{lift} X_q}$  and  $e^{t \text{lift} X}$  are  $\kappa^\times$ -related, that is

$$\kappa^\times \circ e^{t \text{lift} X_q} = e^{t \text{lift} X} \circ \kappa^\times. \quad (69b)$$

Let

$$\sigma^\times : T^*S^1 \rightarrow L^\times : (I, \theta) \mapsto (b(I, \theta), I, \theta)$$

be a smooth section of the bundle  $\pi^\times : L^\times \rightarrow T^*S^1$ , where  $(I, \theta) \mapsto b(I, \theta)$  is a smooth nowhere vanishing complex valued function on  $T^*S^1$ . Then

$$\begin{aligned} (e^{tZ})_* \sigma^\times &= e^{tZ} \circ \sigma^\times \circ e^{-tX} = e^{-tY_{\theta/h}} \circ (e^{t \text{lift} X} \circ \sigma^\times \circ e^{-tX}) \\ &= (e^{-tY_{\theta/h}})_* \circ (e^{t \text{lift} X})_* \sigma^\times \end{aligned} \quad (70)$$

is a smooth section of the bundle  $\pi^\times : L^\times \rightarrow T^*S^1$ . Let

$$\tilde{\sigma}^\times : T^*\mathbb{R} \rightarrow \tilde{L}^\times : (p, q) \mapsto (\tilde{b}(p, q), p, q)$$

be a smooth section of the bundle  $\tilde{\pi}^\times : \tilde{L}^\times \rightarrow T^*\mathbb{R}$ , which is the pull back by the mapping  $\kappa^\times$  of the smooth section  $\sigma^\times$  of the bundle  $\pi^\times : L^\times \rightarrow T^*S^1$ . For every  $(p, q) \in T^*\mathbb{R}$  we have  $\tilde{\sigma}^\times(p, q) = (b(\kappa(p, q)), \kappa(p, q))$ . So

$$\kappa^\times \circ \tilde{\sigma}^\times = \sigma^\times \circ \kappa, \quad (69c)$$

which characterizes  $\tilde{\sigma}^\times$ . We now show that

$$(e^{h\tilde{Z}})_* \tilde{\sigma}^\times = (\kappa^\times)^*(e^{-2\pi i [\theta]} \bullet (e^{h \text{lift} X})_* \sigma^\times), \quad (70)$$

We explain what the symbol  $\bullet$  in the above formula means. For a smooth section  $\sigma^\times : T^*S^1 \rightarrow L^\times : (I, \theta) \mapsto (b(I, \theta), I, \theta)$  of the bundle  $\pi^\times : L^\times \rightarrow T^*S^1$  and a smooth complex valued function  $f$  on  $T^*S^1$  we define  $f \bullet \sigma^\times : T^*S^1 \rightarrow L^\times$  to be the smooth section  $(I, \theta) \mapsto (b(I, \theta)f(I, \theta), I, \theta)$ .

**Proof.** Analogous to (67) we have

$$(e^{t\tilde{Z}})_* \tilde{\sigma}^\times = (e^{-t\tilde{Y}_{\theta/h}})_* \circ (e^{t \text{lift} X_q})_* \tilde{\sigma}^\times, \quad (71)$$

where  $\sigma^\times = (\kappa^\times)^* \sigma^\times$ . To verify equation (70) we first show that

$$(\kappa^\times)^* ((e^{h \text{lift} X})_* \sigma^\times) = (e^{h \text{lift} X_q})_* ((\kappa^\times)^* \sigma^\times). \quad (71a)$$

This follows by applying equations (69a), (69b), and (69c). Next we show that

$$(e^{h\tilde{Y}_{q/h}})_* ((\kappa^\times)^* \sigma^\times) = (\kappa^\times)^* (e^{-2\pi i [\theta]} \bullet \sigma^\times). \quad (71b)$$

Using (68), for every  $(p, q) \in T^*\mathbb{R}$  we get

$$\begin{aligned} e^{2\pi i [\theta](\kappa(p, q))} &= e^{2\pi i [q]} = e^{2\pi i (q+n)}, \text{ for every } n \in \mathbb{Z} \\ &= e^{2\pi i q} = e^{h(2\pi i q/h)}. \end{aligned}$$

So

$$\begin{aligned} (e^{h\tilde{Y}_{q/h}})_* ((\kappa^\times)^* \sigma^\times)(p, q) &= (\kappa^\times)^* ((e^{hY_{q/h}})_* \sigma^\times)(p, q) \\ &= ((e^{hY_{[\theta]/h}})_* \sigma^\times)(\kappa(p, q)) = \sigma^\times(\kappa(p, q)) e^{h(2\pi i q/h)} \\ &= \sigma^\times(\kappa(p, q)) e^{2\pi i [\theta](\kappa(p, q))} = (e^{2\pi i [\theta]} \bullet \sigma^\times)(\kappa(p, q)) \\ &= (\kappa^\times)^* (e^{2\pi i [\theta]} \bullet \sigma^\times)(p, q). \end{aligned}$$

Equation (71) follows from equations (71a) and (71b).  $\blacksquare$

The locally Hamiltonian vector field  $X$  on  $T^*S^1$  with flow  $e^{tX} : T^*S^1 \rightarrow T^*S^1 : (I, \theta) \mapsto (I - t, \theta)$  lifts to a vector field  $\hat{X}$  on  $L$ , whose flow is  $e^{t\hat{X}} : L \rightarrow L : (z, I, \theta) \mapsto (z, I - t, \theta)$ . The map

$$\mu : L = \mathbb{C} \times T^*S^1 \rightarrow L^\times = \mathbb{C}^\times \times T^*S^1 : (z, I, \theta) \mapsto (e^z, I, \theta) = (b, I, \theta) \quad (72)$$

is smooth and  $\mu^* \beta = \lambda$ . Moreover, we have  $\mu \circ e^{t\hat{X}} = e^{t \text{lift} X} \circ \mu$ . Instead of  $e^{t\hat{X}}$  we will write  $\hat{e}^{tX}$ . The mapping  $\mu$  intertwines the (right) action of  $\mathbb{C}^\times$  on  $L^\times$  with the action of  $\mathbb{C}^\times$  on  $L$ , namely,  $\mu(b'z, I, \theta) = \mu(b, I, \theta)(b')^{-1}$  for every  $b, b' \in \mathbb{C}^\times$  and every  $(I, \theta) \in T^*S^1$ . From these remarks it follows that



the operator  $\Phi_h^\times = e^{-2\pi i[\theta]} \bullet (\widehat{e}^{h \text{ lift } X})_*$  on smooth sections of the line bundle  $\pi^\times : L^\times \rightarrow T^*S^1$  becomes the operator

$$\Phi_h = e^{2\pi i[\theta]} (\widehat{e}^{h \text{ lift } X})_* \quad (73)$$

on smooth sections of the line bundle  $\pi : L \rightarrow T^*S^1$ . Clearly,  $\Phi_h$  covers  $e^{hX}$ , that is,  $\pi \circ \Phi_h = e^{hX} \circ \pi$ , and preserves the connection 1-form  $\lambda$ . So  $\Phi_h$  is the desired lowering operator **a**.

We note that  $\Phi_{-h}$  is the raising operator **b**.

## References

- [1] Niels Bohr, On the constitution of atoms and molecules (Part I) , *Philosophical Magazine*, **26** (1913) 1-25.
- [2] Richard H. Cushman and Larry M. Bates, “Global aspects of classical integrable systems”, second edition, Birkhäuser, Basel, 2015.
- [3] Richard Cushman and Jędrzej Śniatycki, Bohr-Sommerfeld-Heisenberg theory in geometric quantization, *J. Fixed Point Theory Appl.* **13** (2013) 3–24.
- [4] Richard Cushman and Jędrzej Śniatycki, Bohr-Sommerfeld Heisenberg quantization of the 2-dimensional harmonic oscillator, [arXiv:math.SG.1207.1477v2](#).
- [5] Richard Cushman and Jędrzej Śniatycki, Bohr-Sommerfeld-Heisenberg theory in geometric quantization, *J. Fixed Point Theory Appl.* **13** (2013) 3–24.
- [6] Richard Cushman and Jędrzej Śniatycki, Shifting operators in geometric quantization, [arXiv:math.SG.1808.04002](#).
- [7] Paul Adrien Maurice Dirac, The fundamental equations of quantum mechanics, *Proc. Roy. Soc. London*, **A 109** (1925) 642–653.
- [8] Paul Adrien Maurice Dirac, “The principles of quantum mechanics”, Clarendon Press, Oxford, UK, 1930
- [9] Holger Dullin, Semi-global symplectic invariants of the spherical pendulum, *J. Differential Equations* **254** (2013) 2942–2963.
- [10] Werner Heisenberg, Über die quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen, *Z. Phys.* **33** (1925) 879–893.
- [11] Jędrzej Śniatycki, *Geometric Quantization and Quantum Mechanics*, Applied Mathematical Series **30**, Springer Verlag, New York, 1980.

- [12] Arnold Sommerfeld, Zur Theorie der Balmerischen Serie, *Sitzungsberichte der Bayerischen Akademie der Wissenschaften (München), mathematisch-physikalische Klasse*, (1915) 425-458.