Bohr-Sommerfeld-Heisenberg quantization of the mathematical pendulum

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Classical mathematical pendulum

config. space: $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Coordinate: α.

phase space: $T^*S^1 = S^1 \times \mathbb{R}$. Coordinates: (α, p).

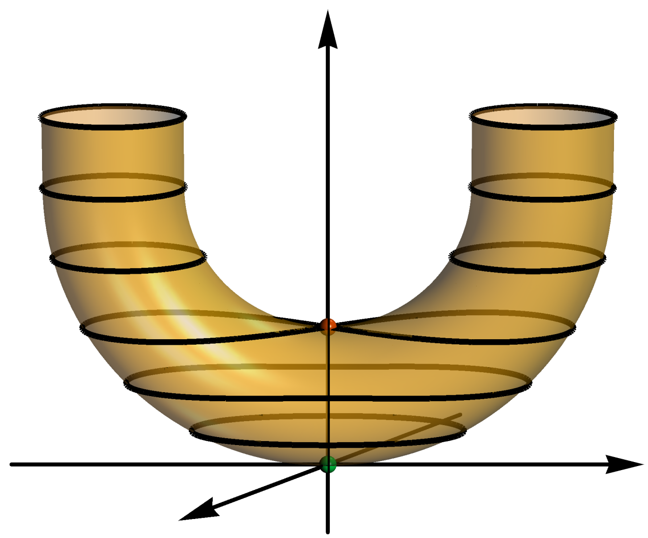
symplectic form: $\omega = \mathrm{d}(p\,\mathrm{d}\alpha) = -\mathrm{d}\Theta$.

Hamiltonian:

$$H : T^*S^1 \to \mathbb{R} : (\alpha, p) \mapsto \tfrac{1}{2}p^2 - \cos\alpha + 1.$$

Hamiltonian vector field: $p\,\frac{\partial}{\partial\alpha} - \sin\alpha\,\frac{\partial}{\partial p}$.

Hamilton's equations: $\frac{\mathrm{d}\alpha}{\mathrm{d}t} = p$ and $\frac{\mathrm{d}p}{\mathrm{d}t} = -\sin\alpha$.



The graph of H

Action angle coordinates

action:

$$I = \frac{1}{2\pi}\int_{C(e)} \Theta = \frac{1}{2\pi}\int_{C(e)} p\,\mathrm{d}\alpha,$$

where $C(e)$ is connected component of $H^{-1}(e)$

angle: $\theta = \frac{2\pi}{T} t$, where $T = 4\int_0^{\pi/2} \frac{1}{\sqrt{1-\frac{e}{2}\sin^2\varphi}} \mathrm{d}\varphi$

is the period of integral curve of X_H parametrizing $C(e)$. I is an integral of X_H.

Poisson bracket: $\{I, \theta\} = -1$.

Symplectic form: $\omega = \mathrm{d}I \wedge \mathrm{d}\theta$.

Quantum mathematical pendulum

quantum line bundle:

$$\rho : L = \mathbb{C} \times T^*S^1 \to T^*S^1 : \big(z, (\alpha, p)\big) \mapsto (\alpha, p).$$

connection 1-form: $\lambda = \mathrm{d}z - \frac{i}{\hbar}\Theta.$

covariant derivative: $\nabla_X \sigma_0 = -i\hbar\,(X \,\lrcorner\, \Theta)\sigma_0.$

$\sigma_0 : T^*S^1 \to L : (\alpha, p) \mapsto \big(1, (\alpha, p)\big)$ and X a vector field on T^*S^1.

polarization D of T^*S^1: integral curves of X_H.

quantum states: smooth sections of L which are covariantly constant along D

Bohr-Sommerfeld quantization

Bohr-Sommerfeld torus is $C(e)$ with $0 < e < 2$ or $e > 2$ such that $\frac{1}{2\pi}\int_\gamma \Theta = n\,\hbar$ with $n \in \mathbf{Z}$. γ is an integral curve of X_H parametrizing $C(e)$. When $e = 0$, $n = 0$.

Bohr-Sommerfeld quantum state is a smooth section σ of $\rho : L \to T^*S^1$ restricted to a union of Bohr-Sommerfeld tori, which is covariantly constant.

$f \in C^\infty(T^*S^1)$ is Bohr-Sommerfeld quantizable if it is constant on each Bohr-Sommerfeld torus.

Hilbert space

Hilbert space $\mathfrak{H}$ has basis $\{\sigma|_C\}$ of Bohr-Sommerfeld quantum states with inner product making distinct states orthogonal.

Subspaces $\mathfrak{H}_{0,\pm}$ spanned by sections $\sigma_{0,\pm}$ with support C in $P_0 = \{(\alpha, p) \in T^*S^1 \big| H(\alpha, p) < 2\}$ or in $P_\pm = \{(\alpha, p) \in T^*S^1 \big| \mp p > 0 \;\&\; H(\alpha, p) > 2\}$.

Bases: For $C \subseteq P_0$ with N the largest integer such that $2\pi\hbar N < I(2)$, $\{\sigma_n^0\}_{n=0}^N$ is a basis of $\mathfrak{H}_0$. For $C \subseteq P_\pm$ with M the smallest integer such that $2M \geq N+1$, $\{\sigma_m^\pm\}_{m=M}^\infty$ is a basis of $\mathfrak{H}_\pm$. $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

Transition operators

raising operator: $\mathbf{b}\sigma_n^0 = \sigma_{n+1}^0$ for $n = 0, 1, \ldots, N-1$;
$\mathbf{b}\sigma_m^{\pm} = \sigma_{m+1}^{\pm}$ for $m = M+1, \ldots$.

lowering operator: $\mathbf{a}\sigma_n^0 = \sigma_{n-1}^0$ for $n = 1, \ldots, N$ & $\mathbf{a}_0 = 0$;
$\mathbf{a}\sigma_m^{\pm} = \sigma_{m-1}^{\pm}$ for $m = M+1, \ldots$.

quantum state transition structure:

$$\sigma_0^0 \leftrightarrows \sigma_1^0 \leftrightarrows \cdots \leftrightarrows \sigma_N^0 \begin{array}{l} \swarrow\!\!\nearrow\ \sigma_M^+ \leftrightarrows \sigma_{M+1}^+ \leftrightarrows \cdots \\ \searrow\!\!\nwarrow\ \sigma_M^- \leftrightarrows \sigma_{M+1}^- \leftrightarrows \cdots \end{array}$$

Corresponding quantum operators

$(I_{0,\pm}, \theta_{0,\pm})$ action angle coordinates on $P_{0,\pm}$.

Commutation relations:

$$
\begin{aligned}
&[Q_{I_0}, \mathbf{b}]\sigma_n^0 = i\hbar\,\sigma_n^0,\ n = 0, \ldots, N-1;\\
&[Q_{I_\pm}, \mathbf{b}]\sigma_m^\pm = i\hbar\,\sigma_m^\pm,\ n = M, \ldots;\\
&[Q_{I_0}, \mathbf{a}]\sigma_n^0 = -i\hbar\,\sigma_n^0,\ n = 1, \ldots, N;\\
&[Q_{I_\pm}, \mathbf{a}]\sigma_m^\pm = -i\hbar\,\sigma_m^\pm,\ m = M+1, \ldots \quad .
\end{aligned}
$$

Because $\{I_{0,\pm}, \theta_{0,\pm}\} = -1$,

$$[Q_{I_{0,\pm}}, Q_{\mathrm{e}^{\mp i\theta_{0,\pm}}}]\sigma_{n,m}^{0,\pm} = \pm i\hbar\, Q_{\mathrm{e}^{\mp i\theta_{0,\pm}}}\sigma_{n,m}^{0,\pm}.$$

Identify $\mathbf{b}$ with $Q_{\mathrm{e}^{-i\theta_{0,\pm}}}$ and $\mathbf{a}$ with $Q_{\mathrm{e}^{+i\theta_{0,\pm}}}$.

$\mathbb{Z}_2$ symmetry

classical $\mathbb{Z}_2$ symmetry of (H, T^*S^1, ω) generated by

$$\zeta : T^*S^1 \to T^*S^1 : (\alpha, p) \mapsto (-\alpha, -p).$$

quantum $\mathbb{Z}_2$ symmetry of $(\rho : L \to T^*S^1, \lambda)$. On bundle space L generated by

$$\mu : L \to L : \big(z, (\alpha, p)\big) \mapsto \big(z, \zeta(\alpha, p)\big).$$

On sections $\Gamma(\rho)$ generated by

$$\mu^* : \Gamma(\rho) \to \Gamma(\rho) : \sigma \mapsto \mu^*\sigma = \mu^{-1} \circ \sigma \circ \zeta.$$

Parity operator

parity operator: $\mathbf{P} : \mathfrak{h} \to \mathfrak{h} : \sigma|_C \mapsto \mu^*(\sigma|_C)$.

properties:

1) μ^* generates a representation of $\mathbb{Z}_2$ on $\mathfrak{H}$.

2) $\mathbf{P}_{0,\pm} = \mathbf{P}|_{\mathfrak{H}_{0,\pm}} : \mathfrak{H}_{0,\pm} \to \mathfrak{H}_{0,\mp}$ is bijective with $\mathbf{P}_0^{-1} = \mathbf{P}_0$ and $\mathbf{P}_\pm^{-1} = \mathbf{P}_\mp$. Adjust inner product on $\mathfrak{H}$ so that $\mathbf{P}\sigma^\pm|_C = \sigma^\mp|_C$.

3) Set $\mathfrak{H}_{\text{even}} = \{\sigma|_C \in \mathfrak{H} \big| \mathbf{P}\sigma|_C = \sigma|_C\}$ and $\mathfrak{H}_{\text{odd}} = \{\sigma|_C \in \mathfrak{H} \big| \mathbf{P}\sigma|_C = -\sigma|_C\}$. Then $\mathfrak{H} = \mathfrak{H}_{\text{even}} \oplus \mathfrak{H}_{\text{odd}}$.

4) if $\sigma|_C \in \mathfrak{H}_{\text{even,odd}} \cap \mathfrak{H}_0$, then its quantum number is even, odd, respectively.

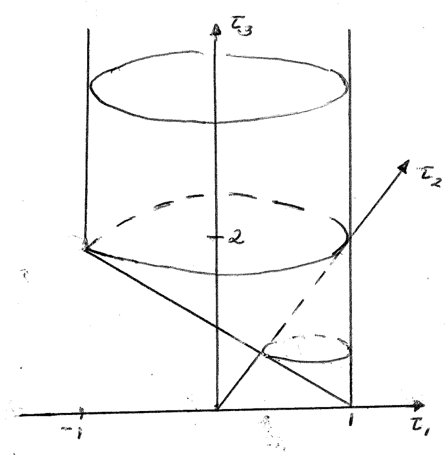
5) invariance of H under $\mathbb{Z}_2$ action gives $[Q_H, \mathbf{P}] = 0$.

$\mathbb{Z}_2$-reduced classical system

algebra of real analytic $\mathbb{Z}_2$-invariant functions generated by
$\tau_1 = \cos\alpha$, $\tau_2 = p\sin\alpha$ and $\tau_3 = \frac{1}{2}p^2 - \cos\alpha + 1$.
relation:

$$0 = \tfrac{1}{2}\tau_2^2 - (\tau_3 + \tau_1 - 1)(1 - \tau_1^2), \quad |\tau_1| \le 1 \;\&\; \tau_3 \ge 0$$

defines the reduced space $P^\vee = T^*S^1/\mathbb{Z}_2$.



The reduced space $P^{\vee}$.

<u>reduced Hamiltonian</u>:

$$H^\vee : P^\vee \subseteq \mathbb{R}^3 \to \mathbb{R} : \tau = (\tau_1, \tau_2, \tau_3) \mapsto \tau_3.$$

<u>reduced Poisson bracket</u>:

$$\begin{aligned} \{\tau_1, \tau_2\} &= \tau_1^2 - 1, \\ \{\tau_2, \tau_3\} &= 2\tau_1(\tau_3 + \tau_1 - 1) + \tau_2^2 - 1, \\ \{\tau_3, \tau_1\} &= \tau_2. \end{aligned}$$

<u>reduced equations of motion</u>:

$$\begin{aligned} \dot{\tau}_1 &= \{\tau_1, \tau_3\} = -\tau_2, \\ \dot{\tau}_2 &= \{\tau_2, \tau_3\} = 2\tau_1(\tau_3 + \tau_1 - 1) + \tau_1^2 - 1, \\ \dot{\tau}_3 &= \{\tau_3, \tau_3\} = 0. \end{aligned}$$

$\mathbb{Z}_2$ reduction map:

$$\pi^\vee : T^*S^1 \to P^\vee \subseteq \mathbb{R}^3 : (\alpha, p) \mapsto \tau(\alpha, p)$$

is 2 to 1 branched covering map branched at $p_0^\vee = (1, 0, 0)$ and $p_2^\vee = (-1, 0, 2)$ corresponding to fixed points $p_0 = (0, 0)$ and $p_2 = (\pi, 0)$ of $\mathbb{Z}_2$ action on T^*S^1.

Reduced action angle coordinates

on $P^{\vee}_{\mathrm{reg}} = P^{\vee} \setminus \{p^{\vee}_0, p^{\vee}_2\}$ with symplectic form $\omega^{\vee} = \mathrm{d}\Theta^{\vee}$

$$\Theta^{\vee} = \mp\sqrt{\tfrac{2(\tau_3+\tau_1-1)}{1-\tau_1^2}}, \quad \pm\tau_2 \geq 0.$$

reduced action:

$$I^{\vee} = \tfrac{1}{2\pi}\int_{C^{\vee}(e)} \Theta^{\vee} = \tfrac{1}{\pi}\int_{\max(1-e,1)}^{1} \sqrt{\tfrac{2(e+\tau_1-1)}{1-\tau_1^2}}\mathrm{d}\tau_1$$

when $0 < e < 2$ or $e > 2$.

period of reduced motion on $C^{\vee}(e)$:

$$T^{\vee} = 2\int_{-1}^{1} \frac{\mathrm{d}\tau_1}{\sqrt{2(e+\tau_1-1)(1-\tau_1^2)}}.$$

reduced angle: $\theta^{\vee} = \frac{2\pi}{T}\, t, \quad t \in [\max(1-e,1), 1].$

$\mathbb{Z}_2$ reduced quantum system

reduced quantum line bundle:

$$\big(\rho^\vee : L^\vee = P^\vee_{\mathrm{reg}} \times \mathbb{C} \to P^\vee_{\mathrm{reg}}, \lambda^\vee = \pi^\vee_*(\lambda)\big).$$

reduced covariant derivative:

$$\nabla^\vee_{X^\vee} \sigma^\vee_0 = -i\hbar\,(X^\vee \mathbin{\lrcorner} \Theta^\vee)\sigma^\vee_0,$$

$\sigma^\vee_0 : P^\vee_{\mathrm{reg}} \to L^\vee : \tau \mapsto (1, \tau)$. $X^\vee$ is a vector field on $P^\vee$.

reduced Bohr-Sommerfeld condition:

$$I^\vee = \tfrac{1}{2\pi} \int_{C^\vee(e)} \Theta^\vee = k\,\hbar, \quad \text{for some } k \in \mathbb{Z}_{\geq 0}.$$

Image of Bohr-Sommerfeld torus in T^*S^1 by reduction map $\pi^\vee$ is Bohr-Sommerfeld torus of reduced system in $P^\vee_{\mathrm{reg}}$.

Reduced quantum states

<u>reduced quantum states</u>: smooth section $\sigma_k^\vee$, $k \geq 0$ of bundle $\rho^\vee$ which are covariantly constant with support a Bohr-Sommerfeld torus $(I^\vee)^{-1}(k\hbar)$.

<u>reduced Hilbert space $\mathfrak{H}^\vee$</u> has basis $\{\sigma_k^\vee\}_{k=0}^\infty$.

<u>reduced raising operator</u>:

$$\mathbf{b}^\vee \sigma_k^\vee = \sigma_{k+1}^\vee \quad \text{for } k \geq 0$$

<u>reduced lowering operator</u>:

$$\mathbf{a}^\vee \sigma_k^\vee = \sigma_{k-1}^\vee \text{ for } k > 0 \text{ and } \mathbf{a}^\vee \sigma_0^\vee = 0.$$

<u>reduced quantum state transition structure</u>:

$$\sigma_0^\vee \leftrightarrows \sigma_1^\vee \leftrightarrows \cdots \leftrightarrows \sigma_k^\vee \leftrightarrows \cdots$$

isomorphism: $R : \mathfrak{H}_{\text{even}} \to \mathfrak{H}^{\vee}$ sends the basis $\{\sigma^0_{2k},\, k = 1, \ldots, K; \sigma^+_m + \sigma^-_m,\, m \geq M\}$ to the basis $\{\sigma^{\vee}_k,\, k = 1, \ldots, K; \sigma^{\vee}_m,\, m \geq M\}$.

$2M = \begin{cases} N+2, \text{ if } N \text{ even} \\ N+1, \text{ if } N \text{ odd} \end{cases}$ and $2K = \begin{cases} N, \quad \text{if } N \text{ even} \\ N-1, \text{ if } N \text{ odd.} \end{cases}$

lift of reduced raising operator: $\mathbf{b}^{\text{even}} = R^{-1}\mathbf{b}^{\vee}R$. $\mathbf{b}^{\text{even}}$ raises the quantum number n by 2 if $0 \leq n+2 \leq N$ and raises the quantum number m by 1 if $m \geq M$.

lift of reduced lower operator: $\mathbf{a}^{\text{even}} = R^{-1}\mathbf{a}^{\vee}R$. $\mathbf{a}^{\text{even}}$ lowers the quantum number n by 2 if $2 \leq n \leq N$ and lowers the quantum number m by 1 if $m \geq M+1$.

shifting quantum states <u>across</u> the singular level $H^{-1}(2)$.

When $N = 2K$ and $M = K + 1$, $\sigma^0_N = \sigma^0_{2K} \in \mathfrak{H}_{\mathrm{even}} \cap \mathfrak{H}_0$ and $\sigma^+_M + \sigma^-_M = \sigma^+_{K+1} + \sigma^-_{K+1} \in \mathfrak{H}_{\mathrm{even}} \cap (\mathfrak{H}_+ \oplus \mathfrak{H}_-)$. So

$$\begin{aligned} R\mathbf{b}^{\mathrm{even}}\sigma^0_N &= R\mathbf{b}_{\mathrm{even}}\sigma^0_{2K} = \mathbf{b}^\vee\sigma^0_{2K} = \mathbf{b}^\vee\sigma^\vee_K \\ &= \sigma^\vee_{K+1} = \sigma^\vee_M = R(\sigma^+_M + \sigma^-_M), \end{aligned}$$

that is, $\mathbf{pr}_\pm(\mathbf{b}^{\mathrm{even}}\sigma^0_N) = \sigma^\pm_M$, where

$$\mathbf{pr}_\pm : \mathfrak{H}_+ \oplus \mathfrak{H}_- \to \mathfrak{H}_\pm : \sigma^+_M + \sigma^-_M \mapsto \sigma^\pm_M.$$

Similarly, $\mathbf{a}^{\mathrm{even}}(\sigma^+_M + \sigma^-_M) = \sigma^0_N$.

When $N = 2K+1$ and $M = K+1$, $\sigma_N^0 = \sigma_{2K+1}^0$ $\in \mathfrak{H}_{\mathrm{odd}} \cap \mathfrak{H}_0$. Then $\sigma_{N-1}^0 = \sigma_{2K}^0 \in \mathfrak{H}_{\mathrm{even}} \cap \mathfrak{H}_0$. Also $\sigma_M^+ + \sigma_M^- \in \mathfrak{H}_{\mathrm{even}} \cap (\mathfrak{H}_+ \oplus \mathfrak{H}_-)$. Thus

$$\mathbf{b}^{\mathrm{even}} \sigma_{N-1}^0 = \sigma_M^+ + \sigma_M^-.$$

So $(\mathbf{b}_{\mathrm{even}}\mathbf{a})\sigma_N^0 = \sigma_M^+ + \sigma_M^-$, that is,

$$\mathbf{pr}_{\pm}\big((\mathbf{b}_{\mathrm{even}}\mathbf{a})\sigma_N^0\big) = \sigma_M^{\pm}.$$