Adiabatic limits, Theta functions, and Geometric Quantization

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Purpose & Main Theorems

Geometric quantization

Geometric quantization $\cdots$ a procedure to construct a representation of the Poisson algebra of certain functions on (M, ω) to a Hilbert space, called a quantum Hilbert space $Q(M, \omega)$ from the given symplectic manifold (M, ω) in the geometric way

Classical mechanics		**Quantum mechanics**
(M, ω)	$\longrightarrow$	$Q(M, \omega)$: Hilbert space
$f \in C^\infty(M)$	$\longrightarrow$	$Q(f)$: operator on $Q(M, \omega)$

Q satisfies $Q(\{f, g\}) = \frac{2\pi\sqrt{-1}}{h}\{Q(f)Q(g) - Q(g)Q(f)\}$

Example (Canonical quantization)

$$\left(\mathbb{R}^{2n}, \omega_0 := \sum_{i=1}^{n} dp_i \wedge dq_i\right) \longrightarrow Q(\mathbb{R}^{2n}, \omega_0) := L^2(\mathbb{R}^n_q)$$

$$p_i, q_i \in C^\infty(\mathbb{R}^{2n}) \longrightarrow \begin{cases} Q(p_i) := \frac{h}{2\pi\sqrt{-1}}\frac{\partial}{\partial q_i} \\ Q(q_i) := q_i \times \end{cases}$$

(M, ω) closed symplectic manifold

(L, ∇^L) prequantum line bundle $\overset{\text{def}}{\Leftrightarrow} \begin{cases} L \to M \text{ Hermitian line bundle} \\ \nabla^L \text{ connection of } L \text{ with } \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega \end{cases}$

In the Kostant-Souriau theory, to obtain the quantum Hilbert space $Q(M, \omega)$, we need a polarization.

Definition

A polarization $\mathcal{P}$ is an integrable Lagrangian distribution of $TM \otimes \mathbb{C}$.

- Let $\mathcal{S}$ be the sheaf of germs of covariant constant sections of L along $\mathcal{P}$.

When a polarization $\mathcal{P}$ is given, $Q(M, \omega)$ is “naively" defined to be

Definition

$$Q(M, \omega) := H^0(M; \mathcal{S})$$

Example (Kähler quantization)

(M, ω, J) closed Kähler manifold

(L, h, ∇^L) holomorphic Hermitian line bundle with Chern connection

$\Rightarrow$ $T^{0,1}M$ can be taken to be a polarization $\mathcal{P}$.

Definition

$$Q_{Kähler}(M, \omega) := H^0(M; \mathcal{O}_L)$$

- When the Kodaira vanishing holds, $\dim Q_{Kähler}(M, \omega) =$ index of the Dolbeault operator with coefficients in L.

Example (Real quantization)

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$ prequantized Lagrangian torus fiber bundle

- $(L, \nabla^L)|_{\pi^{-1}(b)}$ is a flat bundle for $\forall b \in B$.

Definition (Bohr-Sommerfeld (BS) point)

$b \in B$ is *Bohr-Sommerfeld* $\overset{\text{def}}{\Leftrightarrow} \{s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0\} \neq \{0\}$

- BS points appear discretely.
- We denote by B_{BS} the set of BS points

Example (Local model)

$$\left(\mathbb{R}^n \times T^n \times \mathbb{C}, d - 2\pi\sqrt{-1}\sum_{i=1}^{n} x_i dy_i\right) \to (\mathbb{R}^n \times T^n, \omega_0) \xrightarrow{\pi_0} \mathbb{R}^n \ \therefore\ \mathbb{R}^n_{BS} = \mathbb{Z}^n$$

Example (Real quantization) continued

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$ prequantized Lagrangian torus fiber bundle

$\Rightarrow$ The tangent bundle along the fiber $T_\pi M \otimes \mathbb{C}$ can be taken to be a polarization $\mathcal{P}$.

Assume (M, ω) is closed.

Theorem (Śniatycki)

$$H^q(M; \mathcal{S}) = \begin{cases} \bigoplus_{b \in B_{BS}} \left\{ s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0 \right\} & \textit{if } q = \frac{\dim_{\mathbb{R}} M}{2} \\ 0 & \textit{if } q : \textit{otherwise} \end{cases}$$

Definition (Real quantization)

$$Q_{real}(M, \omega) := \bigoplus_{b \in B_{BS}} \left\{ s \in \Gamma(L|_{\pi^{-1}(b)}) \mid \nabla^L s = 0 \right\}$$

Does $Q(M, \omega)$ depend on a choice of polarization?

Question

$$Q_{Kähler}(M, \omega) \cong Q_{real}(M, \omega) \ ?$$

- Several examples show their dimensions agree with each other:
  - $\dim Q_{Kähler}(M, \omega) = \dim Q_{real}(M, \omega)$ (Andersen '97)
  - the moment map μ of a toric manifold (Danilov '78),

$$\dim H^0(M; \mathcal{O}_L) = \#\mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}} = \#\text{BS pts}$$

  - the Gelfand-Cetlin system on the complex flag manifold (Guillemin-Sternberg '83)
  - the Goldman system on the moduli space of flat $SU(2)$-bundles on a Riemann surface (Jeffrey-Weitsman '92)

$Q_{Kähler} \cong Q_{real}$ as a limit of deformation of complex structures

Theorem (Baier-Florentino-Muorão-Nunes '11)

When (M, ω) is a toric manifold, they give a one-parameter family of

- $\{J^t\}_{t>0}$ *compatible complex structures of M*

and for $\forall t > 0$

- $\{\sigma^t_m\}_{m\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}}$ *a basis of holomorphic sections of $L \to (M, \omega, J^t)$*

such that for $\forall m \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$, σ^t_m converges to a delta-function section supported on $\mu^{-1}(m)$ as $t \to \infty$ in the following sense, for any section s of L,

$$\lim_{t\to\infty} \int_M \left\langle s, \frac{\sigma^t_m}{\|\sigma^t_m\|_{L^1}} \right\rangle_L \frac{\omega^n}{n!} = \int_{\mu^{-1}(m)} \langle s, \delta_m \rangle_L \, d\theta_m.$$

- Similar results have been obtained (but only for non-singular fibers):
  - the Gelfand-Cetlin system on the complex flag manifold (Hamilton-Konno '14)
  - smooth irreducible complex algebraic variety with certain assumptions (Hamilton-Harada-Kaveh '16)

For a non-integrable J, we have several generalizations of the Kähler quantization. Among these is the Spin^c quantization.

Theorem (Fujita-Furuta-Y '10)

Let $(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$ be a prequantized Lagrangian torus fiber bundle with compact M. Let J be a compatible almost complex strucutre on (M, ω). For the Spin^c Dirac operator D associated with J, we have

$$\mathrm{ind}\, D = \# BS.$$

Purpose

To generalize BFMN apporach to the Spin^c quantization.

Spin^c quantization – a generalization of the Kähler quantization

$(L, \nabla^L) \to (M, \omega)$ closed symplectic manifold with prequantum line bundle

$\Rightarrow$ By taking a compatible almost complex structure J, we can obtain the Spin^c Dirac operator

$$D\colon \Gamma\left(\wedge^\bullet (T^*M)^{0,1} \otimes L\right) \to \Gamma\left(\wedge^\bullet (T^*M)^{0,1} \otimes L\right).$$

- D is a 1st order, formally self-adjoint, elliptic differential operator.

Definition (Spin^c quantization)

$$Q_{Spin^c}(M, \omega) := \ker(D|_{\wedge^{0,even}}) - \ker(D|_{\wedge^{0,odd}}) \in K(pt) \cong \mathbb{Z}$$

- $\dim Q_{Spin^c}(M, \omega) = \operatorname{ind} D$ depends only on ω and does not depend on the choice of J and ∇^L.
- If (M, ω, J) is Kähler (hence, (L, ∇^L) is holomorphic with Chern connection), then $D = \sqrt{2}(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L)$ and

$$\operatorname{ind} D = \sum_{q \geq 0} (-1)^q \dim H^q(M, \mathcal{O}_L).$$

Deformation of almost complex structure

$\pi\colon (M,\omega) \to B$: Lagrangian torus fiber bundle

J: compatible almost complex structure of (M,ω)

$\Rightarrow \ TM = JT_\pi M \oplus T_\pi M \quad (T_\pi M$: tangent bundle along the fiber of $\pi)$

Definition

For each $t > 0$, define J^t by

$$J^t v := \begin{cases} \frac{1}{t} Jv & \text{if } v \in T_\pi M \\ tJv & \text{if } v \in JT_\pi M. \end{cases}$$

- J^t is still a compatible almost complex structure of (M,ω).
- Assume J is invariant along the fiber of π. Then,

$$J\text{: integrable} \ \Leftrightarrow \ J^t\text{: integrable } \forall t > 0$$

- As $t \to +\infty$, $T_\pi M$ becomes smaller and $JT_\pi M$ becomes larger with respect to $g^t := \omega(\cdot, J^t \cdot)$. (adiabatic-type limit)
- For each $t > 0$, we denote by D^t the Dirac operator with respect to J^t.

Main Theorem

$(L, \nabla^L) \to (M, \omega) \xrightarrow{\pi} B$: prequantized Lagrangian torus fiber bundle

J: compatible almost complex structure of (M, ω) invariant along the fiber of π

$\{J^t\}_{t>0}$: the deformation of J defined as in the previous slide

Theorem (Y '19)

Assume M is closed and B is complete (i.e., $\tilde{B} \cong \mathbb{R}^n$). For each $t > 0$, we give orthogonal sections $\{\vartheta^t_m\}_{m \in B_{BS}}$ on L indexed by B_{BS} such that

1. *each ϑ^t_m converges to a delta-function section supported on $\pi^{-1}(m)$ as $t \to \infty$ in the following sense, for any section s of L,*

$$\lim_{t\to\infty} \int_M \left\langle s, \frac{\vartheta^t_m}{\|\vartheta^t_m\|_{L^1}} \right\rangle_L \frac{\omega^n}{n!} = \int_{\pi^{-1}(m)} \langle s, \delta_m \rangle_L \, |dy|.$$

2. $\lim_{t\to\infty} \|D^t \vartheta^t_m\|_{L^2} = 0.$

Moreover, if J is integrable, then, with a technical assumption, we can take $\{\vartheta^t_m\}_{m \in B_{BS}}$ to be an orthogonal basis of holomorphic sections of $L \to (M, \omega, J^t)$.

Corollary

When $\pi = \mathrm{p}_1 : M = T^n \times T^n \to B = T^n$,

$$\vartheta_m(x, y) = e^{\pi\sqrt{-1}(-m\cdot\Omega m + x\cdot\Omega x)}\vartheta \begin{bmatrix} m \\ 0 \end{bmatrix} (-\Omega x + y, \Omega)\,.$$

Thank you for your attention!