A note on the Fundamental Theorem of Algebra

Theorem. Every nonconstant polynomial over $\mathbf{C}$ has a root.

Proof. Consider the space of polynomials $P(z,a) = z^n + a_{n-1}z^{n-1} + ... + a_0$ of z of degree n with complex coefficients $a_{n-1}, ..., a_0$.

In the space $\mathbf{C}^{n+1} = \{(z, a_{n-1}, ..., a_0)\}$ consider the set Σ cut out by the 2 equations $P(z,a) = Q(z,a) = 0$, where $Q(z,a) := P'(z,a) = \partial P/\partial z = nz^{n-1} + (n-1)a_{n-1}z^{n-2} + ... + a_1$. This is the space of polynomials with coefficients a, that have double root at z.

Lemma 1. Σ is a smooth submanifold in $\mathbf{C}^{n+1}$ of complex codimension 2 (i.e. of complex dimension $n-1$).

Proof. It suffices to check that for the map

$$(P,Q) : \mathbf{C}^{n+1} \to \mathbf{C}^2, \qquad (z,a) \mapsto (P(z,a), Q(z,a))$$

(0,0) is a regular value. Actually, for this map all values in $\mathbf{C}^2$ are regular, since ∇P and ∇Q are everywhere noncollinear in $\mathbf{C}^{n+1}$. Indeed, $\nabla P = (*, ..., *, 1)$, while $\nabla Q = (*, ..., *, 1, 0)$, where the last two derivatives are with respect to a_1 and a_0. $\circ$

Project $\pi : \Sigma^{n-1} \to \mathbf{C}^n_a$ by the "forgetful map" $(z,a) \mapsto a$. The image $\Delta := \pi(\Sigma)$ is the space of all polynomials which have a double root. This image Δ is a surface in $\mathbf{C}^n_a$, which has complex dimension (not greater than) $n-1$, that of Σ. (Actually, this hypersurface Δ^{n-1} is singular, as the forgetful map of Σ^{n-1} to $\mathbf{C}^n_a$ is not an embedding.)

Corollary 2. The surface $\Delta^{n-1} \subset \mathbf{C}^n_a$ is of complex codimension 1, i.e. of real codimension 2 in $\mathbf{C}^n_a$, and therefore its complement in $\mathbf{C}^n_a$ is connected.

Lemma 3. For $a \notin \Delta$, roots of the polynomial $P(z,a)$ depend smoothly on a.

Proof. This is the implicit function theorem: the equation $P(z,a) = 0$ locally defines z as a function of a provided that $\partial P(z,a)/\partial z \neq 0$. But the latter is exactly the condition that the corresponding $a \notin \Delta$, i.e. $(z,a) \notin \Sigma$. $\circ$

Corollary 4. Any two polynomials of degree n, lying outside of the complex hypersurface $\Delta^{n-1} \subset \mathbf{C}^n_a$ have the same number of roots.

Indeed, connect them by a smooth path staying away from Δ^{n-1} (which is possible due to Corollary 2). On the way, the roots change smoothly, i.e. they cannot collide, appear, or disappear.

Finally, note that the polynomial $P_0 := z^n - 1$ has n simple roots. All polynomials outside of Δ^{n-1} can be connected to it, hence they also have the same number of roots (and therefore, at least one).

Finally, it remains to prove the theorem for $P \in \Delta$. But this is evident by definition of Δ: this surface consists of polynomials which have (at least one) double root. QED.