LECTURE NOTES ON HADAMARD-PERRON THEOREM

In these notes we discuss the stable and unstable manifold theorem at a hyperbolic fixed point. The treatment follow Katok-Hasselblatt, (denoted K-H from now on), but the proof of contraction mapping uses the version in Brin-Stuck (B-S), and there are some variations from both sources.

1. Introduction

Our setting is an m-dimensional manifold M, an open set $U \subset M$, and a diffeomorphism $f : U \to M$. Assume that $p \in U$ is a periodic point of f, i.e. $f^N(p) = p$ for some $n \in \mathbb{N}$.

Definition. *A periodic point p of period N is called hyperbolic if the linear map $D(f^N)(p) : T_pM \to T_pM$ has no eigenvalue on the unit circle.*

We also list here the related definition for flows to highlight some differences. Let $f^t : M \to M$ be a flow with vector field $F(x)$.

Definition.
- *An equilibrium point p of the flow f^t is called hyperbolic if the linear map $D(f^t) : T_pM \to T_pM$ has no eigenvalue on the unit circle for all $t \neq 0$.*
- *A periodic point p of f^t with period T is called hyperbolic if $D(f^T)(p) : T_pM \to T_pM$ has a simple eigenvalue $\chi = 1$, and no other eigenvalues on the unit circle.*

In the second case, $F(x)$ is always an eigenvector of $D(f^T)(p)$ with $\chi = 1$. We also have p is a hyperbolic periodic point for the associated Poincaré return map.

Given an $m \times m$ hyperbolic matrix A, let $\mathrm{sp}(A)$ denote the set of all eigenvalues for A, $\mathrm{sp}^-(A) = \{\chi \in \mathrm{sp}(A) : |\chi| < 1\}$, and $\mathrm{sp}^+(A) = \{\chi \in \mathrm{sp}(A) : |\chi| > 1\}$. We define

$$\lambda(A) = \sup\{|\chi| : \chi \in \mathrm{sp}^-(A)\}, \quad \mu(A) = \inf\{|\chi| : \chi \in \mathrm{sp}^+(A)\},$$

and $E^\pm(A) = \oplus_{\chi \in \mathrm{sp}^\pm(A)} E_\chi(A)$, where

$$E_\chi(A) = \{v \in \mathbb{R}^m : (A - \chi I)^k v = 0 \text{ for some } k \in \mathbb{N}\}$$

is the root space for the eigenvalue χ.

In general, given $0 < \lambda < \mu$, we say the spectrum of A admits a (λ, μ)-splitting if

$$\mathrm{sp}(A) = \{\chi \in \mathrm{sp}(A) : |\chi| \leq \lambda\} \cup \mathrm{sp}(A) = \{\chi \in \mathrm{sp}(A) : |\chi| \geq \mu\}.$$

We still denote the splitting of the spectrum $\mathrm{sp}^\pm(A)$ and define $E^\pm(A)$ in the same way.

Proposition 1 (K-H, Proposition 1.2.2)**.** *Assume that A admits a (λ,μ)-splitting. Then for every $\epsilon > 0$, there exists a norm $\|\cdot\|$ on $\mathbb{R}^m$ such that*

$$\|Av\| \le (\lambda+\epsilon)\|v\|, \quad v \in E^-(A),$$
$$\|A^{-1}v\| \le (\mu^{-1}+\epsilon)\|v\|, \quad v \in E^+(A).$$

Proof. We write $PAP^{-1} = J$, where $J = \mathrm{diag}\{J_1, \cdots, J_s\}$ is the Jordan normal form. For each Jordan block J_i of size k, we check that

$$\Lambda_i^{-1} J_i \Lambda_i := \begin{bmatrix} 1 & & & \\ & \epsilon^{-1} & & \\ & & \ddots & \\ & & & \epsilon^{-(k-1)} \end{bmatrix} \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \epsilon & & \\ & & \ddots & \\ & & & \epsilon^{(k-1)} \end{bmatrix}$$
$$= \begin{bmatrix} \lambda & \epsilon & & \\ & \lambda & \ddots & \\ & & \ddots & \epsilon \\ & & & \lambda \end{bmatrix}$$

Let $\Lambda = \mathrm{diag}\{\Lambda_1, \cdots, \Lambda_s\}$, then the norm $\|v\| = \|\Lambda P v\|_\infty$, where $\|\cdot\|_\infty$ is the sup norm on $\mathbb{R}^m$, works. □

Theorem 2 (See K-H, Theorem 6.2.3)**.** *Let p be a hyperbolic fixed point of a C^l-diffeomorphism $f : U \to M$, with $l \ge 1$. Then for each $\delta > 0$, there exists C^l-embeded discs $W_p^+, W_p^- \subset U$ with the following properties:*

1. *$p \in W^+, W^-$, $T_pW^+ = E^+$, $T_pW^- = E^-$.*
2. *W^- is forward invariant and W^+ is backward invariant.*
3. *There exist $C(\delta) > 0$ such that for any $p_1 \in W^-(p)$, $p_2 \in W^+$, $n \ge 0$,*
$$\mathrm{dist}(f^n p_1, p) \le C(\delta)\lambda(df(p))^n d(p_1, p),$$
$$\mathrm{dist}(f^{-n} p_2, p) \le C(\delta)(\mu^{-1}(df(p)) + \delta) d(p_2, p).$$
4. *There exists $r_0 > 0$ such that if $f^n p_1 \in U_{r_0}(p)$ for all $n \ge 0$, then $p_1 \in W^- p$; if $f^{-n}(p) \in U_{r_0}(p)$ for all $n \ge 0$, then $p_2 \in W^+(p)$.*

2. Local maps

By using a coordinate chart $\varphi : \mathbb{R}^m \to V \ni p$, we may consider the map $f_0 = \varphi^{-1} \circ f \circ \varphi : \mathbb{R}^m \to \mathbb{R}^m$. Denote $k = dim E^-$. By taking an additional affine linear coordinate change, we may assume 0 is a hyperbolic fixed point for f_0, and $E^-(Df_0(0)) = \{0\} \times \mathbb{R}^{m-k}$, $E^+(Df_0(0)) = \mathbb{R}^k \times \{0\}$. Then the local map f_0 takes the form

$$f_0(x, y) = (Ax + \alpha(x, y), Bx + \beta(x, y)), \quad x \in \mathbb{R}^k, y \in \mathbb{R}^{m-k},$$

$\|A^{-1}\| \le \mu^{-1}(Df_0(0)) + \epsilon$, $\|B\| \ge \lambda(Df_0(0)) + \epsilon$, α, β vanishes up to order one at $(0,0)$. In particular, for any $\sigma > 0$, there exists $r > 0$, such that $\|\alpha\|_{C^1(B_r)}, \|\beta\|_{C^1(B_r)} < \sigma$.

It suffices to prove Theorem 2 for the map $f_0|_{B_r}$. We will, however, prove a general theorem that also covers the case of "uniformly hyperbolic orbits". Let $f : M \to M$ be a C^1 diffeomorphism of a compact manifold M.

Definition. *Given $0 < \lambda < \mu$, we say the full orbit $\{f^n(p)\}_{n\in\mathbb{Z}}$ of f admits a uniform (λ, μ)-splitting if there exist a Riemannian metric on M with the following properties.*

- *For each $n \in \mathbb{Z}$, there exists a splitting $T_{f^n(p)}M = E_n^+ \oplus E_n^-$ of constant dimensions k and $m - k$, that is invariant under df. This means*

$$df(f^n p)(E_n^{\pm}) = E_{n+1}^{\pm}, \quad n \in \mathbb{Z}.$$

- *Let $\|\cdot\|_n$ denote the norm on tangent vectors given by the Riemannian metric at the nth spot. Then*

$$\|df(f^n p)v\| \le \lambda\|v\|, \quad v \in E_n^-, n \in \mathbb{Z}$$

$$\|(df(f^{n-1}))^{-1}v)\| \le \mu^{-1}\|v\|, \quad v \in E_n^+, n \in \mathbb{Z}.$$

When $\lambda < 1 < \mu$ the orbit is called uniformly hyperbolic.

In the neighborhood of each $f^n(p)$, we consider local coordinate φ_n with $\varphi_n(0) = f^n(p)$, and obtain the local maps $f_n = \varphi_{n+1} \circ f \circ \varphi_n$. We may further change coordinates such that

$$d\varphi_n(0)(\{0\} \times \mathbb{R}^{m-k}) = E_n^-, \quad d\varphi_n(0)(\mathbb{R}^k \times \{0\}) = E_n^+,$$

The spaces $\mathbb{R}^m$ for each φ_n inherits its own norm $\|\cdot\|_n$, however, they are uniformly equivalent to the standard norm. We will omit the subscript in the norm, as which norm to apply will be clear from context. Furthermore, given any $\sigma > 0$, there exists $r > 0$ such that

$$f_n(x, y) = (A_n x + \alpha_n(x, y), B_n y + \beta_n(x, y)),$$

with $\|A_n^{-1}\| \le \mu^{-1}$, $\|B_n\| \le \lambda$, and $\|\alpha_n\|_{C^1(B_r)}, \|\beta_n\|_{C^1(B_r)} < \sigma$. We now state the theorem in terms of local maps.

Theorem 3 (Hadamard-Perron, see K-H Theorem 6.2.8, B-S Proposition 5.6.1)**.** *Let $0 < \lambda < \mu$ with $\mu > 1$. Given C^1-diffeomorphisms $f_n : \mathbb{R}^m \to \mathbb{R}^m$ such that $f_n(x, y) = (A_n x + \alpha_n(x, y), B_n y + \beta_n(x, y))$ with $\|A_n^{-1}\| \le \mu^{-1}$, $\|B_n\| \le \lambda$ and $\alpha(0,0) = \beta(0,0) = 0$.*

Then for any $\delta > 0$, there exists $\sigma > 0$ such that if $\|\alpha_n\|_{C^1(B_r)}, \|\beta_n\|_{C^1(B_r)} < \sigma$, for some $r > 0$, then there exists a unique family of C^1-manifolds

$$W_n^+ = \{(x, \phi_n^+(x)) : x \in \mathbb{R}^k\} = \mathrm{graph}(\phi_n^+)$$

verifying the following:

(1) *(backward invariance)* $(f_{n-1})^{-1}W_n^+ \subset W_{n-1}^+$.
(2) *(backward contraction)* $\|(f_{n-1})^{-1}(z_1)-(f_{n-1})^{-1}(z_2)\| \leq (\mu^{-1}+\delta)\|z_1-z_2\|$ *for* $z_1, z_2 \in W_n^+$.
(3) *(criterion for unstable manifold) If for any* $\lambda < \lambda' < \mu$*, and* $C > 0$ *we have*

$$\|(f_{n-j})^{-1} \circ \cdots (f_n)^{-1}(z)\| \leq \min\{C(\lambda')^{-n}, r\}$$

for all $j \geq 0$*, then* $z \in W_n^+$.

The manifolds W_n^+ *are* C^l*-manifolds if the maps are* C^l, $l \geq 1$.

Remark.
- *Suppose* $r = -\infty$*, i.e. the norm estimates* $\|\alpha_n\|_{C^1}, \|\beta_n\|_{C^1} < \sigma$ *holds globally, then the theorem holds without the assumption* $\mu > 1$*, except item (3) need to be modified. See K-H Theorem 6.2.8 for details.*
- *Suppse* $\lambda < 1 < \mu$ *we can apply the theorem for the map family* $g_{-n} = (f_n)^{-1}$ *to obtain the stable manifolds. Theorem 2 follows from the Hadamard-Perron Theorem.*

3. Cone families

We follow the presentation of K-H. Given a splitting $\mathbb{R}^k \oplus \mathbb{R}^{m-k}$ of $\mathbb{R}^m$ and $0 < \gamma \leq 1$, we can define the standard horizontal and vertical cones in the following way:

$$H_p^\gamma = \{(u,v) \in T_p\mathbb{R}^m : \|v\| \leq \gamma\|u\|\},$$
$$V_p^\gamma = \{(u,v) \in T_p\mathbb{R}^m : \|u\| \leq \gamma\|v\|\}.$$

A cone field K is a collection of cones K_p defined at every point p of a given set. Let K be a cone field on $\mathbb{R}^m$, we say a submanifold N is tangent to K if $T_pN \subset K_p$ for every $p \in N$.

For the map sequence f_n, a cone family is a cone field defined at each copy of $\mathbb{R}^m$. A map family that admits a (λ, μ)-splitting preserves the standard cone families.

Lemma 4. *For any* $0 < \gamma < 1$*, suppose* $(\gamma + \frac{1}{\gamma})\sigma < \frac{\gamma}{1+\gamma}(\mu - \lambda)$*, then for any* $z_1 \in B_r$*,* $(f_m)^{-1}(z_2) \in B_r$*,*

$$Df_m(z)H_z^\gamma \subset \mathrm{int}H_{f_m(z)}^\gamma, \quad (Df_m(z))^{-1}V_z^\gamma \subset \mathrm{int}V_{f_m^{-1}(z)}^\gamma.$$

Proof. Assume $\|v\| \leq \gamma\|u\|$, we write

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = df_m(z_1)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A + D_u\alpha & D_v\alpha \\ D_u\beta & B + D_v\beta \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

Note that all the derivative terms are bounded by σ. Write $\sigma' = (\gamma + \gamma^{-1})\sigma$, we have

$$\|v'\| \leq \|Bv\| + \sigma\|v\| + \sigma\|u\| \leq (\lambda+\sigma)\|v\| + \sigma\|u\| \leq (\gamma\lambda + \sigma\gamma + \sigma)\|u\| \\ < (\gamma\lambda + \sigma')\|u\|$$

$$\gamma\|u'\| \geq \gamma\|Au\| - \gamma\sigma\|u\| - \gamma\sigma\|v\| \geq \gamma(\mu-\sigma)\|u\| - \gamma^2\sigma\|u\| \\ > (\gamma\mu - \gamma\sigma')\|u\|.$$

We need $\lambda\gamma + \sigma' < \gamma\mu - \gamma\sigma'$, which is where we obtained the condition.

For the backward invariance, it is equilvalent to prove forward invariance of $H_p^{1/\gamma}$. This is equivalent to $\sigma' < \frac{1}{1+\gamma}(\mu - \lambda)$. □

Lemma 5. *Write $(u', v') = Df_m(z)(u, v)$, then*

$$\|u'\| > (\mu - 2\sigma)\|u\|, \quad (u,v) \in H_z^\gamma,$$
$$\|v'\| < (\lambda + 2\sigma)\|v\|, \quad (u,v) \in V_z^\gamma.$$

The proof can be read directly from the calculations in Lemma 4, so we omit it.

The conclusions of the last two lemmas are called the cone properties. In fact, these two properties are equivalent to the existence of invariant splittings.

Proposition 6. *Suppose for $0 < \lambda < \mu$, the map family satisfies the conclusions of lemma 4 and 5. We write $\{z_n\}$ (i.e. $f_n(z_n) = z_{n+1}$), we have the following statement.*

(1) *Suppose z_{n-i} are defined (i.e. $z_{n-i} \in B_r$) for all $i \geq 0$, then*

$$E_n^+(z) = \bigcap_{i=0}^{-\infty} Df_{n-1}(z_{n-1}) \circ \cdots \circ Df_{n-i}(z_{n-i}) H_m^\gamma$$

is a k-dimensional subspace.

(2) *Suppose z_{n+i} are defined (i.e. $z_{n+i} \in B_r$) for all $i \geq 0$, then*

$$E_n^-(z) = \bigcap_{i=0}^{\infty} (Df_n(z_{n-1}))^{-1} \circ \cdots \circ (Df_{n-i}(z_{n+i}))^{-1} V_m^\gamma$$

is a $(m-k)$-dimensional subspace.

This proposition can be proved directly using the cone estimates, and we leave it for exercise.

4. Graph transform

Let B_r^+ denote the closed ball in $\mathbb{R}^k \times \{0\}$. Fix a $0 < \gamma \leq 1$, let $\mathcal{L}$ denote the set of all γ-Lipshitz functions $\psi : B_r^+ \to \mathbb{R}^{m-k}$ and let $\mathcal{L}_0 = \mathcal{L} \cup \{\psi(0) = 0\}$. We define a special metric

$$d(\psi_1, \psi_2) = \sup_{x \in B_r^+ \setminus \{0\}} \frac{\|\psi_1(x) - \psi_2(x)\|}{\|x\|},$$

and verifies that $\mathcal{L}_0$ is a complete metric space under this metric.

The following statement under the same assumptions as Lemma 4, f_n defines a mapping on $\mathcal{L}_0$.

Lemma 7. *For any* $0 < \gamma < 1$, *suppose* $(\gamma + \frac{1}{\gamma})\sigma < \frac{\gamma}{1+\gamma}(\mu - \lambda)$, *and* $\mu - 2\sigma \geq 1$. *Then for any* n, $\psi \in \mathcal{L}_0$, *there exists* $\mathcal{G}_{f_n}(\psi) \in \mathcal{L}_0$, *such that*

$$f_n(\text{graph}(\psi)) \cap (B_r^+ \times \mathbb{R}^{m-k}) = \text{graph}(\mathcal{G}_{f_n}(\psi)).$$

We use the notation H^γ and V^γ to denote the standard cone set in $\mathbb{R}^k \times \mathbb{R}^{m-k}$. Note that for any $z_i = (x_i, y_i)$, $i = 1, 2$, $z_2 - z_1 \in H^\gamma$ if and only if $\|y_2 - y_1\| \leq \gamma \|x_2 - x_1\|$. We have the following analog of Lemma 4 and 5.

Lemma 8. *Under the same assumptions as Lemma 7, the following hold.*

(1) *If* $z_1, z_2 \in B_r$ *satisfies* $z_2 - z_1 \in H^\gamma$, *then* $f_n(z_2) - f_n(z_1) \in H^\gamma$. *Moreover,*

$$\|x_2' - x_1'\| > (\mu - 2\sigma)\|x_2 - x_1\|. \tag{1}$$

(2) *If* $z_1, z_2 \in B_r$, *and* $f_n(z_1), f_n(z_2) \in B_r$, *then* $f_n(z_2) - f_n(z_1) \in V^\gamma$ *implies* $z_2 - z_1 \in V^\gamma$. *Moreover,*

$$\|y_2' - y_1'\| < (\lambda + 2\sigma)\|y_2 - y_1\|. \tag{2}$$

Proof. Denote $z_i = (x_i, y_i)$ and $z_i' = f_n(x_i, y_i)$, $i = 1, 2$. Note that

$$\begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix} = \begin{bmatrix} A(x_2 - x_1) + \alpha(x_2, y_2) - \alpha(x_1, y_1) \\ B(y_2 - y_1) + \beta(x_2, y_2) - \beta(x_1, y_1) \end{bmatrix}.$$

Since $\|\alpha(x_2, y_2) - \alpha(x_1, y_1)\|, \|\beta(x_2, y_2) - \beta(x_1, y_1)\| \leq \sigma\|x_2 - x_1\| + \sigma\|y_2 - y_1\|$, by exactly the same calculations as in Lemma 4 and 5, we obtain the conclusions. □

Proof of Lemma 7. Using the cone invariance part of the last lemma, we obtain $f_n(\text{graph}(\psi))$ must be the graph of a γ-Lipshitz function. We only need to show that its domain contains B_r^+. But this follows from the fact that $f_n(0, 0) = (0, 0)$ and (1). □

This map on the space of graphs is usually called the graph transform. Let

$$\Lambda = \left\{ \{\psi_n\}_{n \in \mathbb{Z}} : \quad \psi_n \in \mathcal{L}_0 \right\},$$

equipped the metric

$$d(\{\phi_n\}, \{\psi_n\}) = \sup_n d(\phi_n, \psi_n),$$

and let f_n be the sequence of maps in the assumption of Theorem 3. We now define the graph transform Φ on Λ by

$$\Phi(\{\psi_n\})_{n+1} = \mathcal{G}_{f_n}(\psi_n).$$

Note that if $\{\psi_n^+\}$ is a fixed point of Φ, then

$$f_n(\text{graph}(\psi_n^+)) \supset \psi_{n+1}^+,$$

in other words, the sequence of manifolds $\{\text{graph}(\psi_n^+)\}$ is backward invariant under $\{f_n\}$. We now show that the fixed point exists and is unique.

Proposition 9. *Suppose*

$$\frac{\lambda + 2\sigma}{(\mu - 2\sigma)(1-\gamma^2)} < 1,$$

then $\Phi : \Lambda \to \Lambda$ *is a contraction mapping.*

Proof. It suffice to show that for each $n \in \mathbb{Z}$,

$$d(\mathcal{G}_{f_n}\phi_n, \mathcal{G}_{f_n}\psi_n) < d(\phi_n, \psi_n).$$

We fix $x' \in B_r^+$, and denote $y_1' = \mathcal{G}_{f_n}\phi_n(x')$ and $y_2' = \mathcal{G}_{f_n}\psi_n(x')$. By definition, there exists $x_1, x_2 \in B_r^+$ and $y_1 = \phi(x_1)$, $y_2 = \psi(x_2)$, such that $f_n(x_i, y_i) = (x_i', y_i')$, $i = 1, 2$. We further denote $\tilde{y}_2 = \psi(x_1)$.

Applying (1) to $(0,0)$ and (x_1, y_1), we have $\|x' - 0\| \geq (\mu - 2\sigma)\|x_1\|$; applying (2) to (x_1, y_1) and (x_2, y_2), we have $\|y_2' - y_1'\| \leq (\lambda + 2\sigma)\|y_2 - y_2\|$. Hence

$$\frac{\|\mathcal{G}_{f_n}\phi(x') - \mathcal{G}_{f_n}\psi(x')\|}{\|x'\|} = \frac{\|y_2' - y_1'\|}{\|x'\|} \leq \frac{\lambda + 2\sigma}{\mu - 2\sigma}\frac{\|y_2 - y_1\|}{\|x_1\|}.$$

On the other hand, since $(y_2', x') - (y_1', x') \in V^\gamma$, we know $(y_2, x_2) - (y_1, x_1) \in V^\gamma$, i.e. $\|x_2 - x_1\| \leq \gamma\|y_2 - y_1\|$. Using the γ-Lipshitz property of ψ, we also get $\|\tilde{y}_2 - y_2\| \leq \gamma\|x_2 - x_1\|$. We get

$$\|\tilde{y}_2 - y_1\| \geq \|y_2 - y_1\| - \|\tilde{y}_2 - y_1\| \geq (1 - \gamma^2)\|y_2 - y_1\|,$$

hence

$$\|\phi - \psi\| \geq \frac{\|y_1 - \tilde{y}_2\|}{\|x_1\|} \geq (1-\gamma^2)\frac{\|y_2 - y_1\|}{\|x_1\|}.$$

Combine the estimates obtained we have

$$\|\mathcal{G}_{f_n}\phi - \mathcal{G}_{f_n}\psi\| = \sup_{x' \in B_r^+ \setminus \{0\}} \frac{\|\phi(x') - \psi(x')\|}{\|x'\|} \leq \frac{\lambda + 2\sigma}{(\mu - 2\sigma)(1-\gamma^2)}\|\phi - \psi\|.$$

□

We now prove the statements (1)-(3) of the Hadamard-Perron theorem. Indeed, statements (1) follows immediately, and statement (2) is a easy consequence of Lemma 5.

To show statement (3), we choose σ sufficiently small such that Lemmas 4, 5, 8 holds with $\gamma = 1$. Denote $z_n = z$ and $z_{n-j} = (f_{n-j})^{-1} \circ \cdots (f_n)^{-1}(z_n)$. We claim that if $\|z_{n-j}\| \leq C(\lambda')^{-j}$ for all $j \geq 0$, then all $z_{n-j} \in H^1$. Suppose for some $i > 0$, $z_{n-i} \in V^1$, then due to backward invariance of the vertical cones, $z_{n-j} \in V^1$ for all $j \geq i$. Then Lemma 8 implies that there exists $C' > 0$ such that $\|z_{n-j}\| \geq C'\lambda^{-j}$ for all $j \geq 0$. This is a contradiction. Now all $z_{n-j} \in H^1$ implies that for any $j > 0$, there exists a Lipshitz graph $\psi_{n-j} : \mathbb{R}^k \to \mathbb{R}^{m-k}$ such that $z_{n-j} \in \text{graph}(\psi_j)$, which implies $z \in \mathrm{graph}(\mathcal{G}_{n-1} \circ \cdots \mathcal{G}_{n-j}\psi_{n-j})$. Since the latter graph converges to W_n^+ as $j \to \infty$, this implies $z_n \in W_n^+$.

Now let $\{\psi_n^+\}$ be the unique fixed point of Φ, we write $W_n^+ = \mathrm{graph}\psi_n^+$, these Lipshitz invariant manifolds.

Corollary 10. *The fixed point $\{\psi_n^+\}_{n\in\mathbb{Z}}$ depends continuously on the map family $\{f_n\}_{n\in\mathbb{Z}}$ in the uniform C^1 norm.*

Proof. We use the following observation: the fixed point x^* of a contraction mapping Φ depends continuously on Φ in terms of uniform topology among contraction mappings. □

Now by Proposition 6, the invariant bundles $E_n^+(z)$ is well defined on $z \in W_n^+$. We give a different proof of this statement which at the same time proves that E_n^+ depends continuously on the base point z.

Lemma 11. *For each $z_n \in W_n^+$, and $z_{n-j} = (f_{n-j})^{-1} \circ \cdots \circ (f_{n-1})^{-1}$, there exists a unique invariant family of $k-$dimensional subspaces $E_{n-j}^+ \subset H^\gamma$, namely*

$$df_{n-1}(z_{n-j}) \circ \cdots \circ df_{n-j}(z_{n-j})E_{n-j}^+(z_{n-j}) = E_n^j(z_n).$$

The family $E_n^+(z_n)$ depends smoothly on z_n.

Proof. Consider the linear maps $\{Df_n(z_{n-j})\} : \mathbb{R}^m \to \mathbb{R}^m$, $j \le 0$ as a map family. We verify directly that Proposition 9 applies in the sense that the graph transform on the one-sided sequence is well defined an a contraction mapping.

Furthermore, consider the set of all linear functions $l : \mathbb{R}^k \to \mathbb{R}^{m-k}$ with $\|l\| \le \gamma$. This is a subset of $\mathcal{L}_0$, and note that the graph transform $\mathcal{G}_{Df_n}(z_n)$ now preserves the set of linear maps. It now follows that the unique fixed point for the graph transform under Df_n is a linear map l_n^+, and $E_n^+ = \mathrm{graph}(l_n^+)$. Continuity follows from Corollary 10. □

We complete the C^1 part of the discussion by the following statement.

Proposition 12. *The manifolds W_n^+ are C^1.*

Proof. Fix $z_n = (x_n, y_n) \in W_n^+$, and let $z_{n-j}, j \ge 0$ be the backward orbit of z_n. We then claim that for any $N \in \mathbb{N}$, there exists $\epsilon > 0$ such that for any $\|x_n' - x_n\| < \epsilon$, we have

$$(x_n', \psi_n^+(x_n')) - (x_n, y_n) \in \bigcap_{i=0}^{-N} Df_{n-1}(z_{n-1}) \circ \cdots \circ Df_{n-i}(z_{n-N})H_m^{\gamma+\sigma}.$$

Denote $y_n' = \psi_n^+(y_n')$, and (x_{n-j}', y_{n-j}') the backward orbit of (x_n', y_n'), then (x_{n-j}', y_{n-j}') converges uniformly to (x_{n-j}, y_{n-j}), as $x_n' \to x_n$. Since

$$(x_{n-N}', y_{n-N}') - (x_{n-N}, y_{n-N}) \in H^\gamma$$

and $f_{n_1}\circ\cdots\circ f_{n-N}$ converges to $Df_{n-1}(z_{n-1})\circ\cdots\circ Df_{n-i}(z_{n-i})$ locally as (x'_{n-N}, y'_{n-N}) converges to (x_{n-N}, y_{n-N}), the claim follows.

Since $\bigcap_{i=0}^{-N} Df_{n-1}(z_{n-1})\circ\cdots\circ Df_{n-i}(z_{n-N})H_m^{\gamma+\sigma}$ converges to $E^+(z_n)$ as $N\to\infty$, we conclude that W_n^+ is tangent to E_n^+ at z_n. Since E_n^+ is continuous, W_n^+ is C^1. □

5. Regularity

We now discuss the statement that W_n^+ are C^l when the map is C^l. Given the map family f_n, we define the linear extension

$$F_n : T(\mathbb{R}^m) \simeq \mathbb{R}_m \times \mathbb{R}_m \to \mathbb{R}^m \times \mathbb{R}^m, \quad F_n(z, v) = (f_n(z), df_n(z)v).$$

We note that

$$F_n(z, v) = (df_n(0)z, df_n(0)v) + (f_n(z) - df_n(0)z, (df_n(z) - df_n(0))v)$$

and the second term vanishes at order 1 at $(z, v) = (0, 0)$. Note that the linear part $(df_n(0), df_n(0)v$ admits an invariant (λ, μ)-splitting, and the Hadamard-Perron theorem applies. Write $z = (x, y)$ and $v = (v_x, v_y)$, then there exists $r > 0$ and Lipshitz functions $\Phi_n^+ : B_r^+ \times B_r^+ \subset \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^{m-k} \times \mathbb{R}^{m-k}$, such that

$$V_n^+(z, v) = \{(x, \Pi_y \Phi_n^+(x, v_x), v_x, \Pi_{v_y}\Phi_n^+(x, v_x)), \quad (x, v_x) \in B_r^+\},$$

are the unstable manifolds for F_n. Here Π_y, Π_{v_y} are the standard projections.

Lemma 13. *The function Φ_n^+ described above coincides with the function*

$$(x, v_x) :\mapsto (\phi_n^+(x), l_n^+(x, \phi_n^+(x))v_x),$$

where $l_n^+(x, \phi_n^+(x)) : \mathbb{R}^k \to \mathbb{R}^{m-k}$ is such that $E_n^+(x, \phi_n^+(x)) = \mathrm{graph}(l_n^+(x), \phi_n^+(x))$.

Proof. The proof uses the fact that $\phi_n^+(x)$ and $E_n^+(x)$ are uniquely characterized by their backward contraction properties. We leave the details as an exercise. □

We remark the fact that Φ_n^+ is linear in v_x implies Φ_n^+ is well defined on $x \in B_r^+, v_x \in \mathbb{R}^k$.

The Hadarmard-Perron theorem shows that Φ_n^+ are C^1−functions in x, v_x. This implies that $l_n^+(x, \phi_n^+(x))$ is C^1 in x. Given that l_n^+ coincide with $d\phi_n^+$, we obtain ϕ_n^+ is in fact C^2. This argument can be applied inductively, yielding C^l differentiability of ϕ_n^+.

6. Inclination lemma

In this section, we consider a hyperbolic fixed point p of a C^1 map f. It suffices to consider the map

$$f : U \subset \mathbb{R}^m \to \mathbb{R}^m$$

and the fixed point is 0.

The inclination lemma, sometimes referred to as the λ lemma, is the following statement.

Lemma 14. *Assume that p is a hyperbolic fixed point of $f : U \to M$, with k-dimensional unstable direction and $m-k$-dimensional stable direction. Then for any embeded disk D intersecting W^- transversally at q, and any $\epsilon > 0$, there exists an embedded disk $D_1 \subset D$ containing q and $N > 0$ such that $f^N(D_1)$ is ϵ close to $W^+(p)$ in the C^1-distance.*

To prepare for the proof of this lemma, we first make a coordinate change such that W^+ and W^- becomes the coordinate axes. We also choose an r neighborhood of 0 on which the γ-cones are invariant. We then show that there exists $q \in D_0 D$ and $n_0 \in \mathbb{N}$ such that $f^{n_0}(D)$ is contained in B_r and tangent to the horizontal cones H^γ. It suffices to show that $df^n(q)T_qD$ is contained in the horizontal cone for some n. Arguing by contradiction: assume $df^n(q)T_qD$ is contained in the vertical cone $H^{1/\gamma}$ for all $n \geq 0$ implies T_qD must be contained in the stable subspace $E^-(q)$, this contradicts our assumption.

Step 2 shows if $f^{n_0}D_2$ is tangent to γ-cone, then further iterates of f takes it close to W^+. This is left as an exercise.

7. Comments on the assumption $\mu > 1$

We have been stating our theorem with the assumption $\mu > 1$. A version without this assumption exists, but its application requires caution. We have

Theorem 15. *Let $0 < \lambda < \mu$. Given C^1-diffeomorphisms $f_n : \mathbb{R}^m \to \mathbb{R}^m$ such that $f_n(x,y) = (A_n x + \alpha_n(x,y), B_n y + \beta_n(x,y))$ with $\|A_n^{-1}\| \leq \mu^{-1}$, $\|B_n\| \leq \lambda$ and $\alpha(0,0) = \beta(0,0) = 0$.*

For any $\delta > 0$, there exists $\sigma > 0$ such that if

$$\|\alpha_n\|_{C^1(\mathbb{R}^m)}, \|\beta\|_{C^1(\mathbb{R}^m)} < \sigma,$$

there exists a unique family of C^1-manifolds

$$W_n^+ = \{(x, \phi_n^+(x)) : x \in \mathbb{R}^k\} = \operatorname{graph}(\phi_n^+)$$

verifying the following:

(1) *(backward invariance) $(f_{n-1})^{-1} W_n^+ \subset W_{n-1}^+$.*
(2) *(backward contraction) $\|(f_{n-1})^{-1}(z_1) - (f_{n-1})^{-1}(z_2)\| \leq (\mu^{-1} + \delta)\|z_1 - z_2\|$ for $z_1, z_2 \in W_n^+$.*
(3) *(criterion for unstable manifold) If for any $\lambda < \lambda' < \mu$, and $C > 0$ we have*

$$\|(f_{n-j})^{-1} \circ \cdots (f_n)^{-1}(z)\| \leq C(\lambda')^{-n},$$

for all $j \geq 0$, then $z \in W_n^+$.

The regularity part of the manifolds are trickier and we will not discuss it here.

Given a map family $f_n : U \subset \mathbb{R}^n \to \mathbb{R}^n$ for which df_n admits a (λ, υ)-splitting. One can always modify f outside of B_r such that the assumption of Theorem 15 hold. One has to be very careful, as in the case of $\mu \leq 1$, the invariant manifolds obtained may depend on the modified part of the map, and hence do not reflect properties of the original map.