# NOTES ON KAM THEOREM

## 1. Preliminaries on Hamiltonian systems

A Hamiltonian system is an ODE on $\mathbb{R}^{2m}$ of the form

$$\begin{cases} \dot{q}_i = \partial_{p_i} H(q,p), & i = 1, \cdots, m, \\ \dot{p}_i = -\partial_{q_i} H(q,p), & i = 1, \cdots, m, \end{cases}$$

where $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a smooth function (at least $C^2$). Let us use $X_H(q,p)$ to denote the vector field $(\partial_q H, -\partial_p H)$.

A completely integrable Hamiltonian system (in action-angle coordinates) is an Hamiltonian system on $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m \times \mathbb{R}^m$, such that

$$H(\theta, I) = H_0(I), \quad \theta \in \mathbb{T}^m, I \in \mathbb{R}^m.$$

namely, the Hamiltonian depends only on $I$. Note that in this case, the Hamiltonian equation reads

$$\begin{cases} \dot{\theta} = \partial_I H_0(I), \\ \dot{I} = 0. \end{cases}$$

A nearly integrable system is a perturbation of an integrable one, namely

$$H_\varepsilon(\theta, I) = H_0(I) + \varepsilon H_1(\theta, I).$$

The two-form $\Omega = \sum_{i=1}^m dp_i \wedge dq_i$ is called the standard symplectic form. A diffeomorphism $\Phi : U \subset \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is called symplectic if

$$\Phi_* \Omega = \Omega.$$

If we write $(Q,P) = \Phi(q,p)$, then the above relation is equivalent to $\sum_{i=1}^m dP_i \wedge dQ_i = \sum_{i=1}^m dp_i \wedge dq_i$. Let

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \in M_{2m \times 2m}, \quad I_m = \operatorname{diag}\{1, \cdots, 1\} \in M_{m \times m},$$

we say a $2m \times 2m$ matrix $M$ is symplectic if $M^T J M = J$. Then $\Phi$ is symplectic if and only if $D\Phi$ is symplectic at every point.

We note that treating $\nabla H$ as a row vector and $X_H$ as a column vector, then

$$X_H(q,p) = J(\nabla H(q,p))^T.$$

This formulation of Hamiltonian flow is intrinsic. Given any vector field $X$, we can define a one-form in the following way

$$i_X \Omega(\cdot) = \Omega(X, \cdot).$$

Then the Hamiltonian vector field $X_H$ is the unique vector field such that $i_{X_H}\Omega = dH$.

**Lemma 1.** *The Hamiltonian flow is symplectic.*

*Proof.* Let $A(t) = D\phi_t$, where $\phi_t$ is the Hamiltonian flow. Then

$$\frac{d}{dt}A(t) = DX_H(q(t),p(t))A(t) = \begin{bmatrix} \partial^2_{pq}H & \partial^2_{pp}H \\ -\partial^2_{qq}H & -\partial^2_{pq}H \end{bmatrix}(q(t),p(t))A(t).$$

We now compute

$$\frac{d}{dt}(A^T JA) = A^T DX_H^T JA + A^T J\, DX_H A = 0,$$

since $DX_H^T J + J\, DX_H^T = 0$. Since $A(0) = I_{2m}$ satisfies $A(0)^T JA(0) = J$, so does $A(t)$ for all $t \neq 0$. □

A symplectic coordinate change is called canonical.

**Lemma 2.** *Let $\Phi : (Q,P) \mapsto (q,p)$ be a canonical coordinate change. Then the Hamiltonian flow for $H(q,p)$ under the coordinate change is still Hamiltonian, with the Hamiltonian function*

$$\tilde{H}(Q,P) = H \circ \Phi(Q,P).$$

*Proof.* We note that under the coordinate change $\Phi$, the new ODE should take the form

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = D\Phi^{-1} F \circ \Phi.$$

Let us treat $\nabla H$ as a row vector, then

$$\begin{bmatrix} \partial_Q \tilde{H} \\ -\partial_P \tilde{H} \end{bmatrix} = J\nabla\tilde{H}^T = J(D\Phi)^T(\nabla H \circ \Phi)^T = (D\Phi)^{-1}\, J(\nabla H \circ \Phi)^T,$$

where we used $(D\Phi)^{-1}J = J(D\Phi)^T$ which follows from $(D\Phi)^T J(D\Phi) = J$. □

Given two functions $f, g : \mathbb{R}^{2m} \to \mathbb{R}$, we define the Poisson bracket

$$\{f,g\} = -\sum_{i=1}^m \partial_{q_i} f \partial_{p_i} g + \sum_{i=1}^m \partial_{p_i} f \partial_{q_i} g = -\nabla f J (\nabla g)^T.$$

An intrinsic definition is given by $\{f,g\} = \Omega(X_f, X_g)$. The Poisson bracket satisfies

$$\{f,f\} = 0, \quad \{f,g\} = -\{g,f\}$$

and the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

**Lemma 3.** *Let $\phi_t^g$ be the Hamiltonian flow defined by $g$. Then*

$$\frac{d}{dt}(f \circ \phi_t^g) = \{g,f\} \circ \phi_t^g.$$

*Proof.* We have

$$\frac{d}{dt}(f \circ \phi_t^g) = \nabla f \frac{d}{dt}\phi_t^g = -\nabla f (J\nabla g) \circ \phi_t^g = \{g,f\} \circ \phi_t^g.$$

□

As a directly corollary, any function $G$ with $\{G,H\} = 0$ is constant along the Hamiltonian flow of $H$.

## 2. Examples of nearly integrable systems

The simplest completely integrable system is the Harmonic oscillator, whose equation is

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q, \quad q, p \in \mathbb{R}.$$

The Hamiltonian is $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. Write $q = \frac{\sqrt{I}}{\omega}\cos\theta$ and $p = \sqrt{I}\sin\theta$, we verify that this is a symplectic coordinate change. Then under the new coordinates the Harmonic oscillator has the Hamiltonian $H(\theta, I) = \frac{1}{2}I^2$.

An interesting model is the coupled linear oscillators:

$$H(x_1, \cdots, x_N, p_1, \cdots p_N) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N}(x_{j+1} - x_j)^2,$$

where $x_0 = x_{N+1} = 0$. Then the coordinate change

$$q_j = \sqrt{\frac{2}{N+1}}\sum_{k=1}^{N}\sin(kj\pi\pi/(N+1))Q_k,$$

(same for $p_j$), then takes the system into

$$H(Q,P) = \sum_{k=1}^{N}\frac{1}{2}(P_k^2 + \omega_k Q_k^2), \quad \omega_k = 2\sin(k\pi/(2(N+1))).$$

An nonlinear version of this model is associated with a famous experiment of Fermi-Pasta-Ulam.

Another famous example is the $(N+1)$-body system. This is a system consists of one large celestial body (the Sun) and $N$ bodies of much smaller masses (the planets), evolving according to Newtonian mechanics and universal gravity. It is well known that the two-body problem can be reduced to an integrable Hamiltonian system, then the $(N+1)$-body system may be viewed as

$$\prod_{i=1}^{N}(\text{2BP between Sun and ith planet}) + \text{ (Interactions)},$$

where the first part is integrable, and second part is a small perturbation.

The completely integrable systems are related to integrals of motion. We have the following theorem:

**Theorem 4** (Arnold-Liouville)**.** *Suppose $H = F_1, F_2, \cdots, F_m : U \subset \mathbb{R}^{2m}$ are m smooth functions such that $\{F_i, F_j\} = 0$. For $c \in \mathbb{R}^m$, let $M_c \subset U$ is a compact connected component of $\{F_i = c_i\}$. Assume that the vector fields $X_{F_1}, X_{F_2}, \cdots, X_{F_m}$ are pointwise linearly independent on $M_c$.*

*Then $M_c$ is diffeomophic to $\mathbb{T}^m$, and there exists a neighborhood $V$ of $M_c$ and symplectic coordinate change $\Phi : V \to \mathbb{T}^m \times \mathbb{R}^m$, taking $H|V$ to an completely integrable Hamiltonian.*

There is another place where nearly integrable systems naturally appear, namely, local analysis near a totally elliptic fixed point. Without loss of generality, let us assume that $(0,0)$ is a fixed point of the Hamiltonian system $H(q,p)$. We assume the linearized flow near $(0,0)$

has only purely imaginary eigenvalues. Due to the Hamiltonian nature of the flow, we must have the following eigenvalues

$$\lambda_1 = -i\omega_1, \cdots, \lambda_m = -i\omega_m, \quad \lambda_{m+1} = i\omega_1, \cdots, \lambda_{2m} = \omega_m.$$

According to the discussions on normal forms, we immediately have the following resonances:

$$\lambda_i = \lambda_i + \sum_{j=1}^{m} k_j(\lambda_j + \lambda_{m+j}), \quad i = 1 \cdots, 2m.$$

We assume that the eigenvalues do not have other resonances, this is equivalent to the following condition:

$$k_1\omega_1 + \cdots + k_m\omega_m \neq 0, \quad \forall (k_1, \cdots k_m) \in \mathbb{Z}^m \setminus \{0\}.$$

We call the vector $(\omega_1, \cdots, \omega_m)$ *nonresonant* in this case. Under this assumption, the system admits a local Hamiltonian normal form.

**Theorem 5** (Birkhoff Normal Form)**.** *Let* $(0,0)$ *be a totally elliptic fixed point of a* $C^\infty$ *Hamiltonian flow with Hamiltonian* $H(q,p)$*. Then the flow admits a formal Hamiltonian normal form, namely there exists real formal symplectic coordinate change*

$$\Phi : (Q,P) \mapsto (q,p),$$

*and a formal power series* $N(I_1, \cdots, I_m)$*, called the Birkhoff normal form, such that*

$$H \circ \Phi = N(\frac{1}{2}(Q_1^2 + P_1^2), \cdots, \frac{1}{2}(Q_m^2 + P_m^2)).$$

An further coordinate change

$$Q_i = \sqrt{I_i}\cos\theta_i, \quad P_i = \sqrt{I_i}\sin\theta_i$$

takes the system to action-angle coordinates.

$$\begin{cases} \dot{\theta} = \nabla N(I) \\ \dot{I} = 0 \end{cases}.$$

Let $N_k(I)$ be the terms in $N(I)$ up to degree $k$, this is called the Birkhoff normal form up to degree $k$. We have the following finitely smooth version:

**Corollary 6.** *Under the same assumption as the previous theorem, there exists a* $C^\infty$ *symplectic coordinate change* $\Phi : (\theta, I) \mapsto (q,p)$*, such that*

$$H \circ \Phi = N_k(I) + R_{k+1}(\theta, I),$$

*where* $R_{k+1} = O(I^{k+1})$*.*

Therefore, for a small $\varepsilon$, the system on a $\varepsilon$ neighborhood of a non-resonant elliptic fixed point is a nearly integrable system.

There are analogous versions for symplectic maps with an elliptic fixed point. In this case the normal form of the map is

$$(\theta, I) \mapsto (\theta + V(I) + O(I^{k+1}), I + O(I^{k+1})).$$

## 3. The KAM Theorem

The Hamiltonian $H_0(I)$ is called integrable because all solutions can be easily integrated: indeed, solution with initial condition $(\theta_0, I_0)$ is

$$\theta(t) = \theta_0 + \partial_I H_0(I_0)t, \quad I(t) = I_0.$$

The orbit is contained in the torus $T_{I_0} = \mathbb{T}^m \times \{I_0\}$, and the flow on this torus is linear. We denote $\omega(I) = \partial_I H_0(I)$ and call it the *frequency* of $I$. Evidently, the frequency $\omega(I)$ changes with $I$. We say that $H_0$ is non-degenerate if

$$\det \partial^2_{II} H_0(I) \neq 0.$$

For a non-degenerate Hamiltonian, the frequency map is a local diffeomorphism.

A vector $\omega \in \mathbb{R}^m$ is called resonant if $k \cdot \omega = 0$ for some $k \in \mathbb{Z}^m \setminus \{0\}$. It is called non-resonant otherwise. A vector $\omega$ is called $(\alpha, \tau)$-Diophantine if for $\alpha, \tau > 0$

$$|k \cdot \omega| \geq \alpha |k|^{-\tau}, \quad k \in \mathbb{Z}^m \setminus \{0\},$$

where $|k| = |k_1| + \cdots + |k_m|$. Denote by $\Delta^\tau_\alpha$ the set of all $(\alpha, \tau)$-Diophantine vectors, and write $\Delta^\tau = \bigcup_{\alpha > 0} \Delta^\tau_\alpha$. The Diophantine vectors are sometimes called "strongly non-resonant".

Note that

$$\Delta^\tau_\alpha = \mathbb{R}^m \setminus \bigcup_{k \in (\mathbb{Z}^m \setminus \{0\})} \{\omega \in \mathbb{R}^m : |k \cdot \omega| < \alpha |k|^{-\tau}\}.$$

For any bounded domain $\Omega$, the Lebesgue measure $m(\{\omega \in \Omega : |k \cdot \omega| < \alpha |k|^{-\tau}\}) \leq C\alpha |k|^{-\tau-1}$, and therefore

$$m(\Omega \setminus \Delta^\tau_\alpha) \leq C\alpha \sum_{k \in (\mathbb{Z}^m \setminus \{0\})} |k|^{-\tau-1}.$$

The summation is finite if and only if $\tau > m - 1$. Fix $\tau > m - 1$, we then have

$$m(\Omega \setminus \Delta^\tau_\alpha) = O(\alpha), \quad m(\Omega \setminus \Delta^\tau) = 0.$$

In other words, the set of Diophantine frequencies has full Lebesgue measure.

Recall that for the integrable system $H_0$, we have the invariant tori $T_I$ for $I \in \mathbb{R}^m$. KAM theorem states that an invariant torus $T_I$ with a Diophantine frequency $\omega(I)$ survives a small perturbation to $H_0$.

**Theorem 7** (The KAM Theorem)**.** *For an open bounded domain $D \subset \mathbb{R}^m$, assume that the frequency map $\omega : D \to \Omega$ is a diffeomorphism. Assume that $H_\varepsilon = H_0 + \varepsilon H_1$ is real analytic on $\bar{D} \times \mathbb{T}^m$. Then there exists $\delta > 0$ such that for $|\varepsilon| < \delta\alpha^2$, for each $\omega = \omega(I_0) \in \Omega \cap \Delta^\tau_\alpha$, there exists a real analytic embeded torus $\mathcal{T}_\omega$, invariant under the Hamiltonian flow of $H_\varepsilon$, and is a small deformation of $T_{I_0}$.*

*Moreover, the union of all $\mathcal{T}_\omega$ fill the set $D \times \mathbb{T}^m$ up to a set of measure $O(\alpha)$.*

We will only prove the existence of the invariant tori, and skip the proof of the "moreover" part on measure of the invariant tori.

**Remark.** *Moser showed that the KAM theorem holds for finite regularity as well. The optimal regularity is $C^l$ for $l > 2\tau + 2 > 2m$.*

The following consequences of KAM theorem were historically significant:

(1) A nearly integrable system is never ergodic.
(2) Generically, there are always KAM tori surrounding an elliptic fixed point.

In particular, a generic elliptic fixed point of an area preserving map is stable.

## 4. Ideas of the proof

We need some notations. For $I_0 \in \mathbb{R}^m$, and $r > 0$, let

$$A_r(I_0) = \{I \in \mathbb{C}^m : \|I - I_0\| < r\}.$$

For $s > 0$, we denote

$$D_s = \{\theta \in \mathbb{C}^m : \sup_i |Im\theta_i| < s\}.$$

A complex function $f : A_r \times D_s \to \mathbb{C}$ is called real analytic if $f|D_s \times A_r \cap \mathbb{R}^m \times \mathbb{R}^m$ is real. Define $|f|_{s,r}$ to be the sup-norm of $f$ on $D_s \times A_r$.

Given a frequency $\omega_0 = (\omega_0^1, \cdots, \omega_0^m)$, define a vector field on $\mathbb{T}^m$ by

$$X_{\omega_0} = (\omega_0^1 \partial_{\theta_1}, \cdots, \omega_0^m \partial_{\theta_m}).$$

We will prove the following equivalent theorem.

**Theorem 8.** *Suppose $H_0, H_1$ are real analytic on $D_s \times A_r(I_0)$. Then there exists There exists $\delta, c > 0$ depending only on $r, s, \|H_0\|_{s,r}, \|H_1\|_{s,r}$ such that for $|\varepsilon| < \delta\alpha^2$, for $\omega_0 = \omega(I_0) \in \Omega \cap \Delta_\alpha^\tau$, there exists a real analytic map*

$$\Psi : \mathbb{T}^m \to B \times \mathbb{T}^m,$$

*such that*

$$X_{H_\varepsilon} \circ \Psi = (D\Psi) X_{\omega_0}.$$

*Moreover, $\Psi$ is real analytic on $D_{s/2}$, and*

$$\|\Psi - Id\|_{C^0} \leq c\sqrt{\varepsilon}.$$

To simplify notation, consider a Hamiltonian

$$H_\varepsilon(\theta, I) = H_0(I) + \varepsilon H_1(\theta, I),$$

where $\|H_1\|_{s,r} < 1$. For $r = r_0 > r_1 \cdots > r_n \cdots$ and $s = s_0 > s_1 > \cdots > s_n > \cdots$, $\lim r_n = 0$, $\lim s_n = s/2$, and $I_0, \cdots, I_n, \cdots \in B$, we will define a series of symplectic coordinate changes

$$\Phi^{(n)} : D_{s_n} \times A_{r_n}(I_n) \to D_{s_{n-1}} \times A_{r_{n-1}}(I_{n-1}),$$

such that for $H^{(0)} = H_\varepsilon$ and $H^{(n)} \circ \Phi^{(n+1)} = H^{(n+1)}$ satisfies

$$H^{(n)}(\theta, I) = H_0^{(n)}(I) + \varepsilon_k H_1^{(n)}(\theta, I), \quad \nabla H_0^{(n)}(I_n) = \omega_0,$$

$\|H_1^{(n)}\|_{s_n, r_n} \leq 1$, and $\varepsilon_n \to 0$ as $n \to \infty$.

Let $\Psi^{(n)} = \Phi^{(1)} \circ \cdots \Phi^{(n)}|_{\mathbb{T}^m \times \{I_m\}}$, then we have

$$X_H \circ \Psi^{(n)} = D\Psi^{(n)} X_{H^{(n)}}|_{\mathbb{T}^m \times \{I_m\}} = D\Psi^{(n)}(X_{\omega_0} + X\varepsilon_n X_{H_1^{(n)}})|_{\mathbb{T}^m \times \{I_m\}}.$$

We then define $\Psi = \lim \Psi^{(n)}$ and we see the statement of the theorem is satisfied.

Before proceeding with the proof, we first describe the idea Heuristically. The naive idea comes from the Newton method of solving non-linear equations. We attempt to solve the non-linear equation (for a symplectic map $\Phi$)

$$(H_0 + \varepsilon H_1) \circ \Phi = H_0.$$

To make the problem more tractable, and to ensure $\Phi$ is symplectic, we pick $\Phi$ of a specific form: given a function $G$ on $\mathbb{T}^m \times \mathbb{R}^m$ we define

$$\Phi = \phi_1^{\varepsilon G},$$

where $\phi_t^{\varepsilon G}$ is the Hamiltonian flow of $G$. Then Lemma 1 ensures that $\Phi$ is symplectic. We now try to solve for the function $G$ from

$$(H_0 + \varepsilon H_1) \circ \phi_1^{\varepsilon G} = H_0.$$

In the spirit of the Newton method, we attempt to approximate the equation by a linear equation. Note that by Lemma 3 we have

$$\frac{d}{dt}|_{t=0}(H \circ \phi_t^{\varepsilon G}) = \varepsilon\{G, H\},$$

and applying the lemma repeatedly, we get

$$\frac{d^n}{dt^n}|_{t=0}(H \circ \phi_t^{\varepsilon G}) = \varepsilon^n\{G, \{G\{\cdots\{G, H\}\cdots\}\}\}.$$

Using Taylor expansion

$$H \circ \phi_1^{\varepsilon G} = H + \varepsilon\{G, H\} + \frac{1}{2}\varepsilon^2\{G, \{G, H\}\} + \cdots = H_0$$

Writing $H = H_0 + P$, we have

$$H \circ \phi_1^{\varepsilon G} = H_0 + \varepsilon H_1 + \varepsilon\{G, H_0\} + \cdots = H_0, \tag{1}$$

therefore the linearized equation is

$$H_1 + \{G, H_0\} = 0. \tag{2}$$

It turns out that (2) is still a non-linear PDE that is hard to solve. We further simplify the equation by using $\nabla H_0(I_0) = \omega_0$, and linearize $H_0$ near $I_0$:

$$H_0(I) = H_0(I_0) + \nabla H_0(I - I_0) + O(I - I_0)^2 = \text{const} + \omega_0 \cdot I + O(I - I_0)^2.$$

Consider the system only on the set $A_r(I_0) \times D_s$ with $|r| \le C\sqrt{\delta}$. Then we have

$$H_1 + \{G, H_0\} = H_1 + \{G, \text{const} + \omega_0 \cdot I + O(I - I_0)^2\} = H_1 + \{G, \omega_0 \cdot I\} + O(\sqrt{\varepsilon})$$

Therefore, we can get an approximate solution to (2) by solving

$$H_1 + \{G, \omega_0 \cdot I\} = 0.$$

Note that this is equivalent to

$$H_1 + \omega_0 \cdot \partial_\theta G = 0. \tag{3}$$

We have the following claim that will be precisely stated later.

**Claim.** *The equation* (3) *has a solution under two conditions:*

(1) $\omega_0$ *is Diophantine.*

(2) *The fourier expansion* $H_1(\theta, I) = \sum_{k\in\mathbb{Z}^m} h_k(I)e^{2\pi i k\cdot\theta}$ *satisfies* $h_0(I) = 0$. *This is equivalent to*

$$[H_1] := \int H_1(\theta, I)d\theta_1 \cdots d\theta_m = 0.$$

The proof of the KAM theorem now proceeds as follows:

(1) Solve the equation

$$H_1 - [H_1] + \omega_0 \cdot \partial_\theta G = 0,$$

call the solution $G_1$.

(2) Denote $\Phi^{(1)} = \phi_1^{\varepsilon G_1}$, using (1), we get

$$(H_0 + \varepsilon H_1) \circ \Phi^{(1)} = H_0 + \varepsilon[H_1] + O(\varepsilon^{\frac{3}{2}}) := H_0^{(1)} + \varepsilon_1 H_1^{(1)},$$

where $H_0^{(1)} = H_0 + \varepsilon[H_1]$ and $\varepsilon_1 = O(\varepsilon^{\frac{3}{2}})$.

(3) Pick $I_1$ such that $\nabla H_0^{(1)}(I_1) = \omega_0$, and repeat the first step. Namely, inductively, we solve the equation

$$H_1^{(n)} - [H_1^{(n)}] + \omega_0 \cdot \partial_\theta G_n = 0$$

and let $\Phi^{(n)} = \phi_t^{\varepsilon_n G_n}$. We will get

$$(H_0^{(n)} + \varepsilon_n H_1^{(n)}) \circ \Phi^{(n+1)} = H_0^{(n+1)} + \varepsilon_{n+1} H_1^{(n+1)},$$

with $\varepsilon_{n+1} = O(\varepsilon_n^{\frac{3}{2}})$. The iteration process should converge super-exponentially.

**Remark.** *From this sketch, we observe two features of the KAM scheme:*

(1) *The Newton iteration scheme and fast convergence.*

(2) *Since we can only solve the equation* (3) *for* $H_1 - [H_1]$, *the term* $\varepsilon[H_1]$ *gets added to* $H_0$ *for the next iterate. As a consequence,* $\nabla H_0^{(1)}(I_0) \neq \omega_0$, *and we cannot proceed with the iteration in the original neighborhood. This is sometimes known as "drift of frequency".*

*As a consequence, we are forced to choose a different* $I_1$ *in the neighborhood, to re-focus on the right frequency. This is possible due to the non-degeneracy condition of* $H_0$, *as we can choose* $I_1$ *using inverse function theorem. This shows the importance of the non-degeneracy condition.*

## 5. Solution of the linear problem

The goal of this section is to solve the linearized equation (3). We have the following.

**Proposition 9.** *Suppose* $\omega_0 \in \Delta_\alpha^\tau$, $H_1$ *is real analytic on* $A_r(I_0) \times D_s$, *and* $[H_1] = 0$. *Then for any* $0 < \sigma < s$,

$$\omega_0 \cdot \partial_\theta G + H_1 = 0$$

*has a solution that is real analytic on* $A_r \times D_{s-\sigma}$, *moreover,*

$$\|G\|_{s-\sigma,r} \leq \frac{c}{\gamma\sigma^{\tau+m+1}}\|H_1\|_{r,s}.$$

The proof uses Fourier series, and exploits the following property of functions analytic on $D_s$.

**Lemma 10.** *Suppose $f : \mathbb{T}^m \to \mathbb{R}$ is can be extended to an analytic function on $D_s$, then its Fourier series $\sum_{k\in\mathbb{Z}^m} f_k e^{2\pi i k\cdot\theta}$ satisfies*

$$|f_k| \le \sup_{x\in D_s} |f(x)| e^{-2\pi|k|s},$$

*where $|k| = |k_1| + \cdots + |k_m|$.*

*Proof.* Since $f e^{-2\pi i k\cdot\theta}$ is analytic on $D_s$, for any $\varphi \in \mathbb{R}^m$ with $|\varphi_i| < s$, we have

$$f_k = \int_{\mathbb{T}^m} f(\theta) e^{-2\pi i k\cdot\theta} d\theta = \int_{\mathbb{T}^m + i\varphi} f(\theta) e^{-2\pi i k\cdot\theta} d\theta.$$

Hence

$$|f_k| = \left| \int_{\mathbb{T}^m + i\varphi} f(\theta) e^{-2\pi i k\cdot\theta} d\theta \right| = \left| \int_{\mathbb{T}^m} f(\theta - i\varphi) e^{-2\pi i k\cdot(\theta - i\varphi)} d\theta \right|$$
$$= e^{-2\pi k\cdot\varphi} \left| \int_{\mathbb{T}^m} f(\theta - i\varphi) e^{-2\pi i k\cdot\theta} d\theta \right| \le e^{-2\pi k\cdot\varphi} \sup_{D_s} |f|.$$

For any $0 < \sigma < s$, choose $\varphi = (s - \sigma)(\frac{k_1}{|k_1|}, \cdots, \frac{k_m}{|k_m|})$, then $k \cdot \varphi = |k|(s - \sigma)$. Hence

$$|f_k| \le e^{-2\pi(s-\sigma)|k|} \sup_{D_s} |f|,$$

the lemma follows from taking $\sigma \to 0$. □

*The proof of Proposition 9.* Write

$$H_1(\theta, I) = \sum_{k\in\mathbb{Z}^m} h_k(I) e^{2\pi i k\cdot\theta},$$

with $h_0(I) = 0$. Assume that our solution is $G(\theta, I) = \sum_{k\in\mathbb{Z}^m} g_k(I) e^{2\pi i k\cdot\theta}$, then the equation (3) is equivalent to

$$\sum_{k\in\mathbb{Z}^m} g_k(I)(2\pi i k \cdot \omega_0) e^{2\pi i k\cdot\theta} = -\sum_{k\in\mathbb{Z}^m} h_k(I) e^{2\pi i k\cdot\theta}.$$

Note that the $k = 0$ term of the LHS vanishes, therefore the equation has a solution only if $h_k(I) = 0$. This is satisfied in our case, hence

$$g_k(I) = -\frac{h_k(I)}{2\pi i k \cdot \omega_0}, \quad k \in \mathbb{Z}^m \setminus \{0\}.$$

To guarantee $g_k$ is defined, we need $k \cdot \omega_0 \neq 0$ for all $k \in \mathbb{Z}^m \setminus \{0\}$. Moreover, just having $g_k$ defined does not ensure the convergence of the series $\sum g_k e^{2\pi i k\cdot\theta}$, as the norm of $g_k$ can blow up as the denominator $2\pi i k \cdot \omega_0$ gets very small. This is the so-called *small denominator* problem. This is resolved by using the Diophantine condition, as it provides a lower bound on the small denominator.

More precisely, by Diophantine condition,

$$|g_k(I)| \le |h_k(I)| \frac{1}{2\pi\alpha|k|^{-\tau}} \le \frac{|k|^\tau}{2\pi\alpha} e^{-2\pi|k|s} \|H_1\|_{s,r}.$$

Therefore, for any $0 < \sigma < s$, $\theta \in D_{s-\sigma}$ and $I \in A_r(I_0)$, we have

$$|\sum_{k\in\mathbb{Z}^m} g_k(I)e^{2\pi i\theta}| \le \sum_{k\in\mathbb{Z}^m} |g_k(I)|e^{2\pi|k|(s-\sigma)} \le \sum_{k\in\mathbb{Z}^m} \frac{|k|^\tau}{2\pi\alpha} e^{-2\pi|k|\sigma}\|H_1\|_{s,r}$$
$$\le \frac{1}{2\pi\alpha}\sum_{k\in\mathbb{Z}^m}(\sup_{m\in\mathbb{N}} m^\tau e^{-\pi m\sigma})e^{-\pi|k|\sigma} \le \frac{c}{\alpha\sigma^\tau}\frac{1}{\tau^m}\|H_1\|_{s,r}$$

□

**Remark.** *From Proposition 9 we observe two other features of the KAM scheme. One of which is the "small denominator" phenomenon we mentioned. The other is the fact that* (3) *has no solution in the same space $A_r \times D_s$, and we are forced to use a larger space $A_r \times D_{s-\sigma}$. This is an indication that the solution is "less smooth" than the input. Because we are using a sequence of spaces in the iteration, it is important that our spaces stabilize. This is a non-trivial fact and the fast convergence of the Newton scheme is important in ensuring this.*

## 6. The proof of the KAM theorem

We now proceed to the "real" proof. The only technical tool is Proposition 9 as well as the Cauchy estimate for controlling derivatives, which we list below.

**Lemma 11** (Cauchy estimates)**.** *Let $U \subset \mathbb{C}$ be an open set, and $U_\delta = \bigcup_{z_0\in U}\{z : \|z - z_0\| < \delta\}$ be its $\delta$-neighborhood. Then if $f : U_\delta \to \mathbb{C}$ is analytic, we have*

$$\sup_{z\in U}|df| \le \frac{1}{2\pi\delta}\sup_{z\in U_\delta}|f|.$$

*Proof.* This follows from a direct computation using the Cauchy integral formula. □

We will assume for $D > 0$

$$\sup_{A_r(I_0)} \|H_0\| \le D, \quad \sup_{A_r(I_0)} \|(\partial^2_{II}H_0)^{-1}\| \le D.$$

We also use a somewhat unusual notation convention. Throughout this proof, let $c$ denote a explicit constant, larger than 1, that depends only on the dimesion $m$ and the constant $D$. There will be many such constants in the proof, but *all of them* will be called $c$.

We present the proof in three separate pages.

(1) The first page contains the construction of the sequence of coordinate changes, and the statements we would like to hold.
(2) The second page lists a number of estimates that hold for one coordinate change. Details of the estimates are given later.
(3) The third page list the choice of parameters in the construction, and proves the statements in the first page (using estimates from the second page).

## Cheat sheet I - The construction and the claims

**The coordinate changes.** We start with $H_\varepsilon = H_0 + \varepsilon H_1$ with $H_0, H_1$ analytic on $D_s \times A_r(I_0)$ satisfying $\nabla H_0(I_0) = \omega_0$. Denote $H^{(0)} = H_\varepsilon$ and $H_0^{(0)} = H_0$ , $H_1^{(0)} = H_1$ and $\varepsilon_0 = \varepsilon$.

For each $n \geq 0$ define $G_n$ by the equation

$$\omega_0 \cdot \partial_\theta G_n + H_1^{(n)} - [H_1^{(n)}] = 0,$$

let

$$\Phi^{(n)} = \phi_1^{\varepsilon_n G_n}, \qquad H^{(n+1)} = H^{(n)} \circ \Phi^{(n)},$$

and write

$$H^{(n+1)} = H_0^{(n+1)} + \varepsilon_{n+1} H_1^{(n+1)},$$

where

$$H_0^{(n+1)} = H_0^{(n)} + \varepsilon_n [H_1^{(n)}], \qquad \varepsilon_{n+1} H_1^{(n+1)} = H^{(n+1)} - H_0^{(n+1)}.$$

Here $\varepsilon_{n+1}$ is chosen such that $\|H_1^{(n+1)}\|_{r_{n+1}, s_{n+1}} = 1$.

**The choice of parameters.** There exists $\delta > 0$ such that for each $\varepsilon_0 < \delta\alpha^2$, we can choose parameters

$$r > r_0 = O(\sqrt{\varepsilon}) > r_1 > \cdots r_n > \cdots > 0, \quad r_n \to 0,$$
$$s = s_0 > s_1 > \cdots > s_n > \cdots > s_0/2, \quad s_n \to s_0/2,$$

such that the following claims hold.

**The claims.** For our choice of parameters, there exists $I_n \in \mathbb{T}^m$, such that:

(1) $\nabla H_0^{(n)}(I_n) = \omega_0$.

(2) Each $\Phi^{(n)}$ is a well defined analytic map

$$\Phi^{(n)} : D_{s_{n+1}} \times A_{r_{n+1}}(I_{n+1}) \to D_{s_n} \times A_{r_n}(I_n).$$

(3) $\varepsilon_n \to 0$.

## Cheat sheet II - One step of coordinate change

To simplify notations, we drop the sup-scripts from our notations. Suppose we have $H = H_0 + \varepsilon H_1$ defined on $D_s \times A_r(I_0)$, $\nabla H_0(I_0) = \omega$.

$G$ solves

$$\omega_0 \cdot \partial_\theta G + H_1 - [H_1] = 0,$$

$\Phi = \phi_1^{\varepsilon G}$, and write

$$H^+ = H \circ \Phi = H_0^+ + R,$$

where $H_0^+ = H_0 + \varepsilon[H_1]$.

**Estimates.** For all the following estimates, we fix $0 < \sigma < s/3$ and assume that $r/4 < \sigma$.

- (Lemma 9)

$$\|G\|_{r,\sigma-\sigma} \le \frac{c}{\alpha\sigma^{\tau+m}}\|H_1\|_{s,r}.$$

- (Cauchy estimates) Recall that $\chi_{\varepsilon G}$ denote the Hamiltonian vector field of $\varepsilon G$.

$$\|\chi_{\varepsilon G}\|_{r/2,s-2\sigma} = \frac{c\varepsilon}{\alpha\sigma^{\tau+m}}\frac{1}{\min\{r/2,\sigma\}}\varepsilon \le \frac{c\varepsilon}{\alpha\sigma^{\tau+m}r}\|H_1\|_{s,r} = \frac{c\varepsilon}{\alpha\sigma^{\tau+m}r}$$

- Since

$$\phi_t^{\varepsilon G}(\theta, I) - (\theta, I) = \int_0^t \chi_{\varepsilon G} \circ \phi_s^{\varepsilon G}(\theta, I)ds,$$

  for sufficiently small $\varepsilon$, we can make sure $\|\phi_t^{\varepsilon G}(\theta, I) - (\theta, I)\|$ is so small that for $t \in [0,1]$

$$\phi_t^{\varepsilon G} : D_{s-3\sigma} \times A_{r/4}(I_0) \to D_{s-2\sigma} \times A_{r/2}(I_0)$$

  is well defined. The precise condition is

(A) $$\frac{c\varepsilon}{\alpha\sigma^{\tau+m}r} < \min\{r/4, \sigma\}, \quad \text{or } c\varepsilon < \alpha\sigma^{\tau+m}r^2.$$

- We claim (details will be given later)

$$\|R\|_{s-3\sigma,r/4}$$

$$\le \left(\varepsilon\|\partial_\theta G \cdot (\omega(I) - \omega(I_0))\|_{s-2\sigma,r/2} + \varepsilon^2\|\{G, (tH_1 + (1-t)[H_1])\}\|_{r/2,s-2\sigma}\right)$$

(B) $$\le \varepsilon\left(\frac{cr}{\sigma}\|G\|_{r,s-\sigma} + \frac{c\varepsilon}{r}\|G\|_{s-\sigma,r}\|H_1\|_{s-\sigma,r}\right) \le \varepsilon\left(\frac{cr}{\alpha\sigma^{\tau+m+1}} + \frac{c\varepsilon}{\alpha\sigma^{\tau+m+1}r}\right).$$

- The following estimates keeps track of how much $H_0$ has changed.

(C) $$\|\partial_{II}^2 H_0^+ - \partial_{II}^2 H_0\|_{s-3\sigma,r/4} = \|\varepsilon\partial_{II}^2[H_1]\|_{s-3\sigma,r/4} \le \frac{c\varepsilon}{r^2}.$$

(D) $$\|\nabla H_0^+ - \nabla H_0\|_{s-3\sigma,r/4} = \|\varepsilon\nabla[H_1]\|_{s-3\sigma,r/4} \le \frac{c\varepsilon}{r}.$$

## Cheat sheet III - selection of parameters

- For any $\varepsilon_0$, for a large number $M > 0$, choose

$$r_0 = M\sqrt{\varepsilon_0}, \qquad \sigma_0 = \frac{s_0}{8}.$$

- Choose $M$ so large that (A) hold for $\varepsilon_0$, $\sigma_0$ and $r_0$, i.e.

$$c\varepsilon_0 < \alpha\sigma_0^{\tau+m} r_0^2. \tag{E1}$$

- For $Q > 4$, choose $\varepsilon_0$ so small that the two terms in (B) satisfies the following estimates.

$$\frac{cr_0}{\alpha\sigma_0^{\tau+m+1}} < \frac{1}{2Q^3}, \quad \frac{\varepsilon_0}{\alpha\sigma_0^{\tau+m+1} r_0} < \frac{1}{2Q^3}. \tag{E2}$$

  Note that using $r_0 = O(\sqrt{\varepsilon})$ we get $\sqrt{\varepsilon_0} = O(\alpha)$ (ignoring other parameters).
- Inductively, suppose (E1) and (E2) are satisfied for $\varepsilon_n$, $r_n$, $s_n$, $\sigma_n$, we choose

$$\sigma_{n+1} = \frac{\sigma_n}{4}, \quad s_{n+1} = s_n - 3\sigma_n, \quad r_{n+1} = \frac{r_n}{Q} < \frac{r_n}{4},$$

  then using (B), we have $\varepsilon_{n+1} < Q^{-3}\varepsilon_n$. For $Q > 4^{\tau+m}$, (E1) and (E2) are satisfied for $\varepsilon_{n+1}$, $r_{n+1}$, $s_{n+1}$, $\sigma_{n+1}$, allowing induction to continue.
- We have

$$r_n \to 0, \quad \varepsilon_n \to 0, \quad \lim s_n = s_0 - \sum_{n=1}^{\infty} 3 \cdot 4^{-n}\sigma_0 = s_0/2.$$

- For the moment, by sheet II, suppose $I_n$ is defined, $\Phi^{(n)}$ is well defined as

$$\Phi^{(n)} : D_{s_{n+1}} \times A_{r_n/4}(I_n) \to D_{s_n} \times A_{r_n}(I_n).$$

  We need to show that we can choose $I_{n+1}$ such that $\nabla H_0^{(n+1)}(I_{n+1}) = \omega_0$ and $A_{r_{n+1}}(I_n) \subset A_{r_n/4}(I_n)$.
- Finally, using (C) we have

$$\left\|\partial_{II}^2 H_0^{(n)} - \partial_{II}^2 H_0^{(0)}\right\|_{s_n, r_n} \le \sum_{i=1}^{n-1} \left\|\partial_{II}^2 H_0^{(i)} - \partial_{II}^2 H_0^{(i-1)}\right\|_{s_i, r_i} \le \sum_{i=0}^{n} \frac{c\varepsilon_n}{r_n^2} \le \frac{c\varepsilon_0}{r_0^2} \sum_{i=0}^{n} Q^{-n} \le \frac{c\varepsilon_0}{r_0^2}.$$

  We can choose $M$ large enough such that ($\frac{c\varepsilon_0}{r_0^2}$ is small enough such that)

$$\|(\partial_{II}^2 H_0^{(n)})^{-1}\|_{s_n, r_n} \le D/2, \quad n \ge 0.$$

  Then by (D),

$$\begin{aligned} &\|I_{n+1} - I_n\| = \|(\nabla H_0^{(n+1)})^{-1}(\omega_0) - I_n\| \\ &= \|(\nabla H_0^{(n+1)})^{-1}(\nabla H_0^{(n)}(I_n)) - (\nabla H_0^{(n+1)})^{-1}(\nabla H_0^{(n+1)}(I_n))\| \\ &\le \|\partial_{II}^2 H_0^{(n+1)}\|_{s_{n+1}, r_{n+1}} \|\nabla H_0^{(n)}(I_n) - \nabla H_0^{(n+1)}(I_n)\| \le \frac{cD\varepsilon_n}{r_n} = \frac{\varepsilon_n}{r_n^2} cDr_n < \frac{\varepsilon_0}{r_0^2} cDr_n \end{aligned}$$

  where the last line is by (E1). For $M$ sufficiently large ($\frac{\varepsilon_0}{r_0^2}$ small), we can make sure $A_{r_{n+1}}(I_{n+1}) \subset A_{r_n/4}(I_n)$.