MAT1845HS, PROBLEM SET 1

Due Monday Feb 6th.

(1) Let $A \in SL(n, \mathbb{Z})$, show that the number of fixed points for the Toral automorphism
$$F_A : x \to Ax \mod 1$$
is $\det(A - I)$. Use this result to show that if A is hyperbolic, then the number of n-periodic points grows exponentially fast.

(2) Let
$$f(x, y) = \begin{bmatrix} Ax + \alpha(x, y) \\ By + \alpha(x, y) \end{bmatrix}, \quad (x, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k},$$
with $\|A^{-1}\| < \mu^{-1} < 1$ and $\|B\| < \lambda < 1$. Consider the space of all Lipschitz functions $\phi : \mathbb{R}^k \to \mathbb{R}^{m-k}$ with the norm
$$\|\phi\|_L = \|\phi(0)\| + \sup_{x_1 \neq x_2} \frac{\|\phi(x_1) - \phi(x_2)\|}{\|x_1 - x_2\|},$$
show that the graph transform $\mathcal{G}_f$ is a contraction mapping on this space.

Then use this result to prove the inclination lemma.

(3) Recall that if $\mathbb{R}^m = E_1 \oplus E_2$, then the γ cone around E_1 is defined as
$$\{v = v_1 + v_2 : \quad v_1 \in E_1, v_2 \in E_2, \|v_2\| \leq \gamma \|v_1\|\}.$$

Consider the map
$$f(x, y) = \begin{bmatrix} x + y + \lambda \sin x \\ y + \lambda \sin x \end{bmatrix},$$
and given $0 < a < 1/2$, let $\Lambda_a = \{(x, y) \in \mathbb{R}^2 : \min\{|x - \pi/2|, |x - 3\pi/2| > a\pi\}\}$. Show that for $|\lambda|$ sufficiently large, for every $(x, y) \in \Lambda_a$, there is an invariant cone field around the subspace $(1, 1)\mathbb{R}$ which also expands length.

Use this fact to explain that for $|\lambda|$ large enough, any orbit contained entirely in this region must be hyperbolic.

FYI: This does not mean the standard map is hyperbolic, because Λ_a is not invariant. It does look very chaotic when λ is large, see Figure 1.

(4) (a) Let p be a fixed point of p, and assume that p is partially hyperbolic, in the sense that it has eigenvalues for $|\lambda| < 1$, $|\lambda| = 1$ and $|\lambda| > 1$, and let

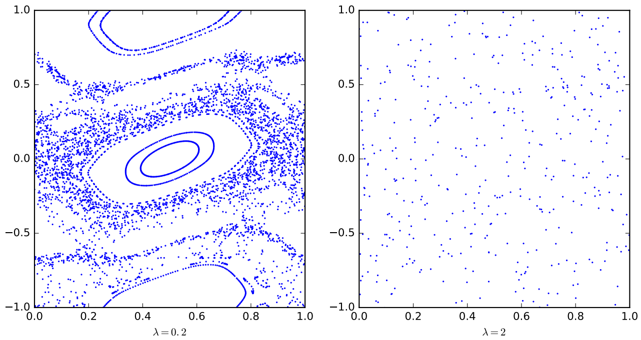

FIGURE 1. Chaos in Standard maps

E^-, E^+, E^0 be the corresponding eigenspaces. Use the Hadamard-Perron theorem to show that there exists an invariant manifold W^0 tangent to E^0.

(Here invariance has a weaker meaning, it means there exists open set $V \ni p$ such that)

$$f^{\pm 1}(W^0 \cap V) \subset W^0.$$

(b) Let f be the time-1-map to the vector field $F(x, y) = (-x, -y^3)$. Explain why f is an example that W^0 obtained above is not unique in general.