MAT1845HS, DYNAMICAL SYSTEMS

1. Introduction

For our course, a dynamical system is either a (invertible or non-invertible) map $f : X \to X$ or a flow (or semi-flow) $f^t : X \to X$. The type of a dynamical system is determined by the structure of the space X (then f should preserve that structure): topological, measure theoretical, smooth, holomorphic, etc. In this course we focus on the smooth case, so $X = M$ is a manifold, and f is C^r, C^∞ or (real) analytic.

2. Course information

Instructor: Ke Zhang
Office: PG 200B
Time and place : M 12-1 and 2-3; W 12-1, BA6180
Office hours: Wednesday 1:30-3:30 or by appointments.

3. Grading

Problem sets and final presentation. I will provide (and update) a list of suggested topics for final presentation. You are welcome to suggest your own related topic.

4. References and lecture notes

Michael Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press.

Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press.

Karl Friedrich Sigburg, *The principle of least action in Geometry and dynamics*. Springer.

The topics covered does not follow particular orders in these textbooks. The goal is to provide lecture notes for a majority of topics covered.

5. List of topics

5.1. **Introduction and examples.**

We introduce the basic concepts by going through a (long) list of examples.

5.2. **Hadamard-Perron Theorem.** In its simplest form, Hadamard-Perron theorem states that a hyperbolic fixed point of a C^1 local diffeomorphism admits a stable and an unstable manifold. However, the nature of the proof is so flexible that almost all aspect of the last statement can be generalized. We will try to give the key ideas that allow application to different settings. Refer to section 6.2 of Katok-Hasselblatt, but we will make some small modifications. We will also discuss the inclination lemma and transverse homoclinic points.

5.3. **Local analysis of fixed points.**

5.3.1. *Hartman-Grobman Theorem.* Hartman-Grobman Theorem states that near a hyperbolic fixed point, a smooth map is *topologically* conjugate to its linear part. Higher regularity is not possible in general. We refer to section 6.3 of Katok Hasselblatt.

5.3.2. *Smooth normal forms.* Under special assumptions, near a hyperbolic fixed point, it is possible to *smoothly* conjugate a map to its linear part. This is the simplest example of a smooth normal form. If time allows, we will also discuss other smooth normal forms. Sternberg's theorem is covered in section 6.6 of Katok-Hasselblatt.

5.3.3. *Elliptic fixed points, a taste of KAM theorem.* The local analysis of elliptic fixed points is much harder. One of the cases when this can be done is if the map is complex analytic. We describe the proof of the Siegel's theorem, as an introduction to KAM phenomenon.

5.4. **Theory of hyperbolic sets.** We start the theory of (uniform) hyperbolicity. Maps admitting a hyperbolic set has many good properties: Anosov closing lemma, shadowing theorem, and structural stability. We also introduce Smale's Axiom A systems and the spectral decomposition theorem.

5.5. **Ergodicity of Anosov diffeomorphisms.** This is one of the main theorems in hyperbolic theory, linking hyperbolicity to ergodicity.

5.6. **Lyapunov exponents and Non-uniform hyperbolicity.** While pure hyperbolicity often lead to strong conclusion about the dynamics, it may be hard to verify in examples. One common approach to detect hyperbolicity in systems is using the Lyapunov exponents. This leads to the notion of non-uniform hyperbolicity.

5.7. **Equilibrium measures and thermodynamic formalism (optional).**

5.8. **Introduction to Hamiltonian systems.** We give an introduction to Hamiltonian systems and Lagrangian systems. We will discuss integrability and Arnold-Liouville Theorem, and the Hamiltonian version of the KAM theorem. (Optional) Nekhoroshev theory of stability.

5.9. **Variational techniques in Hamiltonian systems.** We introduce techniques related to the minimizing orbits in Hamiltonian systems, using the twist map as an example. Various applications to the billiard problem will be given.