Talk on March 31, 2011 [Learning Seminar on Categorification]

... on Section 3 of [LK1]

Goal : To categorify ${}_{\mathcal{A}}\mathbf{f}$, the integral form of the quantum universal enveloping algebra of the negative half of the simply-laced Kac-Moody algebra $\sim\Gamma$.

↳ works for $\hat{sl}_n$ as well.

Main Claim :

Parallel to the story of ${}_{\mathcal{A}}\dot{\mathcal{U}}$ ~ "idempotented", integral form of $\mathcal{U}_q(sl_2)$

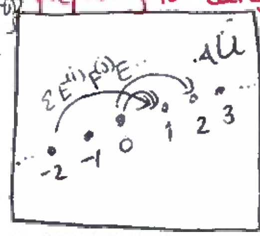
original

Category with 1 object

View ${}_{\mathcal{A}}\mathcal{U}$ as Hom

For basis $\{E^{(a)}F^{(b)}\}$ of ${}_{\mathcal{A}}\mathcal{U}$ add idempotent $\rightsquigarrow$

preparing for categorif.



$\oplus$ Category with multiple ($\mathbb{Z}$ worth of simples) objects

$(\because {}_{\mathcal{A}}\dot{\mathcal{U}} = \bigoplus_{m,n\in\mathbb{Z}} 1_n \dot{\mathcal{U}} 1_m)$

Refining Hom structure of ${}_{\mathcal{A}}\dot{\mathcal{U}}$

categorify the 1-morphisms $\rightsquigarrow$

Categorified

2-category with $\mathbb{Z}$ worth of simple objects (simples = irreps of ${}_{\mathcal{A}}\mathcal{U}$ with highest wt. m)

Enriching : $\mathrm{Hom}_{\mathcal{C}}(m,n) \in \mathrm{Obj}\,\mathcal{D}$

$(\mathrm{Hom}({}_{\mathcal{A}}\mathcal{U}, {}_{\mathcal{A}}\mathcal{U})?)$ getting all of Hom ?

We have for ${}_{\mathcal{A}}\mathbf{f}$

$\underline{i} \in \oplus\mathrm{Seqd}(\mathbb{Z})$

1-morphisms : $\theta_{\underline{i}}$

$*$ relations of $\theta_{\underline{i}}$

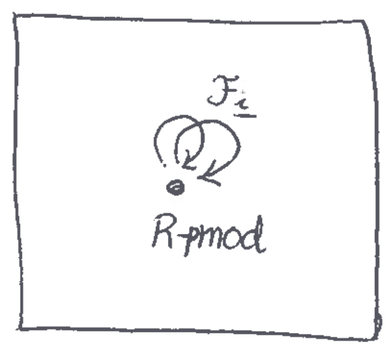
$\rightsquigarrow$ $P_{\underline{i}}$

$\xrightarrow{\gamma}$

K_0 ("decategorify")

$[P_{\underline{i}}]$

$\rightsquigarrow$



goals

1-morphisms : $\mathcal{F}_{\underline{i}}$ are induction functors (by $P_{\underline{i}} \otimes_{R(\nu)} -$)

2-morphisms : natural transformations

$*$ relations of $\mathcal{F}_{\underline{i}}$

Precise Claims : Global

Proposition 3.4 + Surjectivity :

$$\gamma : {}_{\mathcal{A}}f \longrightarrow K_0(R)$$

$$\theta_{\vec{i}} \longmapsto [P_{\vec{i}}] \quad , \quad \vec{i} \in Seqd(\nu) \ , \ \text{any } \nu \in \mathbb{N}_0[\overset{V(\Gamma)}{I}]$$

(i) γ is an isomorphism of $\mathbb{Z}[q,q^{-1}]$-algebras

(ii) Comultiplication of ${}_{\mathcal{A}}f$ $\longmapsto$ comultiplication of $K_0(R)$

Lie algebra type r $\qquad$ $[Res]$

(iii) pairing on ${}_{\mathcal{A}}f$ $\longmapsto$ pairing on $K_0(R)$

$(\ ,\ )$ $\qquad$ $(\ ,\ )$ $\qquad$ ie. $(x,y)_{{}_{\mathcal{A}}f} = (\gamma(x), \gamma(y))_{K_0(R)}$

(iv) bar-involution of ${}_{\mathcal{A}}f$ $\longmapsto$ bar-involution of $K_0(R)$

$\mathbb{Z}[q,q^{-1}]$-antilinear

$q \longmapsto q^{-1}$ $\qquad$ $P_{\vec{i}} \longmapsto \overline{P_{\vec{i}}} := HOM(P_{\vec{i}}, R(\nu))^{\psi}$ $\qquad$ ψ

$\theta_i \longmapsto ?$ $\qquad$ $q \longmapsto q^{-1}$ $\qquad$ as a left module $\qquad$ as a right module

what if $\vec{i} \in Seqd(\nu)$?

Thm 3.21

as 1-morphisms

Any relation $\sum_{\substack{k \\ \text{finite}}} u_k \theta(k) = \sum_{\substack{\ell \\ \text{finite}}} v_\ell \theta'(\ell)$ in ${}_{\mathcal{A}}f$. $\quad u_k, v_\ell \in \mathbb{N}[q,q$

lifts to

as 1-morphisms

relations $\quad \bigoplus_k \mathcal{F}_{\theta(k)}^{\oplus u_k} \cong \bigoplus_\ell \mathcal{F}_{\theta'(\ell)}^{\oplus v_\ell}$ in R-pmod.

(isomorphism of functors)

Definitions: (those appearing directly in Global Claims)

- $\boxed{K_0(R)}$ $\quad R := \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)$ $\quad$ no interactions between different ν's

$R\text{-mod} := \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-mod}$ $\quad$ graded finitely generated

$R\text{-fmod} := \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-fmod}$ $\quad$ graded finitely-dim'l $\rightsquigarrow G_0(R) := \bigoplus_{\nu \in \mathbb{N}[I]} G_0(R(\nu))$

take Grothendieck group

$R\text{-pmod} := \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-pmod}$ $\quad$ graded projectives $\rightsquigarrow K_0(R) := \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$

may be different than what you might first expect

$$(\ , \ ) : K_0(R(\nu)) \times K_0(R(\nu)) \longrightarrow \mathbb{Z}[q^{-1},q][(\nu)_q]$$

$$([P],[Q]) \longmapsto \mathrm{gdim}_{\Bbbk}(P^{\psi} \otimes_{R(\nu)} Q)$$

Here, $(\nu)_q = \mathrm{gdim}_{\Bbbk}(\mathrm{Sym}(\nu)) = \prod_{i \in I}\left(\prod_{a=1}^{\nu_i} \frac{1}{1-q^{2a}}\right)$