IQI Discussion : (2+1)D Extended TQFTs MATH

April 22, 20

Doubled Topological Phases

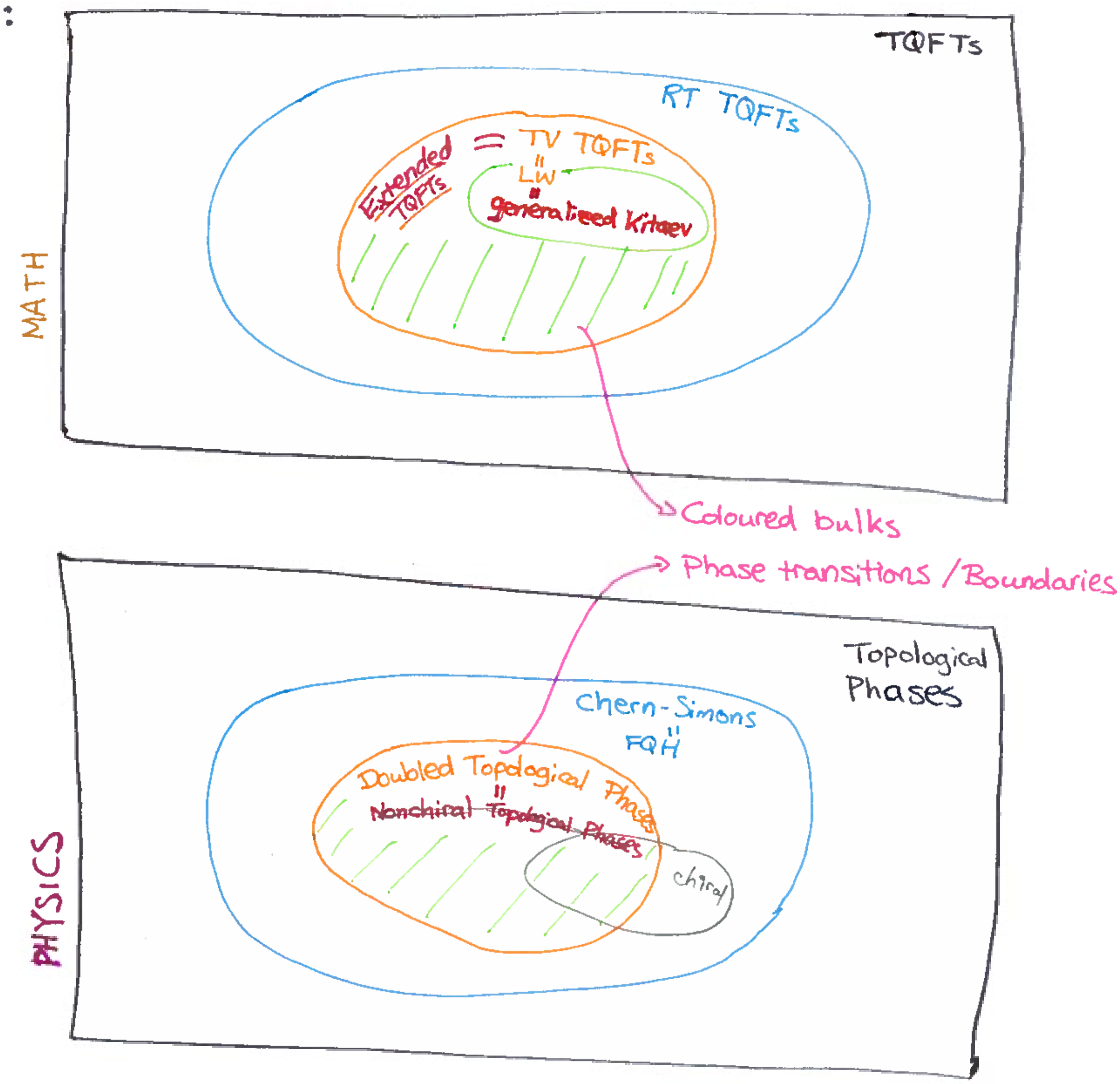
Theories of Excitations PHYSICS

Goals : ① Gain understanding of Coloured, extended TQFTs
- 3 Cob
- 3-categories.

② Raise questions about physical theory vs. mathematical theory.
- some progress / guesses

Comments : Interrupt anytime to ask math/physics questions.

Landscapes : in (2+1)D

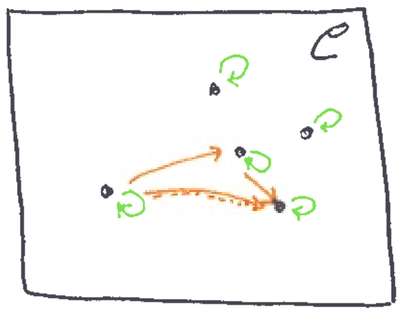


(2+1)D TQFTs (with an increasing amount of structures)

General idea (of TQFTs): "Z" : "$3\mathrm{Cob}$" $\longrightarrow$ "$\mathcal{A}lg$"

- "Z": some functor
- "$3\mathrm{Cob}$": some topological category (cobordisms)
- "$\mathcal{A}lg$": some algebraic category

Step 1 : Categories & Functors

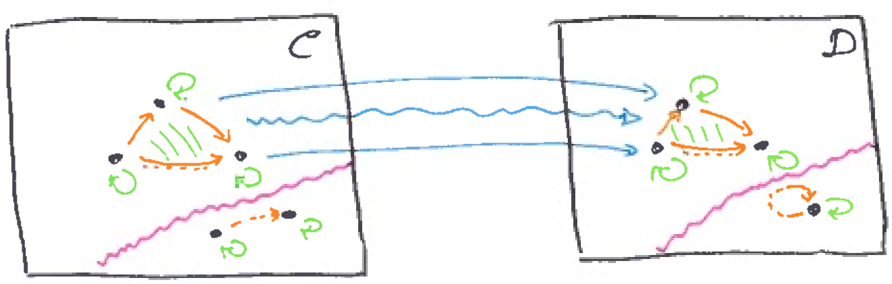


Category $\mathcal{C}$:

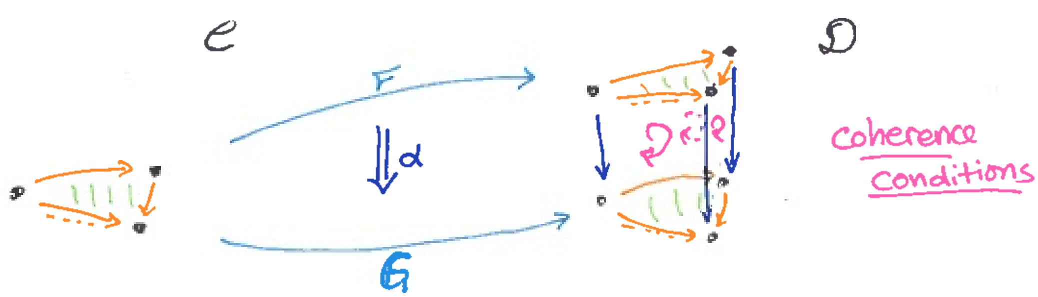
<u>data</u>: Objects $A, B, \dots$; Morphisms $f, g, \dots$ — Source & Target $\mathrm{Hom}(A, \dots)$

<u>Axioms</u>: ① Composition; ② Identity

Functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ maps objects & morph's



Natural Transformation $\alpha : F \longrightarrow G$ maps im(objects) in $\mathcal{D}$ only



Observation: A 1-category resembles a 1-dim'l CW-complex.

Physics : Often we are dealing with concrete categories.

Step 2 : (1+1)D TQFTs

- Need to describe functor

$Z_2 : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$ (Such a "functor" is call a 2D TQFT)

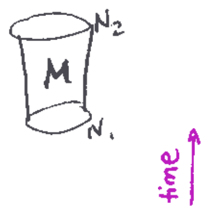
First, describe the categories

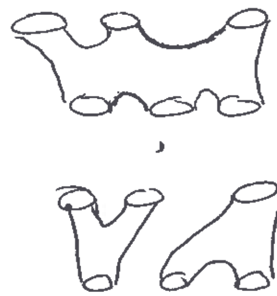
- 2Cob : ← rigid symmetric monoidal category

Objects: the oriented compact, closed ($\partial = 0$) 1D manifolds, ie. ϕ, ○ , ○○ , ○○○ , ...
monoidal unit; either orientation

Morphisms: 2D compact manifolds

$\mathrm{Hom}(\sqcup_k S^1, \sqcup_\ell S^1) =$





2D compact manifolds M with $\partial M = (\sqcup_k S^1) \sqcup (\sqcup_\ell S^1)$

Equivalences between objects: via homeomorphisms of 1-manifolds

Structural morphisms

monoidal $\otimes$

associator

Category $\circ$

id_X $N \times I$

Generating Object S^1

Generating morphisms

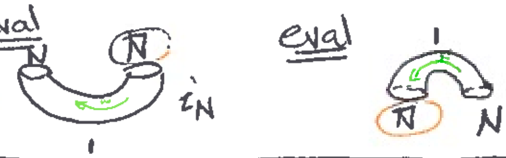
unit i counit tr mult

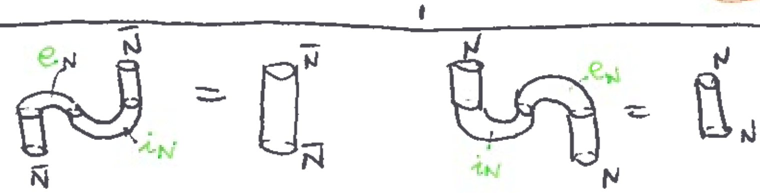
Symmetric B_{N_1,N_2}

$\boxed{B_{N_2,N_1} B_{N_1,N_2} = 1_{N_1 \otimes N_2}}$

rigidity

coeval Duality i_N eval e_N





triangle equality

Eg (of generating morphisms):

- Vect : repeat the above story. ⇝ need A to make all the above work.
special kind of v.sp.

Step 2 : (1+1)D TQFTs (cont'd)

- Let's examine $Z_2 : 2\mathrm{Cob} \longrightarrow \mathrm{Vect}$, $\bigcirc \longmapsto V$,  $\longmapsto \mathrm{Hom}(V \otimes V, V)$

  Since both LHS & RHS categories have
  rigid, symmetric, monoidal structure,
  we demand that Z_2 preserve these structures.

  $\overset{S^1}{\bigcirc} \longmapsto A$ (a f.d. commutative Frobenius algebra)
  equipped with $\varepsilon : A \to \mathbb{C}$
  s.t. $(a,b) = \varepsilon(a \cdot b)$ nondegenerate

   $\longmapsto \mu : A \otimes A \to A$

   $\longmapsto \varepsilon : A \to \mathbb{C}$

   $\longmapsto \eta : \mathbb{C} \to A$

  Call this functor Cor(1+1)DTQFT) $\boxed{Z_2^A}$

Step 3 : (2+1) D TQFTs

A (2+1)DTQFT is a functor

$$Z_3 : 3\mathrm{Cob} \longrightarrow \mathrm{Vect}.$$

- 3Cob :

  Objects 2D manifolds with $\partial = 0$ (compact, oriented), ie. genus g surfaces

  Morphisms 3D manifolds (compact oriented) ; $\mathrm{Hom}(N_1, N_2) = \{ M \text{ 3-manifold} \mid \partial M = \overline{N_1} \sqcup N_2 \}$

  Structures similar to 2Cob.

Remark Reshetikhin – Turaev construction
with input a modular (monoidal) category : $\otimes, *$, braiding, modul
produces a family of 3-2-1 TQFTs (not-0)

e.g. $\mathcal{C} = \mathrm{Rep}\, U_q \mathfrak{g}$, $q = e^{2\pi i/k}$

$Z_{RT}^{\mathcal{C}}(\bigcirc) = \mathcal{C}$

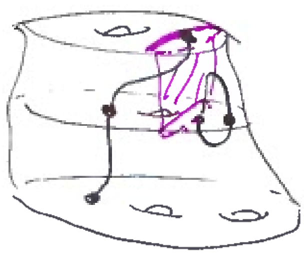
$Z_{RT}^{\mathcal{C}}(\text{(g)}) = \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathrm{Vect}$

Step 4 : (Fully) Extended (2+1)D TQFTs

A (fully) extended (2+1)D is a 3-functor

$$Z_4 : 3\mathrm{Cob}^{ext} \longrightarrow \mathrm{Mon}$$

- $3\mathrm{Cob}^{ext}$



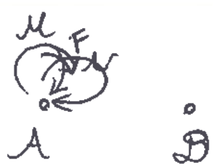
objects: 0-dim topological features

1-morph: 1-dim'l, e.g. worldlines of particles

2-morph: 2-dim'l, e.g. time slice, transparent domain wall

3-morph: 3-dim'l, e.g. spacetime

- Mon



objects: monoidal categories $\mathcal{A}, \mathcal{B}, \ldots$

1-morph: $\mathcal{A}$-$\mathcal{A}$ bimodule categories $\mathcal{M}$

(ie. $\mathcal{M}$ with structure maps
$\mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$
$\mathcal{M} \otimes \mathcal{A} \to \mathcal{M}$)

2-morph: $\mathcal{M} \to \mathcal{N}$ bimodule functors F

3-morph: functorial isomorphism (natural) $\phi : F \to G$

- 3-functor .. ..

Remark Turaev-Viro Construction

with input a spherical (monoidal) category: $\otimes, *$

produces a family of 3-2-1-0 TQFTs (fully extended)

$$Z_{TV}^{\mathcal{A}} : 3\mathrm{Cob}^{ext.} \longrightarrow \mathrm{Mon.}$$

$\bullet \longmapsto \mathcal{A}$

$| \longmapsto {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$

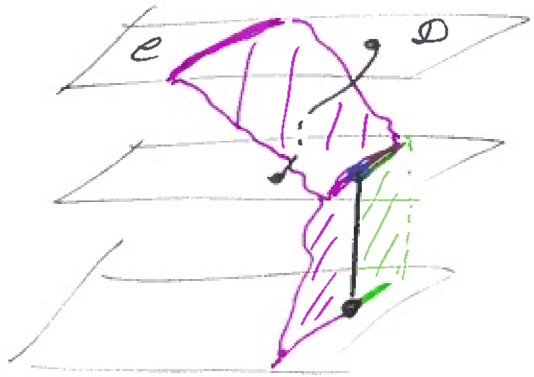
$\bigcirc \longmapsto Z(\mathcal{A})$

completely determined

Conjecture:

3D Extended TQFTs $\longleftrightarrow$ Spherical categories

Step 5: Coloured, Extended (2+1)D TQFTs



(beyond Lurie ~2009)

3-functor

$$Z_5 : 3\mathrm{Cob}^{\text{ext.}}_{\text{coloured}} \longrightarrow \mathcal{B}imod$$

update — bulks coloured — update

[Kitaev & Kong, Kirillov]

Morita equivalence ($Z(\mathcal{C}) = Z(\mathcal{D})$)

Spatial phase transition (Domain walls)

... Discussions ...

Colours vs. Invariants

In Kitaev & Kong,

		colours
(bulk) 3-dim'l	$\rightsquigarrow$	monoidal category $\mathcal{A}, \mathcal{B}$
(time slices) (domain wall) 2-dim'l	$\rightsquigarrow$	$\mathcal{A}$-$\mathcal{B}$ bimodule category
(eg. twists) 1-dim'l	$\rightsquigarrow$	bimodule functor $F : \mathcal{M} \to \mathcal{N}$
(instanton) 0-dim'l	$\rightsquigarrow$	natural isom. $\phi : F \to G$.

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