SOAR2001 — GEOMETRY
SUMMER 2001

1. Introduction to plane geometry

This is the short version of the notes for one of the chapters. The proofs are omitted but some hints are given. Try not to use the hints first, but if you need help look at the hints and pictures provided. The solutions are coming later.

The line joining the midpoints of two sides in a triangle is called a *midline*.

Problem 1. Show that a midline in a triangle is parallel to the base (the third side of the triangle), and is half as long.
Hint: Draw the picture and find similar triangles.

The line joining a vertex to the midpoint of the opposite side is called a *median*.

Problem 2. a) Given a triangle ABC, draw 2 of the three medians, say BB' and CC'. Let them meet at point G. Show that BB' and CC' trisect each other at G, i.e. $2C'G = GC$ and $2B'G = GB$.
Hint: Look at the picture below.

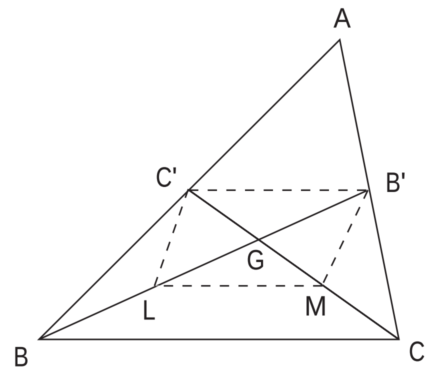


Figure 1. Hint for Problem 2

b) Conclude that all the three medians of any triangle pass through one point.

The point of intersection of the medians G is called the *centroid* of a triangle.

Problem 3. Show that the three bisectors of the three angles of a triangle all pass through one point I.

Hint: Prove and use the fact that the bisector of an angle is the loci of points equidistant from the sides of the angle.

The point I of intersection of the three angle bisectors, is equidistant from all three sides of the triangle, and hence is the center of the inscribed circle(also called *incircle*). We denote the radius of this circle by r and call the point I the *incenter*.

Problem 4. In a triangle ABC let $a = BC$, $b = CA$, $c = AB$. Let $p = \frac{1}{2}(a + b + c)$ be the semiperimeter and S be the area of ABC. Show that $S = pr$.

Problem 5. Show that the perpendicular bisectors of the three sides of a triangle all pass through one point O.
Hint: The perpendicular bisector of a line segment is the loci of the points equidistant from the end points.

This point O is equidistant from all three vertices of a triangle, and hence is the center of the circumscribed circle. This circle is called *circumcircle* and the center O is called the *circumcenter*. We denote the radius of the circumcircle by R.

Problem 6. Prove the following statements:
a) In a circle the angle at the center is double the angle at the circumference, if these two angles are subtended by the same arc, i.e on the picture, $\beta = 2\alpha$.

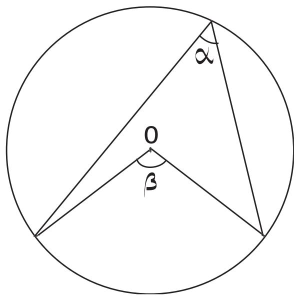


FIGURE 2. 6a)

b) In a circle, if two angles at the circumference are subtended by the same arc, these two angles are equal , i.e. in the picture, $\alpha = \gamma$.
c) In a circle, the angle at the circumference that is subtended by a diameter is equal to $\pi/2$, and vice versa, if a right angle is subtended by a diameter, the vertex of the angle belongs to the circumference.
Hint: The solution is contained in the picture below.
d) The opposite angles of any quadrangle inscribed in a circle are together equal to π, i.e. in the picture, $\alpha + \beta = \pi$.

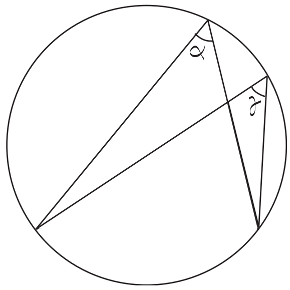


FIGURE 3. 6b)

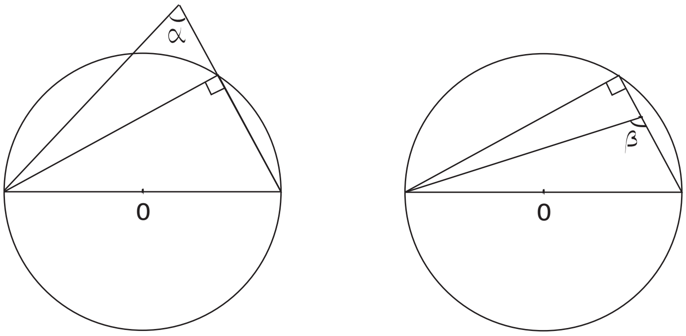


FIGURE 4. Solution to 6c)

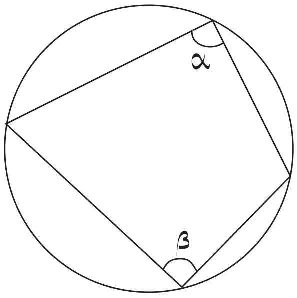


FIGURE 5. 6 d)

Problem 7. Let a,b, c be the sides of triangle ABC, α, β, γ be the opposite angles, R be the circumradius, and S be the area of the triangle. Then

a)

$$S = \frac{1}{2}ab\sin\gamma$$

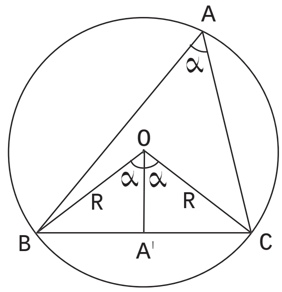
b)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \text{(Sine Law)}$$

Hint: Use a) to prove b).

The line through a vertex perpendicular to the opposite side is called an *altitude*.

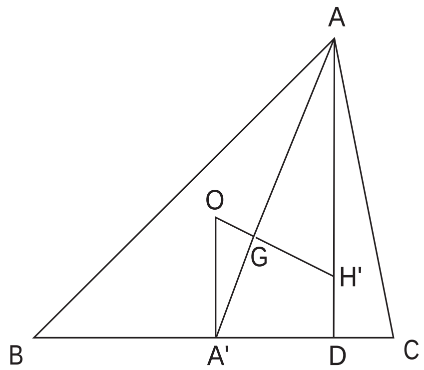
Problem 8. In the notations of the previous problem, $S = \frac{abc}{4R}$.
Hint: See the picture.



Problem 9. The three altitudes of any triangle ABC all pass through one point H.
Hint: Construct the triangle whose sides are the three lines through the three vertices of ABC, each of them parallel to the opposite side of ABC.

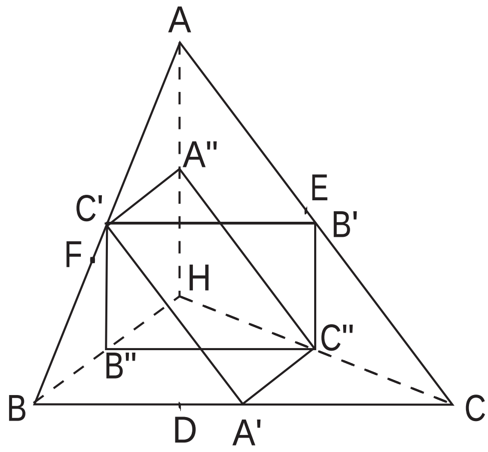
The point of intersection of the three altitudes H is called the *orthocenter* of the triangle.

Problem 10. The circumcenter O, the centroid G, and the orthocenter H, all lie on the same line and $HG = 2GO$. This line is called the *Euler line*.



Hint: Draw a line through the circumcenter O and centroid G. On this line, consider a point H' such that $GH' = 2OG$. Show that AH' is perpendicular to BC. Similar proof will show that BH', CH' are perpendicular to CA, AB respectively, i.e. $H' = H$, the orthocenter.

Problem 11. The midpoints of the three sides of a triangle, the midpoints of the lines joining the orthocenter to the three vertices, and the feet of the three altitudes, all lie on a circle. This circle is called the *nine-point circle*.



Hint: Show that $B'C'B''C''$ and $C'A'C''A''$ are rectangles first, and draw a circle whose diameter is $C'C''$.