MAT 1060H1F
Assignment 3

Prof. McCann Due: Noon on Thursday Sept. 30

Read Evans Appendix D. We have covered Chapter 1, 2.1–2.2 and hope to finish Chapter 2.3 this week and start 2.4 next week.

To be handed in: Evans (Second edition) # 2.8, 2.9, 2.11, 2.12, 2.14 and

1. Let given $U \subset\subset \mathbf{R}^n$ and $g \in C(\partial U)$ let $W = \{u \in C^2(\bar{U}) \mid u = g \text{ on } \partial U\}$. Let

$$A(u) := \int_U \sqrt{1 + |Du(x)|^2}\, dx$$

denote the n-dimensional area of the graph of u, and let $m = \inf_{w \in W} A(w)$ the minimum area of all graphs $w \in W$ satisfying the boundary conditions.

(a) Find a second order nonlinear partial differential equation satisfied by any area minimizing graph $u \in W$ such that $A(u) = m$. (HINT: Use the calculus of variations. An equation derived by finding critical points of a functional is called an *Euler-Lagrange* equation.) This particular PDE is also called the *minimal surface equation*; it represents the equilibrium shape of a soap film whose boundary lies on a wire given by the graph of the function $g \in C(\partial U)$.

(b) Show $A : W \longrightarrow \mathbf{R}$ is strictly convex, meaning $w_0, w_1 \in W$ and $s \in]0, 1[$ imply $A((1-s)w_0 + sw_1) < (1-s)A(w_0) + sA(w_1)$ unless $w_0 = w_1$.

(c) Prove at most one function $u \in W$ satisfies the minimal surface equation. (HINT: First prove that the derivative of a convex function $a : \mathbf{R} \longrightarrow \mathbf{R}$ vanishes *only* at the minimum of a.)