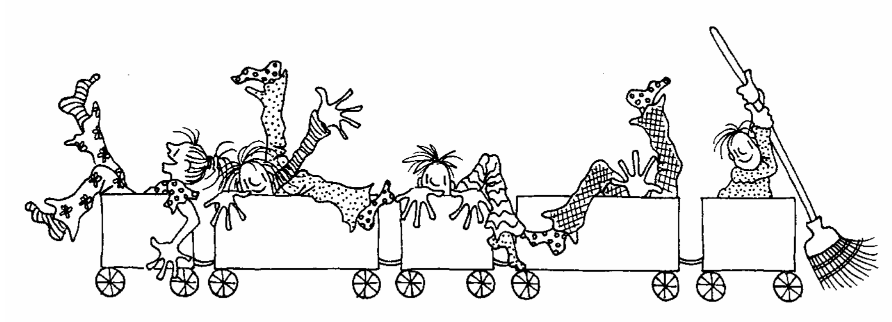
Trains



I want to construct a train of total length 12 units using cars which are either 1 unit long or 2 units long. The question is, how many different trains are there? For example, here is one possibility, using 6 cars of length 1 and 3 cars of length 2.

1	1	1	2	2	1	2	1	1

Other possibilities are obtained by rearranging the cars, or using different numbers of short and long cars. Note that a train has a front and a back, so that the mirror image of the train diagrammed above

1	1	2	1	2	2	1	1	1

is counted as a different train.

Of course, we are not so much interested in trains of length 12 as in trains of any length n. That is we want to discover the relationship between the length of the train and the number of different trains that can be constructed. But 12 serves as a good concrete starting point—small enough that the student will start to look at possibilities, but too large to be easily solved directly.

The students often seem to spend the first ten minutes blasting away at the problem--organizing the different types of trains by some scheme, often according to the number of 2-cars, trotting out the combinatorial coefficients, etc. After a while I say: "what makes the problem hard? *Too many possibilities!* How could we make life simpler? *Have a shorter train?*

Indeed, they recall what they've been always told—find a simpler problem, and work on it first.

Indeed there are some interesting direct approaches which stay with trains of length 12. But my agenda here is to get them to work up from short trains to large. That's often an effective mathematical strategy

Okay: how many trains of length 1? Clearly 1 such train—a single 1-car. How many trains of length 2? Clearly 2 such trains—a single 2-car, or two 1-cars. How many trains of length 3?—of length 4? We have to start making a systematic list of the possibilities. I let them work on that for a moment and collect on the board the results for trains of length up to 6.

For example, here are the trains of length 5

1-1-1-1-1
1-1-1-2
1-1-2-1
1-2-1-1
2-1-1-1
1-2-2
2-1-2
2-2-1

There are a total of 8 of them.

We stare at the results and you can hear the smiles breaking out all over the classroom.

Indeed we have the Fibonacci numbers!

Length of train	Number of trains
1	1
2	2
3	3
4	5
5	8
6	13

If we let t_n be the number of trains of length n, then the sequence t_n appears to be just the Fibonacci sequence u_n. Well not quite—it's shifted by one—for example, the 5th train number t_5 is 8 but that's the 6th Fibonacci number u_6 . But the train numbers do appear to follow the same additive rule as the Fibonacci sequence:

$$t_n = t_{n-1} + t_{n-2}.$$

Well!—if this holds, we have a nice arithmetic way of calculating t_{12} —just iterate the additive rule till we get to n=12. When we do that, we find that there are 233 trains of length 12.

n	t_n
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

So are we finished?—how many think that t_{12} is 233?

My students are wary as usual, and they need to be encouraged a bit, but eventually most of them vote yes.

Well I agree, I think it is too. BUT—can we be sure? Have we shown it beyond doubt?

Some care is needed here. The arithmetic pattern we found in the first six train numbers is certainly compelling, but can we be sure that it will continue to hold? Well not yet. There are nice examples of compelling patterns that suddenly fail (see **Regions**). We really need an argument.

So that's our job: to find some way of convincing ourselves that this simple sum rule should hold for the train sequence. As an example, I give the class the following specific task—convince me that there are 13 = 8+5 trains of length 6 without actually listing them, but using the fact that there are 8 trains of length 5 and 5 trains of length 4.

Well the argument they come up with, briefly stated, is that you can make a 6-train by starting with any 5-train and adding a car of length 1, or by starting with any 4-train and adding a car of length 2, and there will be 8 trains of the first type and 5 trains of the second type, for a total of 13 trains.

Right? Well it's a nice argument, but it requires just a bit more care. What they've given me is two procedures for constructing a 6-train, one that starts with a 5 train, and the other that starts with a 4-train. For this to actually give an exact count of the 6-trains you really have to check first that you get *all* of the 6-trains in one of these two ways, and secondly, that you haven't counted any 6-train *twice*, that is, that you can't get the same 6-train coming out of both routes.

Neither of these are hard to do, but the point is that such a check needs to be made to be sure that the argument works.

To help them “see” how to do this, I give them the following challenge. You want to show that

$$t_6 = t_5 + t_4$$

That is, you want to show that the number of 6-trains is the sum of two known numbers. Well to do this what you really want to do is partition the set of 6-trains into two natural subsets whose size is those two known numbers. For example imagine you’re on top of the CN tower looking down at all the 6 trains arrayed in the railroad yard. You want to find some natural way to sort them into two subsets, one of which has size t_5 and the other of which has size t_4.

Well, here’s how to do it. Let one subset be those trains that begin with a 1-car and let the other subset be those trains that begin with a 2-car.
Every train is clearly in exactly one of those two sets. Now how many trains are there in each set? Clearly there are t_5 in the first (the rest of the train has length 5) and t_4 in the second (the rest of the train has length 4). So t_6 must be the sum of those two numbers.

It’s clear that this argument is quite general, and could be used to show that the number of 7-trains was equal to the sum of the number of 6-trains and the number of 5-trains etc. So the additive rule always holds, and our conclusion that $t_{12} = 233$ is correct.

Now there's something really neat that we can do with this. What we have obtained is a physical "model" of the Fibonacci numbers, and the idea is to see if we can use that to obtain proofs of arithmetic relationships among these numbers.

We have shown that the train numbers are really the same as the Fibonacci numbers. Now there’s an awesome payoff for this connection. Read on.

A "train-theoretic" proof of a Fibonacci identity.

As an example, take the "sum-of-squares" identity, of which an example is

$$u_{11} = u_6{}^2 + u_5{}^2 .$$

There are quite a few ways we might try to show that this pattern holds in general (for example, by induction—see **Fibonacci** problem 2), but here’s a new way—don’t think about Fibonacci numbers at all—*instead, think about trains.* First we formulate the identity in “train language”:

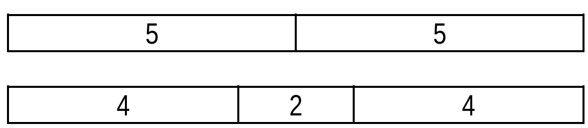
$$t_{10} = t_5{}^2 + t_4{}^2 .$$

Now we ask—can we find a "train-theoretic" argument for this identity?

This time, we want to partition the 10-trains into two disjoint classes so that there are t_5^2 trains of the first kind and t_4^2 trains of the second kind. A clue comes from noting that the subscripts 5 and 4 have the status of being roughly half of 10—this suggest that the classification ought to be based on something like "cutting the 10-trains in half." *Can you take it from there?*

This is another good small group problem for the class, but this time I find they have trouble with it. In fact, as I wander around, I discover a lot of frustration. They know what they have to do—look at half the train—but they just can’t seem to get it together. Typically they try to start with all the 4-trains and all the 5-trains and build 10-trains out of them. “You have to start with 10-trains,” I tell them, “and identify two types, and give me a clear rule by which I’ll be able to classify a 10-train as being of one type or another.

Well, here it is. There are two kinds of 10-trains, depending on whether or not they can be “cut in half” Those that can must have a join in the middle, and those that cannot must have a 2-car in the middle.



Those with a join are constructed of two trains of length 5, and there are t_5 possibilities for the front part and another t_5 possibilities for the back, for a total of t_5^2 trains with a join in the middle. Those with a 2-car in the middle are completed by adding two trains of length 4, and there are t_4^2 possibilities for that. Adding these up, we get

$$t_{10} = t_5{}^2 + t_4{}^2 .$$

Now that's an argument of great beauty!

NOT TO HAND IN: Read Heart 10.1-10.2, 10.5 of 4th ed. (or 9.1-9.2, 9.5 of 3rd ed.)
TO BE HANDED IN: #6, 7, 8, 12 below, plus on Apr 9 final version of independent project.

It has been a pleasure to have you in my freshman seminar. Good luck on your exams and in the rest of your university career. --- RJMc

Problems

1. For the sequence that's begun at the right, each term is equal to the previous term plus twice the term before that. If the first and second terms are both 1, find the 8th term x_8.

n	x_n
1	1
2	1
3	3 =1+2×1
4	5 =3+2×1

2. Use the same rule as 1) but take $x_1 = 1$ and $x_2 = 2$. What now is the 8th term?

3. . For the sequence that's begun at the right, each term is equal to the *magnitude* of the difference between the two preceding terms. If the first term is 4 and the second term is 10, what is the 100th term?

n	x_n
1	4
2	10
3	6 =10–4
4	4 =10–6

4. For the sequence that's begun at the right, each term is equal to twice the previous term minus the term before that. If the first term is 1 and second term is 2, find the 100th term x_{100}.

n	x_n
1	1
2	2
3	3 =2×2–1
4	4 =2×3–2

5. Suppose you take the rule of #4 but start with a different pair of numbers. Experiment a bit and see what kind of patterns you get. Can you formulate a general hypothesis and even perhaps see why it might hold?

6. Let s_n be the number of different symmetric trains of length n which are built out of cars which are either 1 unit long or 2 units long—where, a symmetric train is one that is the same running backwards and forwards. Thus 212 is a symmetric 5-train but 122 is not. For example, $s_5 = 2$ (table at the right). Find a formula for s_n in terms of the Fibonacci numbers. Can you find a general argument for your formula?

The 2 symmetric trains of length 5
1-1-1-1-1
2-1-2

7. Let r_n be the number of different *reversible* trains of length n which are built out of cars which are either 1 unit long or 2 units long. This is the same as the standard train problem, except the 5-trains 122 and 221 are no longer considered different, because one is the same as the other in reverse order. But the train 212 is different from these. For example, $r_5 = 5$ (table at the right). Find a formula for r_n in terms of the Fibonacci numbers. Can you find a general argument for your formula?

The 5 reversible trains of length 5
1-1-1-1-1
1-1-1-2
1-1-2-1
1-2-2
2-1-2

8. Give a "train-theoretic" proof of the Fibonacci identity:
$u_{11} = u_4 \cdot u_6 + u_5 \cdot u_7$.

9. Consider Pascal's triangle and observe that the Fibonacci numbers are the sums of the shallow diagonals (these are in 1-1 correspondance with the left-hand 1's). The combinatorial formula for this is

$$u_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \ldots$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

There is a very elegant "train-theoretic" proof of this formula. Find it.

10. Give a "train-theoretic" proof of each of the following Fibonacci identities. The identities are quite general, but I give particular versions. In (b) there is a different form for odd and even *n*.

(a) $u_{13} = u_{11} + u_{10} + u_9 + \ldots + u_1 + 1$.

(b) $u_{12} = u_{11} + u_9 + u_7 + \ldots + u_1$. and $u_{13} = u_{12} + u_{10} + u_8 + \ldots + u_2 + 1$.

11. Give a "train-theoretic" proof of the Fibonacci identity: $u_{11} = u_4 \cdot u_6 + u_5 \cdot u_7$.

12.(a) Let s_n be the number of sequences of 0's and 1's of length *n* which have no consecutive 1's. For example, there are a total of 8 sequences of 0's and 1's of length 3 (why?), but only 5 of these have no consecutive 1's. Thus $s_3 = 5$. Find the sequence s_n.

(b) A fair coin is tossed until two consecutive heads appear. This might take 2 tosses, or three, or four, etc. Let p_n be the probability that it will take exactly *n* tosses. Thus $p_3 = 1/8$ because only 1 of the 8 possible sequences of H and T gets two consecutive heads for the first time exactly on the 3rd toss (that's THH). Find the sequence p_n.

(c) If you've solved (b) you have a sequence p_n of interesting numbers. Now sooner or later you expect to get two heads in a row, so the sum of all the p_n $(2 \le n \le \infty)$ must equal 1. That gives an interesting summation formula, and a proof of it at the same time. What is it? Can you find an alternative (direct) proof?

13. Give a "train-theoretic" proof of each of the following Fibonacci identities. The identities are quite general, but I give particular versions. In (b) there's a different form for odd or even *n*.

(a) $u_{12}u_{11} = u_{11}^2 + u_{10}^2 + u_9^2 + \ldots + u_1^2$

(b) $u_{13}^2 = u_{13}u_{12} + u_{12}u_{11} + u_{11}u_{10} + \ldots + u_2u_1 + 1$

$u_{12}^2 = u_{12}u_{11} + u_{11}u_{10} + u_{101}u_9 + \ldots + u_2u_1$

[Hint: These are more complicated than #2 in that they involve working with two trains together. For example in (a) we want an 11-train and a 10-train. Park the two trains parallel to one another with the fronts lined up. Now start at the back and walk towards the front between the two trains. Stop when you come to the first 1-car, on either train, and classify the situation according to the position of this car. Do the same thing for (b), except that now the trains are the same length, so that the counting is a bit trickier. It's very satisfying when you see how to make it work.