Cutting space with planes

Suppose I cut space with a plane. How many regions do I get? The answer is 2. Suppose I now use 2 planes––how many regions? Well, it depends––if the planes are parallel, there will be 3, otherwise 4. To remove this ambiguity, let's agree that our planes should be randomly chosen, or "in general position", so that coincidences do not occur. Then the answer is 4 regions. Now try 3 planes, again in general position. The answer is 8 regions––indeed, we can think of the three planes $x=0$, $y=0$ and $z=0$ in the Cartesian coordinate system. The regions which these planes create are called "octants" because there are eight of them.

It's a bit harder to visualize 4 planes, and the question touches off a murmur of debate in the classroom. Can all 8 of the existing regions be cut into by the new plane? The answer is no, but perhaps the argument is not so easy to pin down. But 7 of them can certainly be sliced. One nice construction is to take the three coordinate planes as above, and then to slice everything with the plane that cuts the three axes at a distance +1 from the origin. This cuts a tetrahedron out of the positive octant, and cuts through all the other octants except the "negative" one–– x, y and z all negative. So 7 of the 8 octants are "cut in half," giving 15 regions. The question is, what is the pattern? That is, how does the table at the right continue?

I give the class a few moments to play with 5 planes, but things are very hard to visualize. *Can the new plane pass through one of the four existing intersection points?* No, that would be a coincidence. *With 4 planes, one of the 15 regions is bounded, so it's different from the others. Does it make a difference whether the fifth plane goes through the bounded region?* Donno—that's a good question.

One way of coping with a hard problem is to solve an easier one first. In this case, if we go down to two dimensions we get an easier problem because pictures can be more easily drawn. So our base space is now a plane, and the objects we cut it with are—*lines!* So let's ask how many regions are created in a plane by cutting it with lines. Again we want the lines to be in general position, so we don't have any parallel lines or concurrencies. It is not hard to produce the following table. The case of 5 lines is illustrated at the right.

Now what's the pattern here? The class is quick to notice the interesting difference pattern––successive differences in the right column of the 2-D table are the integers, 1,2,3,4,5, and 6. By this reckoning the next number should be 29, and a careful picture will verify this. But the important task of course is to find a way to "see" why this pattern is expected.

Alas, the class is not really interested in that question—at least not right away. What they clamour to do is to look at the difference pattern in the
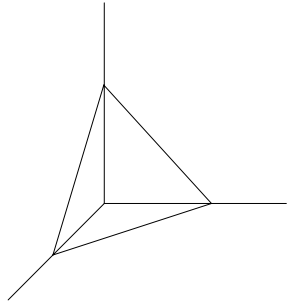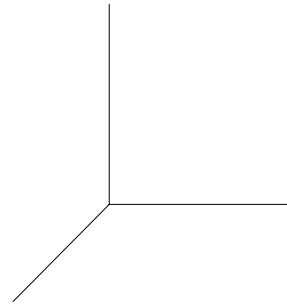
3-D	
planes	regions
0	1
1	2
2	4
3	8
4	15
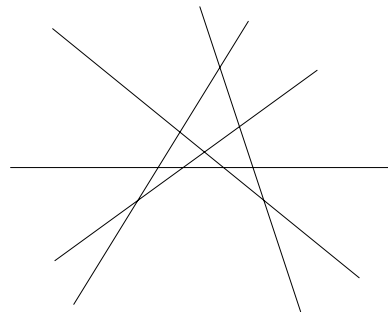
2-D	
lines	regions
0	1
1	2
2	4
3	7
4	11
5	16
6	22

3-D	
planes	regions
0	1
1	2
2	4
3	8
4	15

1

3-D table. This turns out to be the sequence 1,2,4, and 7. Now that's really interesting: these are the first four numbers of the 2-D table! Dare we conjecture that this pattern continues––that therefore the next number of the 3-D table should be 15+11=26? Do 5 planes cut space up into 26 regions?

To answer this question, we really do need some new understanding. Why should the 2-D table provide the set of differences for the 3-D table?

A powerful idea is to go down one dimension. Let's look again at the simple difference pattern for the 2-D table—where does that nice set of 1,2,3,4… really come from? What really *are* those differences?

For example, let's focus on the transition between 5 and 6 lines in the 2-D table. That causes the number of regions to change from 16 to 22, a difference of 6. Now where do those 6 new regions really come from?

Well, if we study the picture, those 6 regions are created by the sixth line slicing through 6 of the old regions and cutting them "in half." But why 6? Is it because it's the 6th line? Not really.

If we stare hard at the picture, we see it is because the 6th line is divided into 6 segments by its intersections with the 5 existing lines, and *each of these segments represents a cutting of one of the existing regions in half!* So what that 6 really is, is the number of segments created on a line by 5 points. The reason I want to say it this way, is because now we see that *this is really the same problem as before but one more dimension down. This is really the problem of how many regions are created on a __line__ if it is "cut" by n points.*

Of course! If the 2-D problem somehow provides the differences for the 3-D table, we might expect the 1-D problem to provide the differences for the 2-D table.

So let's formulate the 1-D problem. What do we cut 1-dimensional space with?––well with points! If we place n points on a line the number of regions (which are really intervals now) we get is n+1. The table is at the right.
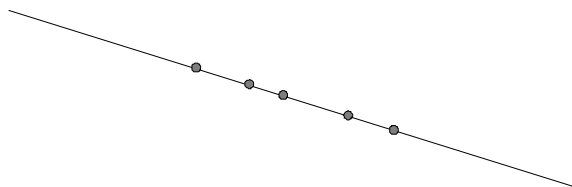
And what we have noticed above is the reason that this table provides the differences for the 2-D table. For example, the difference between 16 and 22 in the 2-D table is 6, but exactly what is that 6? (because there are lots of 6's around here)—it's the 6 in the "regions" column of the 1-D table which sits in the n=5 row. That's how to think of that 6.

2-D	
lines	regions
0	1
1	2
2	4
3	7
4	11
5	16
6	22
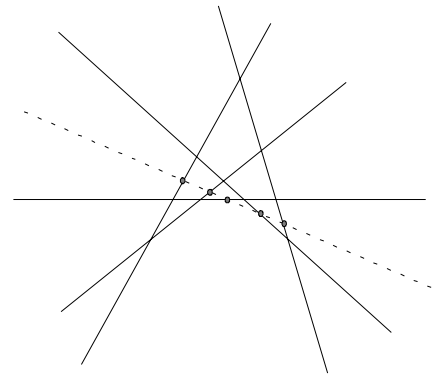
1-D	
points	regions
0	1
1	2
2	3
3	4
4	5
5	
6	

2-D			1-D	
lines	regions		points	regions
0	1		0	1
1	2		1	2
2	4		2	3
3	7		3	4
4	11		4	5
5	16		5	6
6	22		6	7

cutplane 2

Now this should show us the way to argue that the differences in the 3-D table are provided by the 2-D table. Let's see if we can run the whole argument through one dimension up. Take, for example, the addition of the fifth plane. We want to know how many new regions it creates. Well, this plane will intersect all the 4 previous planes (no parallelism) and these intersections will etch a pattern of 4 lines on the plane. These 4 lines must cut the plane into 11 regions (from the 2-D table) and each of these regions really represents a "cutting in half" (by the 5th plane) of each of the original regions in 3-space. So the 5th plane should create 11 new regions in space––that "11" being the $n=4$ entry of the 2-D table. So to get the $n=5$ entry of the 3-D table from the $n=4$ entry, we need to add the $n=4$ entry of the 2-D table. There it is!

It all works out quite beautifully. The 1-D table provides the differences for the 2-D table, and the 2-D table provides the differences for the 3-D table. With this we deduce that 5 planes will indeed cut space into 26 regions, and we can keep going and find the answer for 6 planes, and 7 planes, etc.

I stand regarding the blackboard with awe. The students are silent. They can see how the tables can now be generated. Is it really that easy? It's important to emphasize that along with this computational scheme, we also have an elegant proof that it works––a way to argue with absolute certainty that this scheme will calculate the correct entries. For example, let's go back to our small uncertainties with four planes in 3-space. We know that three planes will determine 8 regions and we now have a nice argument that the 4th plane can't cut through all 8 of the existing regions. Because if it did, we'd contradict the 2-D table–– we'd have a plane with three lines on it defining 8 regions. And in the same way the 2-space table must be right because we know we can trust the 1-space table.

Okay, what's next? Another silence from the class, but this one expectant. A voice from the back catches on: *What about 4-space?* Right! Suppose I slice 4-space with 6 hyperplanes: how many regions?

Well it would seem to be an almost impossible problem, but the arguments behind our results are quite general and would apply to the transition from 3 dimensions to four. Using the entries of the 3-D table as differences, we get the 4-D table at the right. With 6 hyperplanes, we get 57 regions.

3-D	
planes	regions
0	1
1	2
2	4
3	8
4	15
5	26
6	42

This is a superb example of the power of pattern recognition, analogical reasoning, and persistence in getting at the heart of the matter. This was a hard problem! But we have arrived at a convincing analysis.

4-D	
hyper-planes	regions
0	1
1	2
2	4
3	8
4	16
5	31
6	57

This lovely problem was used by the famous mathematician and teacher George Polya in his movie "Let's Teach Guessing".

Problems

1. Find a formula for the nth entry of the 2-D and 3-D tables.

2. Suppose I draw 20 circles in the plane, all passing through the origin, but no two tangent at the origin. Also, except for the origin, no three circles pass through a common point. How many regions are created in the plane?

3. Take a sphere and draw on it a great circle (a great circle is a circle whose centre is the centre of the sphere). There are two regions created. Here, I am referring to regions on the surface of the sphere. Now draw another great circle: there are four regions. Now draw a third, not passing through the points of intersection of the first two. How many regions?

Here's the general question: How many regions are created by n great circles, no three concurrent, drawn on the surface of the sphere?

TO BE HANDED IN: #1, 2, 3 above.

NOT TO BE HANDED IN:
Read Sections 7.1-7.2 of `Heart' 4th ed. (= sections 6.1-6.2 of 3rd ed.)
Do section 7.2 #14 of 4th ed (= 6.2 #14 of 3rd ed.)
Do section 1.1 story #6 `A tight weave'
BE PREPARED, AS USUAL, TO BE QUIZZED ON THIS MATERIAL
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