Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary

A. Burtscher, C. Ketterer, R. McCann, E. Woolgar

S.I.G.M.A. **16** (2020) 131, 29 pages. Special issue dedicated to Gromov's 75th birthday.

University of Toronto

Slides: click on 'Talk' at www.math.toronto.edu/mccann

7 June 2021

Inscribed radius bounds

Inscribed radius $r = r(\Omega)$ of an open subset Ω of a metric space (X, d) is

$$r(\Omega) := \sup\{r \geq 0 \mid B(x; r) \subset \Omega \text{ for some } x \in \Omega\}$$

Inscribed radius bounds

Inscribed radius $r = r(\Omega)$ of an open subset Ω of a metric space (X, d) is

$$r(\Omega) := \sup\{r \geq 0 \mid B(x; r) \subset \Omega \text{ for some } x \in \Omega\}$$

Theorem (classical)

If $(X, d) = (\mathbb{R}^n, |\cdot|)$ and the mean-curvature $H_{\partial\Omega} \geq (n-1)/R$ then

$$r(\Omega) \leq R,$$

with equality iff Ω isometric to the Euclidean ball $B(0; R) \subset \mathbb{R}^n$ ('rigidity')

- this classical bound may be proved using the maximum principle
- it has well-known Lorentzian (Hawking '68) and Riemannian (Kasue '83, Li '14, ...) analogs

Theorem (Kasue '83; mean convex Riemannian mflds with boundary)

If $(X, d) = (M^n, g)$ *has* $Ric_g \geq Kg$ *and* $\Omega \subset M$ *with* $\partial\Omega \neq \emptyset$ *is* C^2*, open, connected, and has* $H_{\partial\Omega}(x) \geq H \in \mathbb{R}$ *with* $\max\left\{K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}}\right\} > 0$*, then*

$$r(\Omega) \leq r_{K,H,n} = r_{\frac{K}{n-1},\frac{H}{n-1},2} < \infty$$

with equality iff Ω *is isometric to an open ball whose boundary has mean curvature* H *in a spaceform of constant (sectional) curvature* $\frac{K}{n-1}$.

- here 'spaceform' refers to a spherical, Euclidean or hyperbolic n-space
- but what if Ω and/or X is less smooth?

Theorem (Kasue '83; mean convex Riemannian mflds with boundary)

If $(X, d) = (M^n, g)$ *has* $Ric_g \geq Kg$ *and* $\Omega \subset M$ *with* $\partial\Omega \neq \emptyset$ *is* C^2*, open, connected, and has* $H_{\partial\Omega}(x) \geq H \in \mathbb{R}$ *with* $\max\left\{K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}}\right\} > 0$*, then*

$$r(\Omega) \leq r_{K,H,n} = r_{\frac{K}{n-1},\frac{H}{n-1},2} < \infty$$

with equality iff Ω *is isometric to an open ball whose boundary has mean curvature* H *in a spaceform of constant (sectional) curvature* $\frac{K}{n-1}$.

- here 'spaceform' refers to a spherical, Euclidean or hyperbolic n-space
- but what if Ω and/or X is less smooth?
- Lott-Villani and Sturm used optimal transport to define lower Ricci bounds in metric spaces (X, d) equipped with a reference measure m;

Theorem (Kasue '83; mean convex Riemannian mflds with boundary)

If $(X,d) = (M^n, g)$ *has* $Ric_g \geq Kg$ *and* $\Omega \subset M$ *with* $\partial\Omega \neq \emptyset$ *is* C^2*, open, connected, and has* $H_{\partial\Omega}(x) \geq H \in \mathbb{R}$ *with* $\max\left\{K, \frac{H}{n-1} - \sqrt{\frac{|K|}{n-1}}\right\} > 0$*, then*

$$r(\Omega) \leq r_{K,H,n} = r_{\frac{K}{n-1},\frac{H}{n-1},2} < \infty$$

with equality iff Ω *is isometric to an open ball whose boundary has mean curvature* H *in a spaceform of constant (sectional) curvature* $\frac{K}{n-1}$.

- here 'spaceform' refers to a spherical, Euclidean or hyperbolic n-space
- but what if Ω and/or X is less smooth?
- Lott-Villani and Sturm used optimal transport to define lower Ricci bounds in metric spaces (X,d) equipped with a reference measure m;
- we extend Kasue's theorem to this metric-measure space ('mms') setting
- our rigidity statement requires the more restrictive *RCD* ('Riemannian curvature dimension') condition of Ambrosio, Gigli, and Savare '14, and equality is attained by truncated cones as well as by balls

Terminology

- (X, d, m) is a metric space with Borel reference measure s.t. $X = \mathrm{spt}\, m$
- *geodesic* refers to a curve $\{x_t\}_{t\in[0,1]} \subset X$ satisfying

$$d(x_s, x_t) = |t - s| d(x_0, x_1) \quad \forall s, t \in [0, 1]$$

- assume geodesics do not branch (or essentially non-branching)

Terminology

- (X, d, m) is a metric space with Borel reference measure s.t. $X = \operatorname{spt} m$
- *geodesic* refers to a curve $\{x_t\}_{t\in[0,1]} \subset X$ satisfying

$$d(x_s, x_t) = |t - s| d(x_0, x_1) \quad \forall s, t \in [0, 1]$$

- assume geodesics do not branch (or essentially non-branching)
- $\mathcal{P}(X, d) := \{\text{Borel probability measures } \mu \text{ on } X\}$, metrized by the L^2-Kantorovich-Rubinstein-Wasserstein distance from optimal transport

$$d_2(\mu, \nu) := \left(\inf_{\{\gamma\in\mathcal{P}(X^2, d\otimes d) \text{ with marginals } \mu \text{ and } \nu\}} \int_{X^2} d(x, y)^2 d\gamma(x, y) \right)^{1/2}$$

Terminology

- (X, d, m) is a metric space with Borel reference measure s.t. $X = \mathrm{spt}\, m$
- *geodesic* refers to a curve $\{x_t\}_{t\in[0,1]} \subset X$ satisfying

$$d(x_s, x_t) = |t-s| d(x_0, x_1) \quad \forall s, t \in [0,1]$$

- assume geodesics do not branch (or essentially non-branching)
- $\mathcal{P}(X, d) := \{\text{Borel probability measures } \mu \text{ on } X\}$, metrized by the L^2-Kantorovich-Rubinstein-Wasserstein distance from optimal transport

$$d_2(\mu, \nu) := \left(\inf_{\{\gamma \in \mathcal{P}(X^2, d\otimes d) \text{ with marginals } \mu \text{ and } \nu\}} \int_{X^2} d(x,y)^2 \mathrm{d}\gamma(x,y) \right)^{1/2}$$

- $\mathcal{P}_2(X, d) := \{\mu \in \mathcal{P}(X, d) \mid d_2(\mu, \delta_x) < \infty\}$ for some (hence all) $x \in X$
- $\mathcal{P}_2^*(X, d, m) := \{\mu \in \mathcal{P}_2(X, d, m) \mid \text{finite rel. entropy } E(\mu \mid m) < \infty\}$

Curvature Dimension (& Measure Contraction) Properties

- Sturm '06: fix curvature and dimension parameters $K \in \mathbb{R}$ and $N \geq 1$
- $K = 0$ and/or $N = \infty$ considered also in Lott-Villani '09

Definition ($CD(K, N)$ after Erbar-Kuwada-Sturm '15)

$(X, d, m) \in CD(K, N) \Leftrightarrow \forall \mu_0, \mu_1 \in \mathcal{P}_2^*(X, d, m)$ $\exists$ geodesic $\{\mu_t\}_{t\in[0,1]}$ s.t.

$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2$$

Curvature Dimension (& Measure Contraction) Properties

- Sturm '06: fix curvature and dimension parameters $K \in \mathbb{R}$ and $N \geq 1$
- $K = 0$ and/or $N = \infty$ considered also in Lott-Villani '09

Definition ($CD(K, N)$ after Erbar-Kuwada-Sturm '15)

$(X, d, m) \in CD(K, N) \Leftrightarrow \forall \mu_0, \mu_1 \in \mathcal{P}_2^*(X, d, m)$ $\exists$ geodesic $\{\mu_t\}_{t\in[0,1]}$ s.t.

$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2 \quad \text{distributionally on } t \in (0,1), \text{ where}$$

$$e(t) = E(\mu_t \mid m) := \begin{cases} \int_X \frac{\mathrm{d}\mu_t}{\mathrm{d}m} \log \frac{\mathrm{d}\mu_t}{\mathrm{d}m} \mathrm{d}m & \text{if } \mu_t << m, \\ +\infty & \text{else,} \end{cases}$$

is the Boltzmann-Shannon entropy along the $(\mathcal{P}(X, d), d_2)$ geodesic.

- $(X, d, m) \in MCP(K, N) \Leftrightarrow$ the same $\forall \mu_0 \in \mathcal{P}_2^*(X, d, m)$ and $\mu_1 = \delta_x$.

- Riemannian mfld $(M^n, d_g, \mathrm{vol}_g) \in CD(K,N) \Leftrightarrow Ric_g \geq Kg$ and $n \leq N$;

e.g. $M = \overline{B(0,1)} \subset \mathbb{R}^n$ is $CD(0,n)$, while $\partial M \in CD(n-2, n-1)$.

- $CD(K,N) \subset MCP(K,N)$ (Sturm '06, Ohta '07)

• Riemannian mfld $(M^n, d_g, \mathrm{vol}_g) \in CD(K, N) \Leftrightarrow Ric_g \geq Kg$ and $n \leq N$;

e.g. $M = \overline{B(0,1)} \subset \mathbb{R}^n$ is $CD(0, n)$, while $\partial M \in CD(n-2, n-1)$.

• $CD(K, N) \subset MCP(K, N)$ (Sturm '06, Ohta '07)

• If $(X, d, m) \in CD(K', N')$ and $\Omega \subset X$, we say $\Omega \in CD_r(K, N)$ if $\mu_0[\Omega] = 1 = \mu_1[\Omega]$ in the previous construction implies

$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2.$$

- Riemannian mfld $(M^n, d_g, \mathrm{vol}_g) \in CD(K,N) \Leftrightarrow Ric_g \geq Kg$ and $n \leq N$;

e.g. $M = \overline{B(0,1)} \subset \mathbb{R}^n$ is $CD(0,n)$, while $\partial M \in CD(n-2, n-1)$.

- $CD(K,N) \subset MCP(K,N)$ (Sturm '06, Ohta '07)
- If $(X,d,m) \in CD(K',N')$ and $\Omega \subset X$, we say $\Omega \in CD_r(K,N)$ if $\mu_0[\Omega] = 1 = \mu_1[\Omega]$ in the previous construction implies

$$e''(t) - \frac{e'(t)^2}{N} \geq K d_2(\mu_0, \mu_1)^2.$$

- restriction $MCP_r(K,N)$ is defined analogously relative to $MCP(K',N')$
- the *signed distance* to the boundary of Ω is defined by

$$d_\Omega^\pm(x) := \begin{cases} d_\Omega(x) & \text{if } x \notin \Omega, \\ -d_{X\setminus\Omega}(x) & \text{if } x \in \Omega, \end{cases}$$

where

$$d_\Omega(x) := \inf_{y \in \Omega} d(x,y)$$

- $Lip(d_\Omega^\pm) \leq 1$

Cavalletti et al's Needle Decomposition / 1d Localization

After discarding a (carefully chosen!) measure zero set from the non-branching MCP space (X, d, m),

$$x \sim y \Leftrightarrow |d_\Omega^\pm(y) - d_\Omega^\pm(x)| = d(x, y)$$

defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically 'normal' to $\partial\Omega$.

Cavalletti et al's Needle Decomposition / 1d Localization

After discarding a (carefully chosen!) measure zero set from the non-branching MCP space (X, d, m),

$$x \sim y \Leftrightarrow |d_\Omega^\pm(y) - d_\Omega^\pm(x)| = d(x, y)$$

defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically 'normal' to $\partial\Omega$.

Let $\tilde{x}$ denote the equivalence class of $x \in X$;
the quotient space $\tilde{X} := X/\sim$ can be identified with $\partial\Omega$,
and inherits the quotient measure $\tilde{m}$ from (X, d, m).
Viewing $(\tilde{x}, d_\Omega^\pm(x)) \in \partial\Omega \times \mathbb{R}$ as 'coordinates' on X,

Cavalletti et al's Needle Decomposition / 1d Localization

After discarding a (carefully chosen!) measure zero set from the non-branching MCP space (X, d, m),

$$x \sim y \Leftrightarrow |d_\Omega^\pm(y) - d_\Omega^\pm(x)| = d(x, y)$$

defines an equivalence relation, whose equivalence classes consist of geodesic segments called needles, heuristically 'normal' to $\partial\Omega$.

Let $\tilde{x}$ denote the equivalence class of $x \in X$;
the quotient space $\tilde{X} := X/\sim$ can be identified with $\partial\Omega$,
and inherits the quotient measure $\tilde{m}$ from (X, d, m).

Viewing $(\tilde{x}, d_\Omega^\pm(x)) \in \partial\Omega \times \mathbb{R}$ as 'coordinates' on X,
Cavalletti and coauthors re-express m relative to these coordinates,
by providing a measure $m_{\tilde{x}}$ on each equivalence class $\tilde{x} \in \partial\Omega$ s.t.

$$m(E) = \int_{\partial\Omega} \mathrm{d}\tilde{m}(\tilde{x}) \int_{\tilde{x} \cap E \subset \mathbb{R}} \mathrm{d}m_{\tilde{x}}(s) \qquad \forall \text{ Borel } E \subset X.$$

Mean convexity and mean curvature

- $(X, d, m) \in CD(K, N) \Leftrightarrow (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)$ $\forall \Omega \subset X$, $\tilde{m}$-a.e. $\tilde{x}$

Mean convexity and mean curvature

- $(X, d, m) \in CD(K, N) \Leftrightarrow (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)$ $\forall \Omega \subset X$, $\tilde{m}$-a.e. $\tilde{x}$
- $(\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \Leftrightarrow \tilde{x} = \{x\}$ or $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds$ and

$$(h_{\tilde{x}}^{\frac{1}{N-1}})'' \leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}$$

distributionally, where $s = d_\Omega^\pm(x) =$ arclength along geodesic segment $\tilde{x}$

Mean convexity and mean curvature

- $(X, d, m) \in CD(K, N) \Leftrightarrow (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)$ $\forall \Omega \subset X$, $\tilde{m}$-a.e. $\tilde{x}$
- $(\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \Leftrightarrow \tilde{x} = \{x\}$ or $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds$ and

$$(h_{\tilde{x}}^{\frac{1}{N-1}})'' \leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}$$

distributionally, where $s = d_{\Omega}^{\pm}(x) =$ arclength along geodesic segment $\tilde{x}$

- Ketterer '20: define the (inner) mean curvature $H_{\partial\Omega}$ at $\tilde{m}$-a.e. $x \in \partial\Omega$ as

$$H_{\partial\Omega}(x) := \frac{\mathrm{d}^{+}}{\mathrm{d}s} \log h_{\tilde{x}}(0^{-}) = \limsup_{s \nearrow 0} \frac{\log h_{\tilde{x}}(s) - \log h_{\tilde{x}}(0)}{s}$$

Mean convexity and mean curvature

- $(X, d, m) \in CD(K, N) \Leftrightarrow (\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N)$ $\forall \Omega \subset X$, $\tilde{m}$-a.e. $\tilde{x}$
- $(\tilde{x}, |\cdot|, m_{\tilde{x}}) \in CD(K, N) \Leftrightarrow \tilde{x} = \{x\}$ or $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds$ and

$$(h_{\tilde{x}}^{\frac{1}{N-1}})'' \leq -\frac{K}{N-1} h_{\tilde{x}}^{\frac{1}{N-1}}$$

distributionally, where $s = d_{\Omega}^{\pm}(x) =$ arclength along geodesic segment $\tilde{x}$

- Ketterer '20: define the (inner) mean curvature $H_{\partial\Omega}$ at $\tilde{m}$-a.e. $x \in \partial\Omega$ as

$$H_{\partial\Omega}(x) := \frac{\mathrm{d}^+}{\mathrm{d}s} \log h_{\tilde{x}}(0^-) = \limsup_{s \nearrow 0} \frac{\log h_{\tilde{x}}(s) - \log h_{\tilde{x}}(0)}{s}$$

- we'll write $H_{\partial\Omega} \geq \lambda$ provided $H_{\partial\Omega}(x) \geq \lambda$ holds $\tilde{m}$-a.e., $m[\partial\Omega] = 0$, and

$$m[\{x \mid \tilde{x} \subset \Omega \cup \partial\Omega\}] = 0$$

(preventing e.g. inward pointing cusps on the boundary)

- c.f. Cavalletti-Mondino '20+ nonsmooth Hawking singularity theorem

Theorem (Extending Kasue's results to nonsmooth spaces)

(a) If $(X, d, m) \in MCP(K', N)$ *for some* $K' \in \mathbb{R}$ *and* $1 < N < \infty$ *and if* $\Omega \subset X$ *open with* $\partial\Omega \neq \emptyset$ *satisfies* $\Omega \in MCP_r(K, N)$ *and* $H_{\partial\Omega} \geq H \in \mathbb{R}$ *with* $\max\left\{K, H - \sqrt{(N-1)|K|}\right\} > 0$, *then the inscribed radius of* Ω

$$r(\Omega) \leq r_{K,H,N} := r_{\frac{K}{N-1}, \frac{H}{N-1}, 2} < \infty$$

where $r_{\kappa,\lambda,2}$ *is again the radius of a circle with curvature* $\lambda := \frac{H}{N-1}$ *in a two-dimensional spaceform of constant curvature* $\kappa := \frac{K}{N-1}$.

Theorem (Extending Kasue's results to nonsmooth spaces)

(a) If $(X, d, m) \in MCP(K', N)$ for some $K' \in \mathbb{R}$ and $1 < N < \infty$ and if $\Omega \subset X$ open with $\partial\Omega \neq \emptyset$ satisfies $\Omega \in MCP_r(K, N)$ and $H_{\partial\Omega} \geq H \in \mathbb{R}$ with $\max\left\{K, H - \sqrt{(N-1)|K|}\right\} > 0$, then the inscribed radius of Ω

$$r(\Omega) \leq r_{K,H,N} := r_{\frac{K}{N-1}, \frac{H}{N-1}, 2} < \infty$$

where $r_{\kappa,\lambda,2}$ is again the radius of a circle with curvature $\lambda := \frac{H}{N-1}$ in a two-dimensional spaceform of constant curvature $\kappa := \frac{K}{N-1}$.

(b) (Rigidity) If also $(X, d, m) \in RCD(K, N)$ with $\kappa \in \{-1, 0, 1\}$ and Ω is connected, then $r(\Omega) = r_{K,H,N} \Leftrightarrow \Omega$ becomes isometric to the ball $B(o, r_{K,H,N})$ around the cone tip in some conical warped product $I \otimes_{\kappa}^{N-1} Y$ of an interval $I := [0, \pi_\kappa)$ with an $RCD(N-2, N-1)$ space (Y, d_Y, m_y), when both Ω and $B(o, r_{K,H,N})$ are equipped with their induced intrinsic distances. (Here Y is a single point when $N < 2$.)

• Kapovitch-Ketterer '20: $RCD(K, N) \subset CD(K, N)$ refers to spaces (X, d, m) for which m-a.e. tangent cone is isometric to Euclidean space

- Kapovitch-Ketterer '20: $RCD(K, N) \subset CD(K, N)$ refers to spaces (X, d, m) for which m-a.e. tangent cone is isometric to Euclidean space
- $I = [0, \pi_\kappa)$ where $\pi_\kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \leq 0 \end{cases}$
- set $\sin_\kappa(t) := \begin{cases} \sin(t) & \text{if } \kappa = 1, \\ t & \text{if } \kappa = 0, \\ \sinh(t) & \text{if } \kappa = -1 \end{cases}$
- the cone $I \otimes_\kappa^{N-1} Y$ refers to the product space $I \times Y$ endowed with measure $sin_\kappa^{N-1}(r)\mathrm{d}r \times \mathrm{d}m_Y$ and

• Kapovitch-Ketterer '20: $RCD(K,N) \subset CD(K,N)$ refers to spaces (X,d,m) for which m-a.e. tangent cone is isometric to Euclidean space

• $I = [0, \pi_\kappa)$ where $\pi_\kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \leq 0 \end{cases}$

• set $\sin_\kappa(t) := \begin{cases} \sin(t) & \text{if } \kappa = 1, \\ t & \text{if } \kappa = 0, \\ \sinh(t) & \text{if } \kappa = -1 \end{cases}$

• the cone $I \otimes_\kappa^{N-1} Y$ refers to the product space $I \times Y$ endowed with measure $sin_\kappa^{N-1}(r)\mathrm{d}r \times \mathrm{d}m_Y$ and the distance d given by

$$d((s,y),(t,z))^2 := s^2 + t^2 - 2st\cos(d_Y(y,z) \wedge \pi) \qquad \text{if } \kappa = 0$$

where $s \wedge t := \min\{s,t\}$

• Kapovitch-Ketterer '20: $RCD(K,N) \subset CD(K,N)$ refers to spaces (X,d,m) for which m-a.e. tangent cone is isometric to Euclidean space

• $I = [0, \pi_\kappa)$ where $\pi_\kappa := \begin{cases} \pi & \text{if } \kappa = 1, \\ \infty & \text{if } \kappa \le 0 \end{cases}$

• set $\sin_\kappa(t) := \begin{cases} \sin(t) & \text{if } \kappa = 1, \\ t & \text{if } \kappa = 0, \\ \sinh(t) & \text{if } \kappa = -1 \end{cases}$

• the cone $I \otimes_\kappa^{N-1} Y$ refers to the product space $I \times Y$ endowed with measure $sin_\kappa^{N-1}(r)\mathrm{d}r \times \mathrm{d}m_Y$ and the distance d given by

$$d((s,y),(t,z))^2 := s^2 + t^2 - 2st\cos(d_Y(y,z) \wedge \pi) \qquad \text{if } \kappa = 0$$

where $s \wedge t := \min\{s,t\}$ and otherwise

$$\sin_\kappa' d((s,y),(t,z)) = \sin_\kappa'(s)\sin_\kappa'(t) - \kappa \sin_\kappa(s)\sin_\kappa(t)\cos(d_Y(y,z) \wedge \pi)$$

• in each case the points $(0,y)$ and $(0,z)$ are identified (the 'cone tip')

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

$$\frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \qquad \forall s \leq 0.$$

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

$$\frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \qquad \forall s \leq 0.$$

- $H_{\partial\Omega}(x) \geq H = N - 1$ implies $\left.\frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds}\right|_{s=0^-} \geq \frac{N-1}{N-1} = 1$

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

$$\frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \qquad \forall s \leq 0.$$

- $H_{\partial\Omega}(x) \geq H = N-1$ implies $\left.\frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds}\right|_{s=0^-} \geq \frac{N-1}{N-1} = 1$
- concavity shows $h_{\tilde{x}}^{\frac{1}{N-1}}$ then becomes negative for $s < -1$

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

$$\frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \qquad \forall s \leq 0.$$

- $H_{\partial\Omega}(x) \geq H = N-1$ implies $\left.\frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds}\right|_{s=0^-} \geq \frac{N-1}{N-1} = 1$
- concavity shows $h_{\tilde{x}}^{\frac{1}{N-1}}$ then becomes negative for $s < -1$
- but $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0$, thus $s := d_{\Omega}^{\pm}(x) \geq -1$ meaning

Idea of proof (a)

Assume $(K, H) = (0, N-1)$ for simplicity, as for the unit ball in $\mathbb{R}^N$:

- on $\tilde{m}$-a.e. geodesic segment $\tilde{x}$ 'normal' to $\partial\Omega$: $\Omega \in CD_r(K, N)$ implies

$$\frac{d^2 h_{\tilde{x}}^{\frac{1}{N-1}}}{ds^2} \leq 0 \qquad \forall s \leq 0.$$

- $H_{\partial\Omega}(x) \geq H = N - 1$ implies $\left.\frac{d \log h_{\tilde{x}}^{\frac{1}{N-1}}}{ds}\right|_{s=0^-} \geq \frac{N-1}{N-1} = 1$
- concavity shows $h_{\tilde{x}}^{\frac{1}{N-1}}$ then becomes negative for $s < -1$
- but $dm_{\tilde{x}}(s) = h_{\tilde{x}}(s)ds \geq 0$, thus $s := d_{\Omega}^{\pm}(x) \geq -1$ meaning
- geodesics $\tilde{x}$ extending further than unit distance into Ω are $\tilde{m}$ negligible
- so in fact no such geodesic can exist.

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave volumetrically as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave volumetrically as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

- we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial\Omega) = 2$.
- we extend Laplacian comparison estimates to $RCD(K, N)$ spaces

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K, N)$ balls behave volumetrically as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

- we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial\Omega) = 2$.
- we extend Laplacian comparison estimates to $RCD(K, N)$ spaces
- where we can combine the strong maximum principle with properties of $\sin_\kappa$ to deduce $r(\Omega) = r_{K,H,N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in Ω depends on $r < r_{K,H,N}$ as it would for a ball in the model space
- (essential) nonbranchingness implies the inscribed ball cannot be disconnected by removing its center

Idea of proof (b)

Relies on a de Philippis and Gigli '16 result which asserts:

whenever two concentric $RCD(K,N)$ balls behave volumetrically as they would in a cone,
then the larger ball is either a metric cone or one-dimensional

• we use $N > 1$ to rule out one-dimensional spaces with $\#(\partial\Omega) = 2$.

• we extend Laplacian comparison estimates to $RCD(K,N)$ spaces

• where we can combine the strong maximum principle with properties of $\sin_\kappa$ to deduce $r(\Omega) = r_{K,H,N}$ implies the volume of (a certain connected component of the punctured) inscribed ball in Ω depends on $r < r_{K,H,N}$ as it would for a ball in the model space

• (essential) nonbranchingness implies the inscribed ball cannot be disconnected by removing its center

Thank you very much!