Causal differential calculus & d'Alembert comparison

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www.math.toronto.edu/mccann/Talk2.pdf

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Overview: Nonsmooth Lorentzian analogs of

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9. nonsmooth (timelike) splitting: extremizing line yields product geometry

(future work)

A nonsmooth framework for gravity

- replace Lorentz manifold (M, g_{ij}) of relativity with *metric spacetime* M (variant on Kunzinger-Sämann's '18 Lorentzian prelength spaces; also Minguzzi–Suhr's '24 bounded Lorentzian metric spaces, [M.24], Müeller)
- $\ell : M^2 \longrightarrow \{-\infty\} \cup [0, \infty)$ is called a *time-separation* function if

$$\ell(x,y) + \ell(y,z) \le \ell(x,z) \qquad \forall x,y,z \in M$$

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- ℓ defines the *transitive* relations *causality* $\le$ and *chronology* $\ll$ by:

$$\le \,:= \{\ell \ge 0\} \qquad \ll \,:= \{\ell > 0\}$$

$$\textit{future} \quad J^+(x) = \{y \in M \mid y \ge x\} \qquad I^+(x) := \{y \in M \mid y \gg x\}$$

$$\textit{past} \quad J^-(z) := \{y \in M \mid y \le z\} \qquad I^-(z) := \{y \in M \mid y \ll z\}$$

- assume $\ell(y,y) = 0$ $\forall y \in M$, so (the preorder) $\le$ is reflexive
- *chronological topology:* the coarsest topology with $I^\pm(y)$ open $\forall y \in M$

• a topology is called *Polish* if it has a complete, separable metrization

Definition (Metric spacetime; time-reversal)

A time-separation function $\ell : M^2 \longrightarrow \{-\infty\} \cup [0, \infty)$ as above makes (M, ℓ) a *metric spacetime* if the chronological topology it induces is Polish. The *time-reversal* (M, ℓ^*) of (M, ℓ) refers to $\ell^*(y, x) = \ell(x, y)$.

• metrizability implies $\leq$ is partial-order: i.e. $(x \leq z \;\&\; z \leq x) \Rightarrow (x = z)$

• $\leq$ is *forward-complete* $\Leftrightarrow$ $x_i \leq x_{i+1} \leq z (\forall i \in \mathbf{N})$ implies $\lim_{i\to\infty} x_i$ exists

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Definition (Forward spacetime — our standing assumption)

A metric spacetime (M, ℓ) (with its causal and chronological relations $\leq$ and $\ll$ and Polish chronological topology) is called *forward* if the partial order $\leq$ is forward-complete and ℓ is upper semicontinuous.

- write (M, ℓ) is *backward* $\Leftrightarrow$ its time-reversal (M, ℓ^*) is forward
- let $J^+(X) := \cup_{x \in X} J^+(x)$ and $J^-(Z) := \cup_{z \in Z} J^-(z)$

Definition (Emeralds)

An *emerald* refers to $J(X, Z) := J^+(X) \cap J^-(Z)$ with $X, Z \subset M$ compact.

Minguzzi: (M, ℓ) is called *globally hyperbolic* if every emerald is compact

Example (Manifolds)

Any smooth Lorentzian manifolds which admits a Cauchy surface is a forward spacetime (as is any globally hyperbolic Lorentzian length space).

Example (Manifolds with boundary)

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Example (Manifolds with boundary)

The closed interval $[-1, 1]$ with the time-separation

$$\ell(x, y) := \begin{cases} y - x & \text{if } y \geq x, \\ -\infty & \text{else,} \end{cases}$$

is a forward spacetime (but not a Lorentzian length space nor a manifold, whereas its open subset $(-1, 1)$ is globally hyperbolic as a Lorentzian manifold, hence also a Lorentzian length space and a forward spacetime).

Calculus of worldlines (i.e. nondecreasing curves)

Definition (Causal curve and speed; c.f. [A90] for (M, d))

$\sigma : [0,1] \longrightarrow M$ is *causal* $\Leftrightarrow \sigma_s := \sigma(s) \leq \sigma(t)$ for all $0 \leq s < t \leq 1$; (it is *timelike* $\Leftrightarrow$ we can replace $\leq$ above with $\ll$). Its *causal speed* refers to the (pointwise a.e.) limit on $(0,1)$

$$|\dot{\sigma}(s)| := \lim_{h \downarrow 0} \frac{\ell(\sigma_{s+h}, \sigma_s)}{h}$$

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- in a metric (resp. forward) spacetime, discontinuities of a causal curve σ are countable (and σ may be taken left-continuous without loss, resp.)
- the set $LCC([0, 1]; M)$ of Left-Continuous Causal curves metrized by

$$D(\sigma, \tau) := d(\sigma_0, \tau_0) + \int_0^1 d(\sigma_s, \tau_s) ds$$

is Polish, if d makes the chronological topology Polish on (M, ℓ)
- Limit-curve theorem: $C \subset M$ compact makes $LCC([0, 1]; C)$ D-compact

q-Lagrangian action and (rough) ℓ-geodesics

Definition (q-Lagrangian action, geodesics [Minguzzi19,M.20,MS23])

Given $0 \neq q < 1$, the *action* of a causal curve refers to

$$A_q[\sigma] := \frac{1}{q} \int_0^1 |\dot{\sigma}(s)|^q ds$$

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Causal curves *maximizing* this action (for given endpoints) are called *rough geodesics*; if $\sigma \in LCC([0,1]; M)$ then simply *geodesic*.

- recall twin paradox

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- maximizers are independent of q;
- the set of geodesics is denoted $CGeo(M)$;
- curves in $TGeo(M) := \{\sigma \in CGeo(M) \mid A_q[\sigma] \neq 0\}$ are called *timelike* or *ℓ-geodesics*.

Nonbranching conditions; characterizing geodesics

Lemma (Indpendence of q; affine parameterization)

A curve $\sigma : [0,1] \longrightarrow M$ is a rough (ℓ-)geodesic iff $\forall 0 \leq s < t \leq 1$,

$$\ell(\sigma(s),\sigma(t)) = (t-s)\ell(\sigma(0),\sigma(1)) \quad (>0).$$

Definition (Nonbranching conditions)

(a) A metric spacetime (M,ℓ) has *no endpoint branching* if any two ℓ-geodesics $\sigma, \tilde\sigma \in TGeo(M)$ that agree on $(0,1)$ also agree on $[0,1]$.
(b) The metric spacetime is called *timelike nonbranching* if

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(b) The metric spacetime is called *timelike nonbranching* if any two ℓ-geodesics $\sigma, \tilde{\sigma} \in TGeo(M)$ that agree on $(\frac{1}{3}, \frac{2}{3})$ also agree on $[0,1]$.

- If a forward spacetime (a) has no endpoint branching, any rough ℓ-geodesic is left-continuous (but not necessarily right-continous).

Fuzzy events: lifting the geometry from events to measures

Optimal transport on forward spacetimes:

$$\ell_q(\mu,\nu) := \sup_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q}$$

defines a time-separation (and a causal relation [EM17]) between Borel probability measures $\mu, \nu \in \mathcal{P}_{em}(M)$ on emeralds in M. Here

$$\Gamma_{\leq}(\mu,\nu) := \Big\{ \gamma \geq 0 \text{ on } M^2 \mid \gamma[\{\ell \geq 0\}] = 1, \quad \mu[Y] = \gamma[Y \times M] \quad \forall Y \subset M, \quad \gamma[M \times Y] = \nu[Y] \Big\}$$

- maximizers γ exist if $\Gamma_{\leq}(\mu,\nu) \neq \emptyset$ and are called q-optimal couplings
- the ℓ_q-speed along any causal curve $(\mu_s)_{s\in[0,1]}$ of measures is

$$|\dot{\mu}_s|_q := \lim_{h \downarrow 0} \frac{\ell_q(\mu_s, \mu_{s+h})}{h}$$

Tangent fields; lifting curves $(\mu_t)_t$ to measures π on curves

Definition (Rough ℓ_q-geodesics can be defined like rough ℓ-geodesics)

Given $0 \neq q < 1$, the action of a causal curve $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$ is

$$\mathcal{A}_q[\mu] := \frac{1}{q}\int_0^1 |\dot\mu_t|_q^q dt \leq \frac{1}{q}\ell_q(\mu_0,\mu_1)^q < \infty \text{ if } \mu_0,\mu_1 \in \mathcal{P}_{em}(M).$$

Define $e_t : LCC([0,1];M) \longrightarrow M$ by $e_t(\sigma) := \sigma(t)$.

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Theorem (Lifting curves of measures in forward spacetimes c.f.[Lis07])

Conversely, if $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$ *is causal, narrowly left-continuous on* $[0,1]$*, and tight on* $(\epsilon, 1-\epsilon)$ *(*$\forall \epsilon > 0$*) then it's induced by a plan* $\pi \in \mathcal{P}(LCC([0,1];M))$ *with expected action*

$$\int A_q[\sigma]d\pi(\sigma) = \mathcal{A}_q[\mu]$$

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$$\int A_q[\sigma]d\pi(\sigma) = \mathcal{A}_q[\mu] \quad (= \ell_q(\mu_0,\mu_1)^q/q \text{ if } \pi \text{ is ``}q\text{-optimal''})$$

Consequences in forward spacetimes

- these measures π on curves (i.e. 'plans') represent tangent fields

Corollary (Optimal plans concentrate on geodesics)

If $\pi \in \mathcal{P}(LCC([0,1]; M))$ *is q-optimal, then* $\pi[CGeo] = 1$.

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Corollary (Narrow forward-completeness in a forward spacetime)

If $\mu_i \leq \mu_{i+1} \leq \nu$ in $(\mathcal{P}(M), \ell_q)$, then $\lim_{i\to\infty} \mu_i$ converges narrowly.

- plays a crucial role in our eventual construction of 'good' *test* plans

q-dualizability and narrow continuity of rough ℓ_q-geodesics

Definition (Strict timelike q-dualizability; c.f. [M.20] [CM24])

The pair $\mu, \nu \in \mathcal{P}_{em}(M)$ is *strictly timelike q-dualizable* iff every q-optimal coupling $\gamma \in \Gamma_{\leq}(\mu, \nu)$ vanishes outside $\{\ell > 0\}$.

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Lemma (Narrow continuity of rough ℓ_q-geodesics)

If (M, ℓ) is a forward spacetime with no endpoint branching and $(\mu_t)_{t\in[0,1]}$ is a rough ℓ_q-geodesic with strictly timelike q-dualizable endpoints $\mu_0, \mu_1 \in \mathcal{P}_{em}(M)$, then $t \in [0,1] \mapsto \mu_t$ is narrowly continuous wherever it is locally tight.

- local tightness can come from e.g., global hyperbolicity or density bounds or narrow forward-completeness...

(Exact, future-directed) cotangent fields; their magnitudes

Definition (Causal functions (nondecreasing); form a convex cone)

$f : M \longrightarrow [-\infty, \infty]$ is *causal* $\Leftrightarrow$ $\ell(x, y) \geq 0$ implies $f(x) \leq f(y)$.

Definition (Metric-measure spacetimes; test plan; maximal subslope)

Fix a Radon measure m on (M, ℓ) assigning finite mass to each emerald.
A plan $\pi \in \mathcal{P}(LCC([0, 1]; M))$ is called (initially) *test* $\Leftrightarrow$

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$$f(\sigma_1) - f(\sigma_0) \geq \int_0^1 g(\sigma_t)|\dot{\sigma}_t|dt$$

for every test plan π and π-a.e. curve σ. They form a stable lattice. Each m-measurable causal f admits a *maximal weak subslope*, denoted $g = |df|$.

- this very general definition, c.f. [AGS14], good for integration-by-parts

Infinitesimal Minkowskianity

Lemma (Examples of weak subslopes; (TMCP $\Rightarrow$ equality))

Continuity of causal f *and* $\ell_+ = \max\{\ell, 0\}$ *imply m-a.e.* y *satisfies*

$$\liminf_{x \ll y} \frac{f(y) - f(x)}{\ell(x,y)} \leq |df(y)|, \qquad \liminf_{z \gg y} \frac{f(z) - f(y)}{\ell(y,z)} \leq |df(y)|.$$

Definition (c.f. infinitesimally Hilbertian [G15] rather than [AGS14d])

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$$|d(f+g)|^2 + |dg|^2 = 2|d(f+2g)|^2 + 2|df|^2 \qquad m\text{-a.e.}$$

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- equivalently, the following polarization is positively bilinear m-a.e.:

$$2((df, dg)) := |d(f+g)|^2 - |df|^2 - |dg|^2$$

- distinguishes Lorentz from Lorentz-Finsler metrics on e.g. $\mathbf{R}^n$ [BO24]

Convex analysis; *horizontal derivatives*; raising indices

Just as causal curves and functions on a smooth Lorentz manifold satisfy

$$\langle df, \dot\sigma\rangle \geq \frac{1}{p}\|df\|_*^p + \frac{1}{q}\|\dot\sigma\|^q \qquad \text{when } p^{-1} + q^{-1} = 1$$

with equality iff $\langle \dot\sigma, \cdot\rangle = \|df\|_*^{p-2} df(\cdot)$, i.e. iff $\dot\sigma = \|\nabla f\|^{p-2}\nabla f$ [M.20],

Theorem (Nonsmooth Fenchel-Young inequality for $0 < q < 1$)

If $(e_s)_\# \pi \to (e_0)_\# \pi$ narrowly, $|df|^p \in L^1(m)$, and π initially test then

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- limit on left called *horizontal* (*inner*, *Lagrangian*) derivative of f along π
- aims at bilinear pairing of π with f; (NB concave p-Dirichlet energy of f)

Definition (Identified tangent and cotangent fields; optimal transport)

If *$\lim_{s\downarrow 0}$* exists and equality holds, we say π *represents the p-gradient* of f.
A nonlinear duality between some tangent and cotangent fields (π and f)

Perturbation & variational derivative of p-Dirichlet energy

• given m-measurable $E \subset M$, write $g \in Pert_p(f, E)$ if for all $\epsilon > 0$ small enough, $f + \epsilon g$ is causal and $|d(f + \epsilon g)|^p \in L^1(E, dm)$.

Theorem (Horizontal dominates vertical derivative; c.f. [G15])

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- last is direction g *vertical* (/ *outer* / *Eulerian*) derivative of p-energy at f
- nonlinear in f but becomes linear in g if two-sided limit in ϵ exists

Corollary (If (M, ℓ, m) is infinitesimally Minkowskian)

and if $-g, g \in Pert_p(f, E)$ then $\lim_{\epsilon \to 0}$ and $\lim_{s \downarrow 0}$ exist & equality holds above!

Curvature bounds via entropy

Given $N \in (1, \infty)$, define N-Renyi (or Boltzmann) entropy of $\mu \in \mathcal{P}(M)$ by

$$S_N(\mu) := -N \int_M [(\frac{d\mu}{dm})^{-1/N} - 1] d\mu \quad (\text{and } S_\infty(\mu) := \lim_{N\to\infty} S_N(\mu))$$

• in the smooth globally hyperbolic setting, convexity properties of $t \in [0,1] \mapsto S_N(\mu_t)$ along ℓ_q-geodesics (or of $S_\infty(\mu_t)$) are well-known to characterize timelike lower Ricci curvature bounds [B23] [MS23] [M.20]; c.f. [RS04][CMS01][OV00][M.94] (or [EKS15])

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$TMCP^\pm$ (or $TMCP_e^+$): a poor man's lower Ricci curvature bounds

• we impose only sublinearity of $S_N(\mu_t)$ only along ℓ_q-geodesics starting or ending at a Dirac point mass — the *timelike measure contraction properties* $TMCP^\pm$ of [B23]; c.f. [CM24] [LV09] [O07] [S06]

• if (μ_0, δ_z) are strictly timelike q-dualizable precisely one ℓ_q-geodesic links μ_0 to δ_z; moreoever $S_N(\delta_z) = 0$ (whereas $S_\infty(\delta_z) := +\infty$.)

A poorer cousin to timelike lower Ricci curvature bounds

Definition (Future timelike measure contraction property; c.f. [B23])

For $K \in \mathbf{R}$ write $(M, \ell, m) \in TMCP^+(K, N)$ if $\forall \mu_0 \in \mathcal{P}_{em}(M) \cap L^1(m)$ and each $z \in \operatorname{spt} m$ with $\mu_0[I^-(z)] = 1$, for some (hence all) $0 \neq q < 1$, there exists a (rough) ℓ_q-geodesic from μ_0 to $\mu_1 := \delta_z$ such that all $t \in [0,1]$ and $N' \geq N$ satisfy

$$S_{N'}(\mu_t) \leq -\int \tau_{K,N}^{(1-t)}(\ell(x,z)) \frac{d\mu_0}{dm}(x)^{1-1/N'} dm(x).$$

Past version: $(M, \ell, m) \in TMCP^-(K, N) \Leftrightarrow (M, \ell^*, m) \in TMCP^+(K, N)$.

- $\tau_{0,N}^{(1-t)}(\ell) := 1 - t$ for $K = 0$;

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- $\tau_{0,N}^{(1-t)}(\ell) := 1 - t$ for $K = 0$; asserts sublinearity of $t \in [0, 1] \mapsto S_{N'}(\mu_t)$, and follows from the strong energy condition, a case of primary interest
- a smooth globally hyperbolic Lorentzian manifold M^n satisfies $TMCP^{\pm}(K, N)$ if $n \leq N$ and $Ric(v, v) \geq Kg(v, v)$ for all timelike $v \in TM$

Test plans: finding ℓ_q-geodesics having density bounds

Theorem (Initial test plans with Dirac targets; c.f. [B23][CM17][R13])

Fix ($K \in \mathbf{R}$ or) $K = 0 \neq q < 1 < N < \infty$, a forward spacetime $(M, \ell, m) \in (TMCP^+ \cap TMCP_e^+)(K, N)$ with no endpoint branching and $z \in M$. If $\mu_0[I^-(z)] = 1$ for $\mu_0 \in L^\infty(m) \cap \mathcal{P}_{em}(M)$ then there exists a q-optimal plan π inducing (an ℓ_q-geodesic) $\mu_t := (e_t)_\# \pi$ from μ_0 to $\mu_1 := \delta_z$ such that $t \in [0,1] \mapsto S_{N'}(\mu_t)$ is (suitably) sublinear for each $N' \geq N$ and

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- $c_{0,N,\ell} = 1$ if $K = 0$ (else $c_{K,N,\ell} := \exp(t\|\ell\|_{L^\infty(\mu_0 \times \mu_1)}\sqrt{K_-(N-1)})$)
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- extends to non-Dirac targets provided

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- Boltzmann version $TMCP_e^+$ can be replaced by global hyperbolicity
- extends to non-Dirac targets provided (μ_0, μ_1) strictly timelike q-dualizable and (M, ℓ, m) is (q-essentially) timelike nonbranching,

COROLLARY (Busemann and Lorentz distance functions have unit slope)
$g(\cdot) = -\ell(\cdot, z)$ satisfies $|dg| = 1$ m-a.e. on $I^-(z)$

When is the p-gradient of f represented by a test plan π?

$$f^{(q)}(z) := \sup_{x \in I^-(z)} f(x) + \frac{\ell(x,z)^q}{q} \qquad g_q(x) := \inf_{z \in I^+(x)}$$

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- then f is causal, upper semicontinuous, and $\partial_{\ell^q/q} f$ relatively closed in $\ll$

$$\partial_{\ell^q/q} f := \{x \ll z \mid f^{(q)}(z) = f(x) + \tfrac{\ell(x,z)^q}{q} \in \mathbf{R}\} \subset M^2, \text{ if } \ell_+ \in C(M)$$

Theorem (A metric Brenier-M. thm; cf.[CM24][MS23][M.20][AGS14])

Fix $0 \neq q < 1$ and $p^{-1} + q^{-1} = 1$. Let (M, ℓ, m) be forward, ℓ_+ continuous and $f = (f^{(q)})_q$.

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$$|df|(\sigma_0) = \ell(\sigma_0, \sigma_1)^{q-1}.]$$

$$\mathrm{dom}\,\partial_{\ell^q/q} f := \{x \in M \mid \partial_{\ell^q/q} f \cap (\{x\} \times M) \neq \emptyset\}$$

Theorem (d'Alembert comparison theorem: $\Box_p f \leq N$ if $K = 0$)

Fix $0 \neq q < 1 = p^{-1} + q^{-1} < N < \infty$, *a forward spacetime* $(M, \ell, m) \in TMCP^{+}_{(e)}(K, N)$ *with no endpoint branching*, $\ell_+ \in C(M)$, $K \in \mathbf{R}$ *and* $f = (f^{(q)})_q$. *Let* (M, ℓ, m) *be (q-essential) timelike nonbranching unless* $\exists z \in M$ *with*

$$f(x) = \begin{cases} -\ell(x,z)^q/q & \forall x \in I^-(z), \\ +\infty & \text{else.} \end{cases}$$

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If $0 \leq \phi \in Pert_p(f) \cap L^\infty$, *compact support and* $m[\mathrm{spt}\,\phi \setminus \mathrm{dom}\,\partial_{\ell^q/q} f] = 0$ *then*

$$\int_M d^+\phi(\nabla f)|df|^{p-2} dm \leq \int_M \tilde{\tau}_{K,N}(|df|^{p-1})\phi\, dm$$

$$\tilde{\tau}_{K,N}(r) := N \frac{\partial \tau^t_{K,N}(r)}{\partial t|_{t=1}} = \begin{cases} N & \text{if } K = 0 \\ 1 + r\sqrt{(N-1)|K|}\cot(r\sqrt{\frac{K}{N-1}}) & \text{else.} \end{cases}$$

Corollary

Same nonsmooth sense and setting with e.g. $K = 0$, a chain rule yields

$$\Box_p(-\ell(\cdot, z)) \leq \frac{N-1}{\ell(\cdot, z)} \qquad \text{on } I^-(z)$$

• Analogous results also hold true in backward spacetimes and $K \neq 0$. After time-reversing them, the forward $(M, \ell, m) \in TMCP^-(K, N)$ satisfies

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• [BGMOS25+] extends his timelike splitting theorem to $g_{ij} \in C^1(M)$

Defining the p-d'Alembertian

- thus even on smooth globally hyperbolic manifolds we obtain new results

- functional analysis: $\Box_q f$ is a measure, nonunique unless infinitesimally Minkowskian, $TMCP^{\pm}_{(e)}(K, N)$, and $Pert_p(f, E)$ is dense; c.f. [G15]

- localization: [B24+] establishes many fundamental properties of $\Box_p f$ by developing an approach based on needle decompositions; c.f. [CM20]

Selected references (apologies for omissions/oversights):

Beran-Braun-Calisto-Gigli-M.-Ohanyan-Rott-Sämann arXiv:2408.15968 Octet

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[A90] Ambrosio (Ann SNS Pisa Cl Sci)

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[B23,B23,B24+] Braun (Nonlinear Analysis, JMPA, arXiv:2408.16525)

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[EKS15] Erbar–Kuwada–Sturm (Inventiones)

[EM17] Eckstein–Miller (AIHP)

[G04,G15,G24+] Gigli (PhD, MAMS, forthcoming)

[Lis07] Lisini (CVPDE)

[KS18] Kunzinger–Sämann (Ann Global Anal Geom)

[M.94,M.20,M.24] M. (PhD, Cambridge J Math, CMP)

[MS23] Mondino–Suhr (J Euro Math Soc)

[O07,O14] Ohta (CM Helvetica, Anal Geom Metric Spaces)

[OV00,LV09] Otto–Villani (J. Funct. Anal.), Lott–Villani (Annals)

[R12,R13] Rajala (JFA, DCDS)

[RS04,S06] von Renesse–Sturm (CPAM), Sturm (Acta)

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THANK YOU!