

Maximizing angle sums between Euclidean lines

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Find a globally stable configuration of k charges on projective sphere \mathbf{RP}^n , which repel each other pairwise with a force independent of distance.

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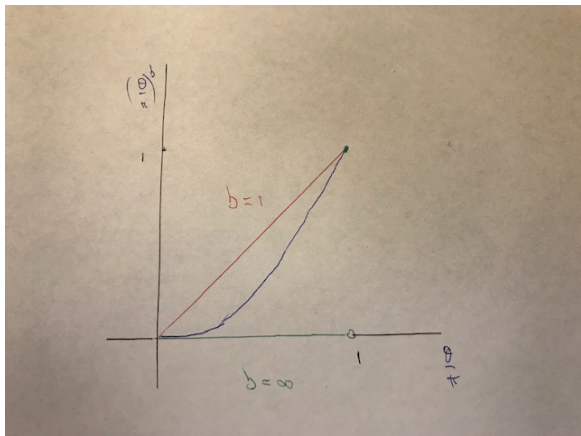
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- [Fejes Tóth conjecture \(1959\)](#): Distribute the lines / charges as evenly as possible over an orthonormal basis of \mathbf{R}^{n+1}
- obvious for $k \leq n + 1$; also resolved affirmatively for all k with $n = 1$ but remains open otherwise
- for $n = 1$ and $k > 2$ there are many inequivalent maximizers as well (some of which accumulate to the uniform distribution)

To make progress, let the force increase with a power $b - 1 > 0$ of the distance, so we maximize $\sum \left(\frac{\theta}{\pi}\right)^b$ instead



Identifying $\pm x \in \mathbf{S}^n$ yields $\mathbf{RP}^n = \frac{2}{\pi} \mathbf{S}^n / \{+, -\}$ scaled to diameter 1. Let

$$d_{\mathbf{RP}^n}(x, y) = \frac{2}{\pi} \min\{d_{\mathbf{S}^n}(x, y), \pi - d_{\mathbf{S}^n}(x, y)\}$$

$$\mathcal{P}(\mathbf{RP}^n) = \{0 \leq \mu \text{ on } \mathbf{RP}^n \mid \int_{\mathbf{RP}^n} d\mu = 1\}$$

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$$\mathcal{P}_k^{\overline{}}(\mathbf{RP}^n) = \{\mu \in \mathcal{P}(\mathbf{RP}^n) \mid \mu = \frac{1}{k} \sum_{i=1}^k \delta_{x_i}, \quad x_i \in \mathbf{RP}^n\}$$

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Fejes Tóth's problem becomes the case $b = 1$ of

$$\max_{\mu \in \mathcal{P}_k^{\overline{}}(\mathbf{RP}^n)} E_b(\mu)$$

where

$$E_b(\mu) = \frac{1}{2} \iint d_{\mathbf{RP}^n}(x, y)^b d\mu(x) d\mu(y)$$

F.T. Conjecture: standard basis $\{\hat{e}_1, \dots, \hat{e}_{d+1}\}$ (i.e. maximal projective simplex) $\bar{\mu}_k = \frac{1}{k} \sum_{i=1}^k \delta_{\hat{e}_{(k \bmod d+1)}}$ achieves maximum for all $b \in [1, \infty]$.

Theorem (Discrete threshold for simplex maxima)

(a) For $k > n + 1$, there exists $b_{\Delta^n}(k) \in [1, \infty)$ such that $\bar{\mu}_k$ maximizes $E_b(\mu)$ on $\mathcal{P}_k^=(\mathbf{RP}^n)$ if and only if $b \geq b_{\Delta^n}(k)$.

(b) $\bar{\mu}_k$ maximizes uniquely up to rotations if and **only if** $b > b_{\Delta^n}(k)$.

RMK: Fejes Tóth conjecture $\iff b_{\Delta^n}(k) = 1$ for all $k > n + 1$

Note $2E_b(\bar{\mu}) = \frac{n}{n+1}$ for $\bar{\mu} = \bar{\mu}_k$ when $n + 1$ divides k .

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Theorem (Continuum threshold for simplex maxima)

- If $n + 1$ divides k then there exists $b_{\Delta^n} \in [1, \infty)$ such that
- (c) $\bar{\mu} = \mu_k$ maximizes E_b over the full set $\mathcal{P}(\mathbf{RP}^n)$ iff $b \geq b_{\Delta^n}$,
- (d) $\bar{\mu}$ maximizes **uniquely** on $\mathcal{P}(\mathbf{RP}^n)$ up to rotations iff $b > b_{\Delta^n}$

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- (e) $b_{\Delta^n} < 2$

RMK: (e) improves $b_{\Delta^n} \leq 2$ (Bilyk, Glazyrin, Matzke, Park, Vlasiuk)

$b = 2$ is the threshold for *mild repulsion* in attractive-repulsive models

ASIDE: a different but related energy optimization

Balagué, Carrillo, Laurent, Raoul '13; Attractive-repulsive potentials

If

$$\mu \in \arg \min_{\mu \in \mathcal{P}(\mathbf{R}^n)} E_{a,b}(\mu)$$

for $b \in (2 - n, a)$ where

$$E_{a,b}(\mu) := \iint_{\mathbf{R}^n \times \mathbf{R}^n} \left(\frac{|x|^a}{a} - \frac{|x|^b}{b} \right) d\mu(x) d\mu(y)$$

then $\text{spt } \mu$ has Hausdorff dimension $\geq 2 - b$

e.g. $a = 2$

A proof sketch of Balagué et al:

Setting $W_{a,b}(x) := \frac{|x|^a}{a} - \frac{|x|^b}{b}$, the Euler-Lagrange equation asserts a minimizer μ vanishes outside

$$\arg \min(W_{a,b} * \mu) \subset \mathbf{R}^n,$$

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so

$$\begin{aligned} (b-n+2) \iint_{\mathbf{R}^n \times \mathbf{R}^n} |x|^{b-2} d\mu(x) d\mu(y) &\leq \int_{\mathbf{R}^n} \Delta_{\mathbf{R}^n} \left(\frac{|x|^b}{b} * \mu \right) d\mu \\ &= \int_{\mathbf{R}^n} \Delta_{\mathbf{R}^n} \left(\frac{|x|^a}{a} * \mu \right) d\mu \\ &< \infty \end{aligned}$$

But **finiteness of this singular integral** is known to imply μ cannot be supported on any set of Hausdorff dimension less than $2 - b$. QED

Back to our main results:

Theorem (Discrete threshold for simplex maxima)

- (a) For $k > n + 1$, there exists $b_{\Delta^n}(k) \in [1, \infty)$ such that $\bar{\mu}_k$ maximizes $E_b(\mu)$ on $\mathcal{P}_k^{\neq}(\mathbf{RP}^n)$ if and only if $b \geq b_{\Delta^n}(k)$.
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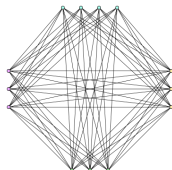
Proof (apart from 'only if'):

- $0 \leq d_{\mathbf{RP}^n}(x, y) \leq 1$ with equality iff $x = y$ or $x \perp y$
- $d_{\mathbf{RP}^n}(x, y)^b \geq d_{\mathbf{RP}^n}(x, y)^{b+\epsilon}$ with the same conditions for equality
- $(\text{spt } \hat{\mu}_k) = \{\hat{e}_1, \dots, \hat{e}_{n+1}\}^2$ lies in the equality set
- if $\hat{\mu}_k$ maximizes E_b for some b , it also maximized for all $b + \epsilon > b$
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- and every measure not supported in the equality set does strictly worse
- nonunique $n = b = 1 = \frac{k}{3}$ maximizer implies $b_{\Delta^n}(k) \geq 1$ if $k > n + 1$

- we argued $b_{\Delta^n}(k) < \infty$ in an earlier work, simplified by Bilyk et al, who also observed $k < \infty$ follows as a consequence of Turan's (1941) theorem, which identifies the graph $T(k, n+1) = K_{\lceil \frac{k}{n+1} \rceil, \dots, \lceil \frac{k}{n+1} \rceil, \lfloor \frac{k}{n+1} \rfloor, \dots, \lfloor \frac{k}{n+1} \rfloor}$ free of $(n+2)$ -cliques that has the maximal number of edges; see e.g. $T(13, 4)$ (image from Wikipedia)



- it remains to show 'only if', i.e. $\operatorname{argmax} E_\infty \subset \operatorname{argmax} E_{b_{\Delta^n}(k)}$ strictly

'only if' follows from a local optimality result $\forall b > 1$:

OPTIMAL TRANSPORT:

L^p -Kantorovich-Rubinstein-Wasserstein distance d_p on $\mu, \nu \in \mathcal{P}(\mathbf{RP}^n)$

$$d_p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \|d_{\mathbf{RP}^n}(X, Y)\|_{L^p} \quad p \in [1, \infty]$$

$$= \min_{0 \leq \gamma \in \Gamma(\mu, \nu)} \left(\iint_{\mathbf{RP}^n \times \mathbf{RP}^n} d_{\mathbf{RP}^n}(x, y)^p d\gamma(x, y) \right)^{1/p} \quad p \in [1, \infty)$$

- $p < \infty$: metrizes weak-* convergence of measures
- $d_\infty = \lim_{p \rightarrow \infty} d_p$ metrizes a much finer topology
- in this finer topology, $E_b(\mu)$ can have more local maxima
- c.f. McCann (PhD 1994, HJM 2006) stable binary stars

Weighted maximal projective simplices

For $0 \leq m_1 \leq m_2 \leq \dots \leq m_{n+1}$ with $\sum_{i=1}^{n+1} m_i = 1$ set $\vec{m} = (m_1, \dots, m_{n+1})$

$$\mu_{\vec{m}} = \sum_{i=1}^{n+1} m_i \delta_{\hat{e}_i}$$

Theorem (Weighted simplices are strict d_∞ -locally maxima $\forall b > 1$)

Given $\epsilon > 0$ and $n \in \mathbf{N}$ there exists $r = r_n(\epsilon)$ such that if $b \geq 1 + \epsilon$, $m_1 \geq \epsilon$ and $\mu \in \mathcal{P}(\mathbf{RP}^n)$ satisfy $d_\infty(\mu, \mu_{\vec{m}}) < r$ then $E_b(\mu) \leq E_b(\mu_{\vec{m}})$ and the inequality is strict unless μ is a rotate of $\mu_{\vec{m}}$

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- crucial for proving $\operatorname{argmax} E_\infty \subset \operatorname{argmax} E_{b_{\Delta^n}(k)}$ **strictly** (i.e. 'only if')
- e.g. $\mu_\epsilon \in \operatorname{argmax}_{\mathcal{P}_k^{\neq}(\mathbf{RP}^n)} E_{b_{\Delta^n}(k)-\epsilon}$ d_2 -accumulates to $\mu_0 \in \operatorname{argmax}_{\mathcal{P}_k^{\neq}(\mathbf{RP}^n)} E_{b_{\Delta^n}(k)}$
- on the discrete measures $\mathcal{P}_k^{\neq}(\mathbf{RP}^n)$, d_∞ gives the **same** topology as d_2

- On the full space $\mathcal{P}(\mathbf{R}^n)$, the argument is more subtle: if $d_2(\mu_\epsilon, \bar{\mu}) \rightarrow 0$ the Euler-Lagrange equation satisfied by μ_ϵ implies $d_\infty(\mu_\epsilon, \mu_{\vec{m}}) \rightarrow 0$ for *spt* $\mu_{\vec{m}}$ an approximately maximal simplex with approximately uniform weights $\vec{m} = (m_1, \dots, m_{n+1})$; energetic maximality of μ_ϵ implies both approximations are exact

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- The d_∞ -local maximizer theorem relies on the following estimate, which follows approximately from control of the ℓ_2 norm on \mathbf{R}^n by the ℓ_1 norm

Lemma

Given $C \in (0, 1)$ there exists $r = r_n(C) > 0$ such that $d_\infty(\nu_i, \delta_{\hat{e}_i}) < r$ for all $i = 1, \dots, n$ implies $\exists \bar{x} = \bar{x}(\nu_1, \dots, \nu_n)$ such that

$$\sum_{i=1}^n \int (1 - d_{\mathbf{RP}^n}(x, y)) d\nu_i(y) \geq C d_{\mathbf{RP}^n}(x, \bar{x}) \quad \text{whenever } d_{\mathbf{RP}^n}(x, x_{n+1}) < r$$

For $b > 1$, if $d_\infty(\mu, \mu_{\vec{m}}) < r$, this can be used to show we gain more energy than we lose by collapsing any mass of μ near \hat{e}_{n+1} to \bar{x}

Further evidence for Fejes Tóth's conjecture?

Corollary (Discontinuous bifurcation unless $b_{\Delta^n} = 1$)

Fix $n \in \mathbf{N}$. No curve $(\mu_b)_{b>0}$ of optimizers

$$\mu_b \in \operatorname{argmax}_{\mathcal{P}(\mathbf{RP}^n)} E_b$$

can be d_∞ -continuous at $b = b_{\Delta^n}$ except possibly if $b_{\Delta^n} = 1$.

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Corollary (On connectedness of the threshold set of optimizers)

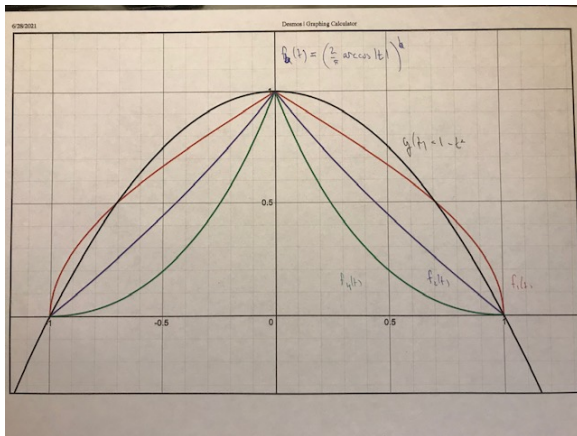
Fix $n \in \mathbf{N}$. If the set of threshold optimizers

$$\operatorname{argmax}_{\mathcal{P}(\mathbf{RP}^n)} E_{b_{\Delta^n}}$$

forms a d_∞ -connected subset of $\mathcal{P}(\mathbf{RP}^n)$, then $b_{\Delta^n} = 1$.

Bilyk-Glazyrin-Matzke-Park-Vlasiuk bound: $b_{\Delta^n} \leq 2$

Set $f_b(t) := \left(\frac{2}{\pi} \arccos(t)\right)^b$ and $g(t) := 1 - t^2$



Proof: $f_b(t) := (\frac{2}{\pi} \arccos(t))^b$ yields

$$E_b(\mu) = \iint_{\mathbf{RP}^n \times \mathbf{RP}^n} f_b(x \cdot y) d\mu(x) d\mu(y) =: F^{f_b}(\mu)$$

For $b \geq 2$ we have $f_b(t) \leq g(t) := 1 - t^2$ with equality iff $t \in \{-1, 0, 1\}$.

Thus

$$E_2(\mu) \leq F^g(\mu) \leq F^g(\sigma) = F^g(\bar{\mu}) = E_2(\bar{\mu})$$

where $\sigma = \text{normalized surface area}$ and $\bar{\mu} = \frac{1}{n+1} \sum \delta_{\hat{e}_i}$. The first inequality is **strict** unless $\mu = \mu_{\vec{m}}$ for some $\vec{m} = (m_1, \dots, m_{n+1})$.

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Here the **blue (in)equalities** are known and follow from Cauchy-Schwartz

$$\iint_{\mathbf{S}^n \times \mathbf{S}^n} (x \cdot y)^2 d\mu(x) d\mu(y) = \text{Tr}[I(\mu)^2] \geq \frac{1}{n+1} (\text{Tr } I(\mu))^2 = \frac{1}{n+1}$$

for the moment of inertia tensor

$$I(\mu) = (I_{ij}(\mu))_{i,j=1}^{n+1} \quad I_{ij}(\mu) = \int_{\mathbf{R}^n} x_i x_j d\mu(x).$$

QED

Thank you!