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A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities

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Abstract

We show that mass transportation methods provide an elementary and powerful approach to the study of certain functional inequalities with a geometric content, like sharp Sobolev or Gagliardo–Nirenberg inequalities. The Euclidean structure of $\mathbb{R}^n$ plays no role in our approach: we establish these inequalities, together with cases of equality, for an arbitrary norm.

1. Introduction

The goal of the present paper is to discuss a new approach for the study of certain geometric functional inequalities, namely Sobolev and Gagliardo–Nirenberg inequalities with *sharp* constants. More precisely, we wish to

(a) give a unified and elementary treatment of sharp Sobolev and Gagliardo–Nirenberg inequalities (within a certain range of exponents);
(b) illustrate the efficiency of mass transportation techniques for the study of such inequalities, and by this method reveal in a more explicit manner their geometrical nature;

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(c) show that the treatment of these sharp Sobolev-type inequalities does not even require the Euclidean structure of $\mathbb{R}^n$, but can be performed for arbitrary norms on $\mathbb{R}^n$;
(d) exhibit a new duality for these problems;
(e) as a by-product of our method, determine all cases of equality in the sharp Sobolev inequalities.

Before we go further and explain these various points, a little bit of notation and background should be introduced. Whenever $n \geqslant 1$ is an integer and $p \geqslant 1$ is a real number, define the Sobolev space

$$W^{1,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n);\ \nabla f \in L^p(\mathbb{R}^n) \}.$$

Here $L^p(\mathbb{R}^n)$ is the usual Lebesgue space of order p, and ∇ stands for the gradient operator, acting on the distribution space $\mathscr{D}'(\mathbb{R}^n)$. When $p \in [1, n)$, define

$$p^\star = \frac{np}{n-p}. \tag{1}$$

Then the (critical) Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$ asserts the existence of a positive constant $S_n(p)$ such that for every $f \in W^{1,p}(\mathbb{R}^n)$

$$||f||_{L^{p^\star}} \leqslant S_n(p) \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p}, \tag{2}$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. For the great majority of applications, it is not necessary to know more about the Sobolev embedding, apart maybe from explicit bounds on $S_n(p)$. However, in some circumstances one is interested in the exact value of the smallest admissible constant $S_n(p)$ in (2). There are usually two possible motivations for this: either because it provides some geometrical insights (as we recall below, a sharp version of (2) when $p = 1$ is equivalent to the Euclidean isoperimetric inequality), or for the computation of the ground-state energy in a physical model. Most often, the determination of $S_n(p)$ is in fact not as important as the identification of *extremal functions* in (2).

Similar problems have been studied at length in the literature for very many variants of (2): one example discussed by Del Pino and Dolbeault, which we also consider here, is the Gagliardo–Nirenberg inequality:

$$||f||_{L^r} \leqslant G_n(p, r, s) ||\nabla f||_{L^p}^{\theta} ||f||_{L^s}^{1-\theta}, \tag{3}$$

where $n \geqslant 2$, $p \in (1, n)$, $s < r \leqslant p^\star$, and $\theta = \theta(n, p, r, s) \in (0, 1)$ is determined by scaling invariance. Note that inequality (3) can be deduced from (2) with the help of Hölder's inequality.

The identification of the best constant $S_n(p)$ in (2) for $p > 1$ goes back to Aubin [2] and Talenti [30]. The proofs by Aubin and Talenti rely on rather standard techniques (symmetrization, solution of a particular one-dimensional problem). For $p = 1$, it

has been known for a very long time that (2) is equivalent to the classical Euclidean isoperimetric inequality which asserts that, among Borel sets in $\mathbb{R}^n$ with given volume, Euclidean balls have minimal surface area (see [28,29] for references about this problem). Also the case $p = 2$ is particular, due to its conformal invariance, as exploited in Beckner [5]. In Lieb [21], this case was derived by (rather technical) rearrangement arguments. Carlen and Loss have pointed out the crucial role of "competing symmetries" in this problem and used it to give a simpler proof [11], reproduced in [22]. Recently, Lutwak et al. [23] and Zhang [32] combined the co-area formula and a generalized version of the Petty projection inequality (related to the new concept of affine L^p surface area) to obtain an affine version of the Sobolev inequalities, which implies the Euclidean version (2).

Considerable effort has been spent recently on the problem of optimal Sobolev inequalities on Riemannian manifolds, see the survey [17] and references therein. In the present work however, we shall concentrate on the situation where the problem is set on $\mathbb{R}^n$. We do not know whether our methods would still be as efficient in a Riemannian setting. Note however that nonsharp Sobolev Riemannian inequalities can easily be derived by mass transportation techniques, as shown in [12].

For inequality (3), the computation of sharp constants $G_n(p, r, s)$ is still an open problem in general. Very recently, Del Pino and Dolbeault [15,16] made the following breakthrough: they obtained sharp forms of (3) in the case of the one-parameter family of exponents:

$$\begin{cases} p(s-1) = r(p-1) & \text{when } r, s > p, \\ p(r-1) = s(p-1) & \text{when } r, s < p. \end{cases} \tag{4}$$

Inequality (2) is actually a limit case of (3) when $r = p^\star$ (in which case $\theta = 1$). Note that an L^p version of the usual logarithmic Sobolev inequality also arises as a limit case of (3) when $r = s = p$ (see [16]; the usual inequality would be $p = 2$).

The proofs by Del Pino and Dolbeault for (3) rely on quite sophisticated results from calculus of variations, including uniqueness results for nonnegative radially symmetric solutions of certain nonlinear elliptic or p-Laplace equations. This work by Del Pino and Dolbeault has been the starting point of our investigation. We shall show in the present work how their results can be recovered (also in sharp form) by completely different methods.

Unlike the above-mentioned approaches, our arguments do not rely on conformal invariance or symmetrization, nor on Euler–Lagrange partial differential equations for related variational problems. Instead, we shall use the tools of mass transportation, which combine analysis and geometry in a very elegant way. Let us briefly recall some relevant facts from the theory of mass transportation. If μ and ν are two nonnegative Borel measures on $\mathbb{R}^n$ with same total mass (say 1), then a Borel map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to *push-forward* (or *transport*) μ onto ν if, whenever B is a Borel subset of $\mathbb{R}^n$, one has

$$\nu[B] = \mu[T^{-1}(B)], \tag{5}$$

or equivalently, for every nonnegative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$,

$$\int b(y)\, dv(y) = \int b(T(x))\, d\mu(x). \tag{6}$$

The central ingredient in our proofs is the following result of Brenier [6], refined by McCann [25]:

Theorem 1. *If μ and ν are two probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to Lebesgue measure, then there exists a convex function φ such that $\nabla\varphi$ transports μ onto ν. Furthermore, $\nabla\varphi$ is uniquely determined $d\mu$ almost everywhere.*

Observe that φ is differentiable almost everywhere on its domain since it is convex; in particular, it is differentiable $d\mu$ almost everywhere. The (monotone) map $T = \nabla\varphi$ will be referred to as the *Brenier map*. By construction, it is known to solve the Monge–Kantorovich minimization problem with quadratic cost between μ and ν, but here we shall not need this optimality property explicitly. See [31] for a review, and discussion of existing proofs.

From now on, we assume that μ and ν are absolutely continuous, with respective densities F and G. Then (6) takes the form

$$\int b(y)G(y)\, dy = \int b(\nabla\varphi(x))F(x)\, dx, \tag{7}$$

for every nonnegative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$. If φ is of class C^2, the change of variables $y = \nabla\varphi(x)$ in (7) shows that φ solves the *Monge–Ampère equation*

$$F(x) = G(\nabla\varphi(x)) \det D^2\varphi(x). \tag{8}$$

Here $D^2\varphi(x)$ stands for the Hessian matrix of φ at point x. Caffarelli's deep regularity theory [8–10] asserts the validity of (8) in classical sense when F and G are Hölder-continuous and strictly positive on their respective supports and G has convex support. In the present paper, we shall use a much simpler measure-theoretical observation, due to McCann [26, Remark 4.5] which asserts the validity of (8) in the $F(x)\, dx$ almost everywhere sense, without further assumptions on F and G beyond integrability. In Eq. (8), $D^2\varphi$ should then be interpreted in Aleksandrov sense, i.e. as the absolutely continuous part of the distributional Hessian of the convex function φ. Of course, $D^2\varphi$ is only defined almost everywhere. An alternative, equivalent way of defining $D^2\varphi$ is to note (see [18]) that a convex function φ admits almost everywhere a second-order Taylor expansion

$$\varphi(x+h) = \varphi(x) + \nabla\varphi(x)\cdot h + \tfrac{1}{2} D^2\varphi(x)(h)\cdot h + o(|h|^2).$$

Where defined, the matrix $D^2\varphi$ is symmetric and nonnegative, since φ is convex.

Mass transportation (or *parameterization*) techniques have been used in geometric analysis for quite a time. They somehow appear in all known proofs of the Brunn–Minkowski inequality,

$$|A+B|^{1/n} \geqslant |A|^{1/n} + |B|^{1/n}, \tag{9}$$

where $A, B \subset \mathbb{R}^n$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$ (see [19,29]). The isoperimetric inequality easily follows from (9). An important source of inspiration for us has been the direct mass transportation proof by Gromov [27, Appendix] of the (functional) isoperimetric inequality, namely inequality (2) in the case $p = 1$; we shall recall his argument below. Closely related to our work is also the mass transportation proof by McCann [26] of functional versions of (9) known as Prékopa–Leindler and Borell–Brascamp–Lieb inequalities (see [19]). More recently, Barthe has exploited all the power of Brenier's theorem to prove deep Gaussian inequalities (see [4] or the reviews [19,31, Chapter 6]). Our proof has many common points with Barthe's work, which is surprising since the inequalities under study here and there look quite different. As far as tools and methods are concerned, the present paper can be seen as the continuation of the very recent works [13,14]. Until recently, it was believed that those techniques could not be adapted to general Sobolev-type inequalities besides the $p = 1$ case. Here we shall demonstrate that this guess was wrong.

Among the main advantages of our proof, we note that it is extremely simple (apart from nonessential technical subtleties linked to the lack of smoothness of the Brenier map). In addition to the existence of the Brenier map, our proof makes use of just two ingredients: the *arithmetic–geometric inequality* on one hand (domination of the geometric mean by the arithmetic mean), and on the other hand the standard *Young inequality* for convex conjugate functions, in the very particular case of Eq. (10) below, or equivalently *Hölder's inequality* (11).

Our proof avoids any compactness argument, and has the great merit to allow room for quantitative versions, which are often important in problems coming from physics: for instance, if a function is far enough from the optimizers in (2), how to give a lower bound on how far the ratio $||\nabla f||_{L^p}/||f||_{L^{p\star}}$ departs from the optimal value $S_n(p)$? Here we will not investigate such questions (to do so, it would be desirable to have a more precise formulation of the problem), but it will be clear from our arguments that their constructive nature makes them a plausible starting point for such an investigation, at least when f is strictly positive on $\mathbb{R}^n$.

Finally, our proof will cover non-Euclidean norms. It clearly shows that the treatment of optimal Sobolev inequalities, and the resulting extremal functions, do not depend on the Euclidean structure of $\mathbb{R}^n$. As far as Sobolev inequalities are concerned, such versions for arbitrary norms are not new. The $p = 1$ case was contained in Gromov's treatment. For $p > 1$, the inequalities can be obtained by using a symmetrization procedure and Aubin and Talenti's argument; this was done recently by Alvino et al. [1]. As mentioned, our approach is completely different since we will not solve any variational problem and since our proof will be carried on $\mathbb{R}^n$ till the end.

As we just discussed, the only two ingredients which lie behind our proof of Sobolev inequalities are the arithmetic–geometric inequality, and Hölder's inequality. By tracing carefully cases of equality in these two inequalities, we shall manage to identify all cases of equality in the Sobolev inequalities. Though this problem has been solved in the case of the Euclidean norm, the result seems to be new in the case of arbitrary norms; in [1] this problem was left open. And even in the Euclidean case, we believe that our approach is simpler than the classical one based on sharp rearrangement inequalities.

The plan of the paper is as follows. First, in the next section, we give a proof of optimal Sobolev inequalities. Then, in Section 3, we shall give the adaptations which enable to turn this proof into a proof of optimal Gagliardo–Nirenberg inequalities. Even though we could have treated directly the general case of Gagliardo–Nirenberg inequalities with general norms, we have chosen to present Sobolev inequalities separately because they are popular and of independent interest. Finally, Section 4 contains some comments, and the identification of all minimizers in the Sobolev inequalities.

2. Sharp Sobolev inequalities

Stating and proving our main results for general norms will be hardly any longer than for Euclidean norms, so let us consider general norms from the beginning. Let $(E, ||\cdot||)$ be an n-dimensional normed space, with dual space $(E^*, ||\cdot||_*)$. Let λ be an invariant Haar measure on E (unique up to a multiplicative constant). We shall prove a sharp version of the Sobolev inequality

$$\left(\int_E |f|^{p^\star}\, d\lambda\right)^{1/p^\star} \leqslant S_{E,\lambda}(p)\left(\int_E ||df||_*^p\, d\lambda\right)^{1/p}.$$

Here $df : E \to E^*$ denotes the differential map of $f : E \to \mathbb{R}$.

For convenience and without loss of generality we assume that $E = (\mathbb{R}^n, ||\cdot||)$ where $||\cdot||$ is an arbitrary norm on $\mathbb{R}^n$. Then the dual space is $E^* = (\mathbb{R}^n, ||\cdot||_*)$ where, for $X \in E^*$,

$$||X||_* := \sup_{||Y||\leqslant 1} X \cdot Y$$

and $X \cdot Y := \sum X_i Y_i$. The duality can also be expressed through *Young's inequality*

$$X \cdot Y \leqslant \frac{\lambda^{-p}}{p}||X||_*^p + \frac{\lambda^q}{q}||Y||^q \tag{10}$$

for $\lambda > 0$. Here and throughout the paper $q = p/(p-1)$ denotes the dual exponent of $p > 1$ (we hope this notation will avoid confusions with $p^\star$ defined in (1)). For $X : \mathbb{R}^n \to E^*$ in L^p and $Y : \mathbb{R}^n \to E$ in L^q, integration of (10) and optimization in λ

gives *Hölder's inequality* in the form

$$\int X \cdot Y \leqslant \left(\int ||X||_*^p \right)^{1/p} \left(\int ||Y||^q \right)^{1/q}. \tag{11}$$

This inequality expresses the well-known fact that the dual space of $L^p(\mathbb{R}^n, E)$ coincides with $L^q(\mathbb{R}^n, E^*)$.

The norm $||\cdot||$ is Lipschitz and therefore differentiable almost everywhere. Whenever $x \in \mathbb{R}^n \backslash \{0\}$ is a point of differentiability, the gradient of the norm at x is the unique vector $x^* = \nabla(||\cdot||)(x)$ such that

$$||x^*||_* = 1, \quad x \cdot x^* = ||x|| = \sup_{||y||_* = 1} x \cdot y. \tag{12}$$

Of course, in the usual case of the Euclidean norm $|\cdot|$, $x^* = x/|x|$.

For $1 \leqslant p < n$, we define the function h_p as follows:

$$\begin{cases} h_p(x) := \dfrac{1}{(\sigma_p + ||x||^q)^{\frac{n-p}{p}}} \quad (p > 1), \\ h_1(x) := \dfrac{1_B(x)}{|B|^{\frac{n-1}{n}}}, \end{cases} \tag{13}$$

where $\sigma_p > 0$ is determined by the condition

$$||h_p||_{L^{p^\star}} = 1, \tag{14}$$

and B stands for the unit ball of $(\mathbb{R}^n, ||\cdot||)$,

$$B := \{x \in \mathbb{R}^n;\ ||x|| \leqslant 1\}.$$

These functions will turn out to be extremal in the Sobolev inequalities. Of course, this property is well-known in the Euclidean case ($||\cdot|| = |\cdot|$): for $p > 1$ it is due to Aubin and Talenti and for $p = 1$ it is the classical isoperimetric inequality. As mentioned, the case of arbitrary norms was considered in [1].

The natural space to look for extremal functions in the Sobolev inequality is the *homogeneous* Sobolev space

$$\dot{W}^{1,p}(\mathbb{R}^n) := \{f \in L^{p^\star}(\mathbb{R}^n);\ \nabla f \in L^p(\mathbb{R}^n)\}.$$

This space coincides with the space of functions f whose distributional gradient lies in L^p and verifying that $|\{|f| \geqslant a\}|$ is finite for every $a > 0$. It is homogeneous in the same sense inequality (2) is homogeneous under the rescaling $f \mapsto f_\lambda \equiv f(\cdot/\lambda)$. This space is better adapted to the study of inequality (2) than $W^{1,p}$; indeed, for $p > 1$, extremal functions will always exist in $\dot{W}^{1,p}(\mathbb{R}^n)$ but will not belong to $W^{1,p}(\mathbb{R}^n)$ when $p \geqslant \sqrt{n}$.

If $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, it is natural to consider the dual norm of the ∇f. Thus, we define

$$||\nabla f||_{L^p} := \left(\int ||\nabla f||_*^p \right)^{1/p}. \tag{15}$$

For notational reasons, we will separate the case $p = 1$ from the rest. Let us start with $p > 1$.

Theorem 2. *Let $p \in (1,n)$ and $q = p/(p-1)$. Whenever $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ and $g \in L^{p^*}(\mathbb{R}^n)$ are two functions with $||f||_{L^{p^\star}} = ||g||_{L^{p^\star}}$, then*

$$\frac{\int |g|^{p^\star(1-1/n)}}{\left(\int ||y||^q |g(y)|^{p^\star}\, dy\right)^{1/q}} \leqslant \frac{p(n-1)}{n(n-p)} ||\nabla f||_{L^p} \tag{16}$$

with equality if $f = g = h_p$.

As immediate consequences we have

(i) *The duality principle*

$$\sup_{||g||_{L^{p^\star}}=1} \frac{\int |g|^{p^\star(1-1/n)}}{\left(\int ||y||^q |g(y)|^{p^\star}\, dy\right)^{1/q}} = \frac{p(n-1)}{n(n-p)} \inf_{||f||_{L^{p^\star}}=1} ||\nabla f||_{L^p} \tag{17}$$

with h_p extremal in both variational problems;

(ii) *The sharp Sobolev inequality*: *if $f \neq 0$ lies in $\dot{W}^{1,p}(\mathbb{R}^n)$, then*

$$\frac{||\nabla f||_{L^p}}{||f||_{L^{p^\star}}} \geqslant ||\nabla h_p||_{L^p}. \tag{18}$$

The variant for $p = 1$ of (18), for general norms, can be found in Gromov [27, Appendix]. Below we shall shortly reproduce his argument, with minor modifications which will make it look just like the proof of Theorem 2 above. Extremal functions for $p = 1$ do not exist in $W^{1,1}(\mathbb{R}^n)$, and should rather be searched for in the space of functions with bounded variation.

Theorem 3 (Isoperimetry). *If $f \neq 0$ is a smooth compactly supported function, then*

$$\frac{||\nabla f||_{L^1}}{||f||_{L^{n/(n-1)}}} \geqslant n|B|^{\frac{1}{n}}.$$

This inequality extends to functions with bounded variation, with equality if $f = h_1$.

Remark. (1) Inequality (16) is interesting only when $\int ||y||^q |g(y)|^{p^\star}\, dy < +\infty$, in which case (16) forces g to belong to $L^{p^\star(1-1/n)}(\mathbb{R}^n)$.

(2) The crucial property of h_p here is that, for almost every x, there is *equality* in Young's inequality (10) when $X = -\nabla h_p(x),\ Y = h_p^{p^\star/q}(x)x$ and

$$\lambda = \lambda_p := \left(\frac{n-p}{p-1}\right)^{1/q}. \tag{19}$$

Indeed, after a few computations and using (12), we are led to the straightforward equality

$$\left(\frac{n-p}{p-1}\right)\frac{||x||^q}{(\sigma_p + ||x||^q)^n} = \frac{1}{p\lambda_p^p}\left(\frac{n-p}{p-1}\right)^p \frac{||x||^q}{(\sigma_p + ||x||^q)^n} + \frac{\lambda_p^q}{q}\frac{||x||^q}{(\sigma_p + ||x||^q)^n}.$$

As a consequence (or by a direct computation), the same choice of X and Y gives an equality in Hölder's inequality (11):

$$-\int \nabla h_p(x)\cdot[h_p^{p^\star/q}(x)x]\,dx = ||\nabla h_p||_{L^p}\left(\int ||x||^q h_p^{p^\star}(x)\,dx\right)^{1/q}. \tag{20}$$

Let us now give the proof of Theorem 2.

Proof of Theorem 2. First of all, it is well-known that whenever $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, then $\nabla|f| = \pm\nabla f$ almost everywhere, so f and $|f|$ have equal Sobolev norms. Thus, without loss of generality, we may assume that f and g are nonnegative and, by homogeneity, satisfy $||f||_{L^{p^\star}} = ||g||_{L^{p^\star}} = 1$. Moreover, we shall prove (16) only in the special case when f and g are smooth functions with compact support; the general case will follow by density.

Introduce the two probability densities

$$F(x) = f^{p^\star}(x), \quad G(y) = g^{p^\star}(y)$$

on $\mathbb{R}^n$; let $\nabla\varphi$ the Brenier map which transports $F(x)\,dx$ onto $G(y)\,dy$. In a first step, we shall establish that

$$\int G^{1-\frac{1}{n}} \leqslant \frac{1}{n}\int F^{1-\frac{1}{n}}\Delta\varphi, \tag{21}$$

where $\Delta\varphi(x) := \operatorname{tr} D^2\varphi(x)$ appears as the absolutely continuous part of the distributional Laplacian.

As explained in the introduction (8), we have, for $F(x)\,dx$ almost every $x\in\mathbb{R}^n$,

$$F(x) = G(\nabla\varphi(x))\det D^2\varphi(x). \tag{22}$$

Therefore, for $F(x)\,dx$ almost every x,

$$G^{-1/n}(\nabla\varphi(x)) = F^{-1/n}(x)(\det D^2\varphi(x))^{1/n}$$
$$\leqslant F^{-1/n}(x)\frac{\Delta\varphi(x)}{n}, \tag{23}$$

where we used the arithmetic–geometric inequality. By integrating inequality (23) with respect to $F(x)\,dx$, we find

$$\int G^{-1/n}(\nabla\varphi(x))\,F(x)\,dx \leqslant \frac{1}{n}\int F(x)^{1-\frac{1}{n}}(x)\Delta\varphi(x)\,dx.$$

The proof of (21) is completed by using the definition of mass transport (7).

Here we shall go a little bit into nonessential technical subtleties. In the inequality (21), $\Delta\varphi = \operatorname{tr} D^2\varphi$ is to be understood in the almost everywhere sense. It is well-known that $\Delta\varphi$ can be bounded above by $\Delta_{\mathscr{D}'}\varphi$, which denotes the distributional Laplacian of φ, viewed as a nonnegative measure on the set where φ is finite (see for instance [18, pp. 236–242] or [13]). On the other hand, since f and g are compactly supported, we know that $\nabla\varphi$ is bounded on $\operatorname{supp}(f)$, the support of f, since $\nabla\varphi(\operatorname{supp}(f))\subset\operatorname{supp}(g)$ (see [31, Theorem 2.12]). Extending φ if necessary outside of the support of f, we can assume that the support of f lies within an open set where φ is finite, and then we can apply the integration by parts formula

$$\frac{1}{n}\int F^{1-\frac{1}{n}}\Delta\varphi \leqslant \frac{1}{n}\int F^{1-\frac{1}{n}}\Delta_{\mathscr{D}'}\varphi = -\frac{1}{n}\int\nabla(F^{1-\frac{1}{n}})\cdot\nabla\varphi. \tag{24}$$

Back to our original notations $F = f^{p^\star}$ and $G = g^{p^\star}$, we have just shown, combining (21) and (24), that

$$\int g^{p^\star(1-1/n)} \leqslant -\frac{p(n-1)}{n(n-p)}\int f^{\frac{n(p-1)}{n-p}}\nabla f\cdot\nabla\varphi = -\frac{p(n-1)}{n(n-p)}\int f^{p^\star/q}\nabla f\cdot\nabla\varphi. \tag{25}$$

We now apply our second crucial tool: Hölder's inequality (11) with the choice $X = -\nabla f$ and $Y = f^{p^\star/q}\nabla\varphi$. This gives

$$-\int f^{p^\star/q}\nabla f\cdot\nabla\varphi \leqslant ||\nabla f||_{L^p}\left(\int f^{p^\star}||\nabla\varphi||^q\right)^{1/q}. \tag{26}$$

But, by definition of mass transport (7), $\int f^{p^\star}||\nabla\varphi||^q = \int ||y||^q g^{p^\star}(y)\,dy$. Therefore, the combination of (25) and (26) concludes the proof of inequality (16).

Let us now choose $f = g = h_p$, and check that equality holds at all the steps of the proof, and therefore in (16). Of course this function is not compactly supported, but in this particular case the Brenier map reduces to the identity map $\nabla\varphi(x) = x$, and all the steps can be checked explicitly. Indeed $\nabla\varphi(x) = x$ leads to an equality in (21)

and in (24) (via integration by parts). Then Eq. (20) ensures the equality in (26). This ends the proof of Theorem 2. □

Remark. Following the terminology of McCann [26], inequality (21) can be rephrased by saying that the functional

$$\rho \mapsto -\int \rho(x)^{1-\frac{1}{n}}\,dx$$

is *displacement convex*. This fact is well-known to specialists, and rests on the concavity of the map $M \mapsto (\det M)^{1/n}$, defined on the set of nonnegative symmetric matrices; see in particular [31, Section 5.2].

Proof of Theorem 3. Gromov's original proof [27] relied on the Knothe map [20], but the proof also works with the Brenier map as it was pointed out to us some time ago by Michael Schmuckenschläger.

Without loss of generality, we prove the theorem only when f is a nonnegative function, such that $||f||_{L^{n/(n-1)}} = 1$. We introduce the Brenier map $\nabla\varphi$ which pushes forward $F(x)\,dx = f^{n/(n-1)}(x)\,dx$ onto $G(y)\,dy = h_1^{n/(n-1)}(y)\,dy$. Reasoning as in the proof of Theorem 2, we write, after (21),

$$|B|^{1/n} \leqslant \frac{1}{n}\int f\Delta\varphi \leqslant -\frac{1}{n}\int \nabla f\cdot\nabla\varphi.$$

The justification of the integration by parts goes as in (24). By definition of h_1, for almost every x in the support of f, $\nabla\varphi(x)\in B$. In particular $-\nabla f\cdot\nabla\varphi \leqslant ||\nabla f||_*$, and thus

$$n|B|^{\frac{1}{n}} \leqslant \int ||\nabla f||_* = ||\nabla f||_{L^1}. \tag{27}$$

By a standard approximation argument, one can express this inequality in terms of an isoperimetric inequality: whenever A is some closed (say) subset of $\mathbb{R}^n$, we have

$$m^+(\partial A) \geqslant n|B|^{\frac{1}{n}}|A|^{\frac{n-1}{n}}, \tag{28}$$

where m^+ stands for the surface measure with respect to the metric $||\cdot||$ (not necessarily Euclidean),

$$m^+(\partial A) := \liminf_{\varepsilon\to 0} \frac{|A+\varepsilon B| - |A|}{\varepsilon}.$$

Note that $A+\varepsilon B$ is the ε-neighborhood of A with respect to the metric $||\cdot||$. Now, there is equality in (28) when A is an affine image of B. So this inequality has to be sharp, and so has to be (27). □

We conclude this section with a few remarks about the way we have proven and stated our results.

Remark. (1) A classical way to attack the problem of optimal constants for Sobolev inequalities is to look at the Euler–Lagrange equation and to identify its solutions. Here, on the contrary, we have established that h_p is an optimizer without establishing any Euler–Lagrange equation. Neither did we use the co-area formula or a rearrangement procedure.

(2) The best constant $\tilde{S}_n(p) := ||\nabla h_p||_{L^p}^{-1}$ in the sharp Sobolev inequality (18) can easily be expressed as a function of $|B|$ since h_p is radially symmetric with respect to the norm $||\cdot||$. In particular, we have

$$\tilde{S}_n(p) = \left(\frac{|B_2^n|}{|B|}\right)^{1/n} S_n(p),$$

where B_2^n is the standard Euclidean ball and $S_n(p)$ is the best constant in the Euclidean Sobolev inequality (2). We stress however that the extremal function h_p depends on $||\cdot||$ and not just on $|B|$.

(3) If we exploit the left-hand side maximization in (17), we immediately obtain, after setting $h = g^{p^\star}$, the following sharp inequality: there exists a constant $C_n(p) > 0$ such that for every $h \in L^1$,

$$\int |h|^{1-1/n} \leqslant C_n(p) \left(\int ||y||^q |h(y)|\, dy\right)^{1/q} \left(\int |h|\right)^{1/p^\star}$$

with equality if $h(y) = h_p^{p^\star}(y) = (\sigma_p + ||y||^q)^{-n}$. It would be interesting to understand why this inequality appears as a dual of the Sobolev inequality.

(4) The right-hand side of (16) is invariant under dilations and translations (for fixed $L^{p^\star}$ norm), whereas the left-hand side is only invariant under dilations. If we define $\mathrm{Var}_q(G) := \inf_{y_0} \int ||y - y_0||^q G(y)\, dy$, then inequality (16) can obviously be replaced by the following dilation–translation invariant version: for $||f||_{L^{p^\star}} = ||g||_{L^{p^\star}} = 1$,

$$\frac{\int |g|^{p^\star(1-1/n)}}{\mathrm{Var}_q(|g|^{p^\star})^{1/q}} \leqslant \frac{p(n-1)}{n(n-p)} ||\nabla f||_{L^p} \tag{29}$$

with equality if $f = g = h_p$.

(5) What happens if in the proof of Theorem 2, in Eq. (26), we use, instead of Hölder's inequality (11), the simpler Young inequality (10) ? In view of the remark before (19), we obtain the following (equivalent) form of the theorem: *whenever* $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$ *are two functions with* $||f||_{L^{p^\star}} = ||g||_{L^{p^\star}} = 1$, *then, for all* $\lambda > 0$,

$$\frac{n(n-p)}{p(n-1)} \int |g|^{p^\star(1-1/n)} - \frac{\lambda^q}{q} \int ||y||^q |g(y)|^{p^\star}\, dy \leqslant \frac{1}{p\lambda^p} \int ||\nabla f||_*^p \tag{30}$$

with equality if $f = g = h_p$ *and* $\lambda = \lambda_p$ (19). *As a consequence we have the duality principle*

$$\sup_{||g||_{L^{p^\star}}=1}\left[\frac{n(n-p)}{p(n-1)}\int |g|^{p^\star(1-1/n)} - \frac{\lambda_p^q}{q}\int ||y||^q |g(y)|^{p^\star}\,dy\right] = \frac{1}{p\lambda_p^p}\inf_{||f||_{L^{p^\star}}=1}\int ||\nabla f||_*^p,$$

with h_p *extremal in both variational problems.*

This formulation was our original one. Clearly, the duality seems to be expressed in a much more satisfactory way in (16) than in (30). Furthermore, the extremal function h_p appears more naturally via (16), and one need not choose $\lambda = \lambda_p$ in a seemingly arbitrary manner. On the other hand, (30) has the advantage to separate the integrals in an additive way, and this form will appear more convenient to deal with more sophisticated integral expressions, as we shall see in the next section.

3. Gagliardo–Nirenberg inequalities

In this section, we give a treatment of some Gagliardo–Nirenberg inequalities in sharp form. As we explained in the introduction, the results in the Euclidean case were recently obtained, with a different method, by Del Pino and Dolbeault (the case $p = 2$ is treated in [15] and the general case in [16]). Here again, we shall consider an arbitrary norm $||\cdot||$ on $\mathbb{R}^n$.

Let us introduce a new family of functions, which will turn out to be optimal in a more general family of inequalities: for $\alpha \geqslant 0$, we define

$$h_{\alpha,p}(x) := (\sigma_{\alpha,p} + (\alpha-1)||x||^q)_+^{\frac{1}{1-\alpha}}.$$

As before we write $q = p/(p-1)$, and $\sigma_{\alpha,p} > 0$ is chosen in such a way as to turn $h_{\alpha,p}^{\alpha p}$ into a probability density. Note that for $\alpha < 1$, $h_{\alpha,p}$ has compact support, while for $\alpha > 1$ it is positive everywhere, decaying polynomially at infinity. The L^p norm of the gradient is again considered in the sense of (15). We stress that the statement will include $L^r(\mathbb{R}^n)$-spaces with $r \in (0,1)$, for which $||\cdot||_{L^r}$ is no longer a norm. We shall prove

Theorem 4. *Let* $n \geqslant 2$, $p \in (1,n)$ *and* $\alpha \in (0, \frac{n}{n-p}]$, $\alpha \neq 1$. *Let* f *and* g *be such that* $||f||_{L^{\alpha p}} = ||g||_{L^{\alpha p}} = 1$. *Then, for all* $\mu > 0$,

$$\begin{aligned}\frac{\alpha p}{(\alpha-1)(\alpha p-(\alpha-1))}\int |g|^{\alpha(p-1)+1} - \frac{\mu^q}{q}\int ||y||^q |g(y)|^{\alpha p}\,dy \\ \leqslant \frac{1}{p\mu^p}\int ||\nabla f||_*^p + \frac{\alpha p - n(\alpha-1)}{(\alpha-1)(\alpha p-(\alpha-1))}\int |f|^{\alpha(p-1)+1}.\end{aligned} \tag{31}$$

Moreover, when

$$\mu = \mu_p := q^{1/q}, \tag{32}$$

then

(i) *equality holds in* (31) *when* $f = g = h_{\alpha,p}$; *in particular, one has the duality principle*

$$\sup_{||g||_{L^{\alpha p}}=1}\left[\frac{\alpha p}{(\alpha-1)(\alpha p-(\alpha-1))}\int |g|^{\alpha(p-1)+1} - \frac{\mu_p^q}{q}\int ||y||^q |g(y)|^{\alpha p}\,dy\right]$$
$$= \inf_{||f||_{L^{\alpha p}}=1}\left[\frac{1}{p\mu_p^p}\int ||\nabla f||_*^p + \frac{\alpha p-n(\alpha-1)}{(\alpha-1)(\alpha p-(\alpha-1))}\int |f|^{\alpha(p-1)+1}\right] \quad (33)$$

and $h_{\alpha,p}$ *is extremal in both variational problems*;

(ii) *as a corollary, whenever* $f \neq 0$ *lies in* $\dot{W}^{1,p}(\mathbb{R}^n)$, *then for* $\alpha > 1$,

$$\frac{||\nabla f||_{L^p}^{\theta}||f||_{L^{\alpha(p-1)+1}}^{1-\theta}}{||f||_{L^{\alpha p}}} \geqslant ||\nabla h_{\alpha,p}||_{L^p}^{\theta}||h_{\alpha,p}||_{L^{\alpha(p-1)+1}}^{1-\theta}, \quad (34)$$

where

$$\theta = \frac{n(\alpha-1)}{\alpha(np-(\alpha p+1-\alpha)(n-p))} = \frac{p^\star(\alpha-1)}{\alpha p(p^\star-\alpha p+\alpha-1)};$$

for $\alpha < 1$,

$$\frac{||\nabla f||_{L^p}^{\theta}||f||_{L^{\alpha p}}^{1-\theta}}{||f||_{L^{\alpha(p-1)+1}}} \geqslant \frac{||\nabla h_{\alpha,p}||_{L^p}^{\theta}}{||h_{\alpha,p}||_{L^{\alpha(p-1)+1}}}, \quad (35)$$

where

$$\theta = \frac{n(1-\alpha)}{(\alpha p+1-\alpha)(n-\alpha(n-p))} = \frac{p^\star(1-\alpha)}{(p^\star-\alpha p)(\alpha p+1-\alpha)}.$$

Remark. (1) Note that when $\alpha < 1$, the terms in (31) not containing μ are nonpositive; while they are all nonnegative when $\alpha > 1$. This change of sign corresponds to a change of the sign of $1-\gamma$ in (36) below.

(2) Theorem 4 includes Theorem 2 (in the form (30)) as a particular case, namely when $\alpha = n/(n-p)$, in which case $\theta = 1$. In the interesting limit $\alpha \to 1$, the function $h_{\alpha,p}$ would look like $e^{-||x||^q}$, and the Gagliardo–Nirenberg inequality would reduce to an L^p analogue of the logarithmic Sobolev inequality (see [16] for the Euclidean case). It was pointed out to us by Michel Ledoux that when $\alpha \to 0$, inequality (35) reduces to the following Faber–Krahn-type sharp inequality: for every compactly supported $f \in \dot{W}^{1,p}(\mathbb{R}^n)$, and every $p \in (1,n)$,

$$||f||_1 \leqslant C|\mathrm{supp}(f)|^{1-1/p^\star}||\nabla f||_{L^p}$$

for some numerical constant $C > 0$, where $\mathrm{supp}(f)$ stands for the support of f; moreover, equality holds when f is of the form

$$(\sigma - ||x||^q)_+.$$

(3) Theorem 4 yields exactly the same one-parameter family (4) of Gagliardo–Nirenberg inequalities as the one obtained in [15,16] (for the Euclidean norm). This may seem surprising, since the methods here and there are completely different. This one-parameter family may however have a particular geometrical meaning, as suggested by a tensorization argument due to Bakry [3]. Combining sharp Sobolev inequalities in $\mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$ with a clever tensorization argument, he was able to recover again the same family of sharp Gagliardo–Nirenberg inequalities in $\mathbb{R}^n$.

Proof of Theorem 4. The proof will follow the same scheme as in the previous section. The basic inequality replacing (21) will be the following: whenever $\gamma \geqslant 1 - 1/n$, $\gamma \neq 1$,

$$\frac{1}{1-\gamma}\int G^\gamma \leqslant \frac{1-n(1-\gamma)}{1-\gamma}\int F^\gamma + \int F^\gamma \Delta\varphi. \tag{36}$$

Inequality (21) corresponds to the case $\gamma = 1 - 1/n$. Here and as in the proof of Theorem 2, F and G denote two probability densities, and $\nabla\varphi$ is the Brenier map pushing $F(x)dx$ forward to $G(y)\,dy$. In [31, Chapter 5], inequality (36) is shown to be an immediate consequence of the displacement convexity (in the terminology of McCann [26]) of the functional

$$\rho \mapsto \frac{1}{1-\gamma}\int \rho^\gamma(x)dx. \tag{37}$$

Again, for the sake of completeness we shall give a short proof which does not rely explicitly on this concept. It proceeds exactly in the same way that we followed to prove (21). From the Monge–Ampère equation (8) we deduce that for $F(x)\,dx$ almost every $x \in \mathbb{R}^n$ we have

$$G^{\gamma-1}(\nabla\varphi(x)) = F^{\gamma-1}(x)(\det D^2\varphi(x))^{1-\gamma}. \tag{38}$$

The function $M \mapsto (\det M)^k$ is concave (resp. convex) on the set of nonnegative symmetric $n \times n$ matrices when $k \in [0, 1/n]$ (resp. $k < 0$). In other words, the function

$$M \mapsto \frac{1}{1-\gamma}(\det M)^{1-\gamma}$$

is concave on the set of nonnegative symmetric $n \times n$ matrices whenever $\gamma \geqslant 1 - 1/n$. (The case $\gamma = 1$, not needed here, is defined in the limit as the log-concavity of the determinant and can be used for proving logarithmic Sobolev inequalities [13]). Thus, for a nonnegative symmetric matrix M, we have

$$\begin{aligned}(1-\gamma)^{-1}(\det M)^{1-\gamma} &= (1-\gamma)^{-1}(\det(I + (M-I)))^{1-\gamma}\\ &\leqslant (1-\gamma)^{-1} + \operatorname{tr}(M-I)\\ &= (1-\gamma)^{-1}(1-n(1-\gamma)) + \operatorname{tr}(M).\end{aligned}$$

We then deduce from (38) that

$$\frac{1}{1-\gamma}\, G^{\gamma-1}(\nabla\varphi(x)) \leqslant \frac{1-n(1-\gamma)}{1-\gamma}\, F^{\gamma-1}(x) + F^{\gamma-1}(x)\Delta\varphi(x).$$

Integrating this inequality with respect to $F(x)dx$ and using the definition of mass transport (7), we conclude to (36).

Now let us go on with the proof of (31). Define

$$\gamma := \frac{\alpha(p-1)+1}{\alpha p} = 1 - \frac{\alpha-1}{\alpha p}$$

and note that $\gamma \geqslant 1-1/n$ precisely when $\alpha \in (0, n/(n-p)]$. Reasoning exactly as in Theorem 2, we deduce from (36) that whenever F and G are two smooth, compactly supported probability densities and $\nabla\varphi$ is the corresponding Brenier map, then

$$\frac{\alpha p}{\alpha-1}\int G^{\gamma} \leqslant \frac{\alpha p - n(\alpha-1)}{\alpha-1}\int F^{\gamma} - \int \nabla F^{\gamma}\cdot\nabla\varphi. \tag{39}$$

Choosing $F = f^{\alpha p}$ and $G = g^{\alpha p}$ in this inequality, we obtain

$$\frac{\alpha p}{\alpha-1}\int g^{\alpha(p-1)+1} \leqslant \frac{\alpha p - n(\alpha-1)}{\alpha-1}\int f^{\alpha(p-1)+1} - (\alpha(p-1)+1)\int f^{\alpha(p-1)}\,\nabla f\cdot\nabla\varphi.$$

Next we apply Young's inequality (10) with $X = -\nabla f(x)$ and $Y = f^{\alpha(p-1)}(x)\,\nabla\varphi(x)$, to find

$$\begin{aligned}\frac{\alpha p}{(\alpha-1)(\alpha p-(\alpha-1))}\int g^{\alpha(p-1)+1} \leqslant{}& \frac{\alpha p - n(\alpha-1)}{(\alpha-1)(\alpha p-(\alpha-1))}\int f^{\alpha(p-1)+1} \\ &+ \frac{1}{p\mu^p}\int ||\nabla f||_*^p + \frac{\mu^q}{q}\int f^{\alpha p}||\nabla\varphi||^q.\end{aligned}$$

To conclude the proof of (31), it suffices to apply the identity (7) to the last integral.

We now turn to the proof of part (i) of Theorem 4. Just as in Theorem 2, it is a direct consequence of the observation that if we set $f = g = h_{\alpha,p}$, and thus $\nabla\varphi(x) = x$, in the previous proof, then all the steps can be computed explicitly and lead to equalities. The crucial point here, which ensures a pointwise equality in Young's inequality (10) is that, for almost all $x \in \mathbb{R}^n$,

$$-\nabla h_{\alpha,p}(x)\cdot[h_{\alpha,p}^{\alpha(p-1)}(x)x] = \frac{1}{p\mu_p^p}||\nabla h_{\alpha,p}(x)||_*^p + \frac{\mu_p^q}{q}||h_{\alpha,p}^{\alpha(p-1)}(x)x||^q.$$

Indeed, after a little bit of computation, this identity reduces to the straightforward equality

$$q\frac{||x||^q}{(\sigma_{\alpha,p}+(\alpha-1)||x||^q)^{\frac{\alpha p}{\alpha-1}}}=\frac{q^p}{p\mu_p^p}\frac{||x||^q}{(\sigma_{\alpha,p}+(\alpha-1)||x||^q)^{\frac{\alpha p}{\alpha-1}}}+\frac{\mu_p^q}{q}\frac{||x||^q}{(\sigma_{\alpha,p}+(\alpha-1)||x||^q)^{\frac{\alpha p}{\alpha-1}}}.$$

Finally, let us prove part (ii) of Theorem 4. To show that part (i) of the theorem implies part (ii), we use a scaling argument, more or less standard in problems of this kind. Assume for instance $\alpha>1$, and let us see how to establish (34). From part (i) we have the inequality, when $||f||_{L^{\alpha p}}=1$,

$$\frac{1}{p\mu_p^p}\int||\nabla f||_*^p+\frac{\alpha p-n(\alpha-1)}{(\alpha-1)(\alpha p-(\alpha-1))}\int|f|^{\alpha(p-1)+1}\geqslant C:=\left[\frac{\alpha p}{(\alpha-1)(\alpha p-(\alpha-1))}\int|h_{\alpha,p}|^{\alpha(p-1)+1}-\frac{\mu_p^q}{q}\int||y||^q|h_{\alpha,p}(y)|^{\alpha p}\,dy\right],\quad(40)$$

with equality when $f=h_{\alpha,p}$. Thus, for every $f\in\dot{W}^{1,p}(\mathbb{R}^n)$,

$$\frac{||f||_{L^{\alpha(p-1)+1}}^{\alpha(p-1)+1}}{||f||_{L^{\alpha p}}^{\alpha(p-1)+1}}+C_1\frac{||\nabla f||_{L^p}}{||f||_{L^{\alpha p}}^p}\geqslant C_2,\quad(41)$$

where C_1 and C_2 are positive constants. Here we do not write down the precise values of C_1 and C_2; anyway this is not necessary, to carry on the argument till the end it will be sufficient to know that $h_{\alpha,p}$ is optimal in this inequality.

Next, we apply (41) with f replaced by $f_\lambda=f(\cdot/\lambda)$ $(\lambda>0)$. We find

$$\lambda^{\frac{n(\alpha-1)}{\alpha p}}\frac{||f||_{L^{\alpha(p-1)+1}}^{\alpha(p-1)+1}}{||f||_{L^{\alpha p}}^{\alpha(p-1)+1}}+C_1\lambda^{\frac{\alpha(n-p)-n}{\alpha}}\frac{||\nabla f||_{L^p}}{||f||_{L^{\alpha p}}^p}\geqslant C_2,\quad(42)$$

and we can now optimize with respect to $\lambda>0$, to recover

$$||f||_{L^{\alpha p}}\leqslant C||\nabla f||_{L^p}^{\theta}||f||_{L^{\alpha(p-1)+1}}^{1-\theta},$$

with equality when $f=(h_{\alpha,p})_\lambda$, with the optimal choice of λ. As expected, θ is determined by scaling invariance. The same scaling invariance guarantees that there is also equality when $f=h_{\alpha,p}$, which is the content of (34).

The case $\alpha<1$ is obtained exactly in the same way. This concludes the proof of Theorem 4. □

4. Further remarks and equality cases

The mass transportation method appears to be extremely efficient in the treatment of sharp Gagliardo–Nirenberg inequalities, as illustrated by the short length and simplicity of the proofs above. Among the other advantages of our method, we note that it provides a common framework to *all* the family of Sobolev inequalities, making the link with isoperimetric estimates clearer. It also emphasizes a strong connection between the Brunn–Minkowski inequality (9) (and more generally convex geometry) and sharp Sobolev inequalities. Finally, we should mention that the use of the Brenier map is not compulsory: we could as well have worked with the Knothe map [20].

Certainly, one of the most irritating open problems remaining in the field, is the fact that we do not understand how to get sharp inequalities and extremal functions in the rest of the range of the Gagliardo–Nirenberg family (3). The solution to this problem may go through a better understanding of the duality principle which was displayed in the present paper.

Another natural problem is that of the identification of *all* cases of equality in Sobolev or Gagliardo–Nirenberg inequalities. In the case of a Euclidean norm, it is known that the functions h_p are the only minimizers, up to translation, dilation and multiplication by a constant. But even in this case, the known proofs of this result are far from being straightforward; they first use the Brothers–Ziemer theorem [7] to reduce to the one-dimensional case, after which a somewhat tedious analysis is performed.

From our proof, it is possible to determine all cases of equality, even when dealing with arbitrary norms. We restrict the discussion to the sharp Sobolev inequalities. A similar proof would solve the problem for the Gagliardo–Nirenberg inequalities, at least in the case $\alpha > 1$.

Theorem 5. *A function $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ is optimal in the Sobolev inequality* (18) *if and only if there exist $C \in \mathbb{R}$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that*

$$f(x) = Ch_p(\lambda(x - x_0)). \tag{43}$$

It is enough to prove Theorem 5 for nonnegative functions f. Indeed, for an arbitrary optimal function f, $|f|$ will also be optimal and then the conclusion of the theorem will force f to have constant sign on $\mathbb{R}^n$.

Let f and g be two nonnegative measurable functions; we say that f is a *dilation–translation image* of g if there exists $C > 0$, $\lambda \neq 0$ and $x_0 \in \mathbb{R}^n$ such that $f(x) = Cg(\lambda(x - x_0))$. If $\int f^k = \int g^k$ for some $k > 0$, then necessarily $f(x) = |\lambda|^{n/k} g(\lambda(x - x_0))$. This is equivalent to saying that the Brenier map $\nabla\varphi$ pushing $f^k(x)\,dx$ forward to $g^k(y)\,dy$ is a *dilation–translation map*, in the sense that $\nabla\varphi = \lambda(\mathrm{Id} - x_0)$. Note that f is a dilation–translation image of g if and only if g is a dilation–translation image of f.

Of course, the Sobolev inequality is invariant under dilation–translation maps. Thus, it suffices to prove Theorem 5 when $f^{p^\star}$ is a probability density. In view of

Theorem 2, we just have to set $g = h_p$, and prove that all f's which achieve equality in (16) are dilation–translation images of h_p. Then, Theorem 5 is an immediate consequence of

Proposition 6. *Let $p \in (1, n)$, and let f and g be two nonnegative functions satisfying the assumptions of Theorem 2. If equality holds in (16), then f is a dilation–translation image of g.*

The proof of Proposition 6 will not rely on any sharp rearrangement inequality, but on rather standard tools from distribution theory, combined with careful approximation procedures. Let us start with an informal discussion. Our derivation of the optimal Sobolev inequality only relied on

(i) Theorem 1, together with the Monge–Ampère equation (22) and the definition of mass transport;
(ii) the arithmetic–geometric inequality (23), $(\det D^2\varphi)^{1/n} \leqslant \Delta\varphi/n$, integrated with respect to $f^{p^\star(1-1/n)}(x)\,dx$;
(iii) the integration by parts formula (24);
(iv) Hölder's inequality (11), in the form of Eq. (26).

If φ was smooth and f positive everywhere, equality in the arithmetic–geometric inequality (ii) would imply that $D^2\varphi$ is a pointwise multiple of the identity, from which it could be shown that it is in fact a constant multiple of the identity, so that $\nabla\varphi$ is a dilation–translation map. However, we do not know a priori that φ is smooth, neither that f is positive almost everywhere. Moreover, it is definitely not clear that the integration by parts formula (24) applies to the minimizer: we proved it only in the case when f and g are compactly supported! This restriction on f and g had no consequence on the generality of the final inequality, since a density argument could be applied; but it prevents us to go anywhere as far as equality cases are concerned. Therefore, our proof will be performed in two steps: (1) generalize the proof of (16) in order to directly obtain the inequality for all admissible f's and g's, *not necessarily smooth and compactly supported*; (2) trace back cases of equality in the proof of this inequality, without assuming extra smoothness on f, g or φ.

To carry out step 1, it is sufficient to generalize the proof of the integration by parts (24) to more general functions f and g. This is the content of the following:

Lemma 7. *Let $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ and $g \in L^{p^\star}(\mathbb{R}^n)$ be two nonnegative functions such that $\|f\|_{L^{p^\star}} = \|g\|_{L^{p^\star}} = 1$ and $\int g^{p^\star}(y)\|y\|^q\,dy < +\infty$. Let $\nabla\varphi$ denote the Brenier map pushing $f^{p^\star}(x)\,dx$ forward to $g^{p^\star}(y)\,dy$. Then, $f^{p^\star/q}\nabla\varphi \in L^q(\mathbb{R}^n)$ and*

$$\int f^{p^\star(1-1/n)}\Delta\varphi \leqslant -\int \nabla[f^{p^\star(1-1/n)}]\cdot\nabla\varphi = -\frac{p(n-1)}{(n-p)}\int f^{p^\star/q}\nabla f\cdot\nabla\varphi, \quad (44)$$

where $\Delta\varphi = \operatorname{tr} D^2\varphi \geqslant 0$ denotes the absolutely continuous part of the distributional Laplacian.

To achieve step 2, and eventually prove Proposition 6, we shall have to overcome a few more technical difficulties. Our first task will be to establish that f is positive; the proof of this fact was given to us by Almut Burchard, who is warmly thanked. As we shall see, the argument eventually relies on the fact that there should be equality in Hölder's inequality (iv) above. From this strict positivity we shall deduce that the distributional Hessian $D^2_{\mathscr{D}'}\varphi$ is absolutely continuous, and therefore coincides with $D^2\varphi$, defined almost everywhere. Once we have introduced the distributional Hessian in our problem, we will use a standard regularization argument to conclude the proof.

A subtle point in the argument is the following: for our proof to work out, it is not sufficient to prove that f is positive almost everywhere. Indeed, if f would vanish at some place, then we could not exclude the possibility that $D^2\varphi$ has some singular part, living precisely on the set where f vanishes. On the other hand, f is not a priori continuous, so discussing the positivity of f *everywhere* does not seem to make much sense. To avoid this contradiction, we shall show that f is positive everywhere in the sense that it is, locally, bounded from below almost everywhere by a positive constant.

After these explanations, we can go on with the proofs of Lemma 7 and of Proposition 6.

Proof of Lemma 7. By definition of mass transport (7), we know that $\int f^{p^\star}||\nabla\varphi||^q = \int g^{p^\star}||y||^q\,dy$ and so $f^{p^\star/q}\nabla\varphi \in L^q(\mathbb{R}^n)$. The proof of (44) will be done by approximation and regularization; there is no fundamental difficulty, but one has to be careful enough.

Let Ω be the interior of the convex set where $\varphi < +\infty$. Note that $\overline{\Omega}$ contains the support of f, and that $\partial\Omega$ is of zero measure. Without loss of generality we assume that $0 \in \Omega$. Whenever $\varepsilon > 0$ is a (small) positive number, we define

$$f_\varepsilon(x) = \min\Big[f\Big(\frac{x}{1-\varepsilon}\Big),\ f(x)\,\chi(\varepsilon x)\Big], \tag{45}$$

where χ is a C^∞ cut-off function with $0 \leqslant \chi \leqslant 1$, $\chi(x) \equiv 1$ for $|x| \leqslant 1/2$, $\chi(x) \equiv 0$ for $|x| \geqslant 1$. Note that the support of f_ε is compact and contained within Ω (here we use the fact that Ω is starshaped with respect to 0).

Both functions in the right-hand side of (45) are bounded in $\dot{W}^{1,p}(\mathbb{R}^n)$, uniformly in ε. This is clear for the first one; for the second one this is a consequence of

$$\begin{aligned}\int_{\mathbb{R}^n} f^p(x)|\nabla[\chi(\varepsilon x)]|^p\,dx &= \varepsilon^p \int_{\mathbb{R}^n} f^p|\nabla\chi(\varepsilon x)|^p\,dx \\ &\leqslant \left(\int_{\mathbb{R}^n} f^{p^\star}\right)^{1-\frac{p}{n}}\left(\int_{\mathbb{R}^n} \varepsilon^n|\nabla\chi(\varepsilon x)|^n\,dx\right)^{\frac{p}{n}} \\ &= \left(\int_{\mathbb{R}^n} f^{p^\star}\right)^{1-\frac{p}{n}}\left(\int_{\mathbb{R}^n} |\nabla\chi(x)|^n\,dx\right)^{\frac{p}{n}},\end{aligned}$$

where we used Hölder's inequality and the change of variables $x \to \varepsilon x$. Thus (by the formula $\min(f,g) = (f+g)/2 - |f-g|/2$), f_ε lies in $\dot{W}^{1,p}$, and in fact ∇f_ε is bounded in L^p as $\varepsilon \to 0$.

We now fix $\varepsilon > 0$, and let Ω_ε be a bounded open set whose closure is contained within Ω, and which contains the support of f_ε. It is standard that f_ε can be approximated in $\dot{W}^{1,p}(\mathbb{R}^n)$ by a sequence $f_\varepsilon^\delta \to f_\varepsilon$ of smooth nonnegative functions compactly supported inside Ω_ε; for this one just has to regularize f_ε by convolution with a kernel whose support is contained within a ball of radius δ, δ small enough and going to 0. Then we can use the fact that $\Delta\varphi$ (in the sense of Aleksandrov) is bounded above by the distributional Laplacian of φ in Ω (see [18, pp. 236–242] or [12]), and write

$$\int (f_\varepsilon^\delta)^{p^\star(1-1/n)} \Delta\varphi \leqslant -\int \nabla\left[(f_\varepsilon^\delta)^{p^\star(1-1/n)}\right] \cdot \nabla\varphi = -c_{n,p} \int (f_\varepsilon^\delta)^{p^\star/q} \nabla f_\varepsilon^\delta \cdot \nabla\varphi \quad (46)$$

where $c_{n,p} := p(n-1)/(n-p) > 0$. We know that f_ε^δ converges to f_ε in $L^{p^\star}$ (by convergence in $\dot{W}^{1,p}(\mathbb{R}^n)$) and since $\nabla\varphi$ remains essentially bounded within Ω_ε, we conclude that $(f_\varepsilon^\delta)^{p^\star/q}\nabla\varphi$ converges to $(f_\varepsilon)^{p^\star/q}\nabla\varphi$ in L^q. On the other hand we know that $\nabla f_\varepsilon^\delta$ converges to ∇f_ε in L^p. We then deduce from (46) by Fatou's lemma that

$$\int (f_\varepsilon)^{p^\star(1-1/n)} \Delta\varphi \leqslant -c_{n,p} \int (f_\varepsilon)^{p^\star/q} \nabla f_\varepsilon \cdot \nabla\varphi. \quad (47)$$

It now remains to pass to the limit in (47) as $\varepsilon \to 0$. For this we argue as follows. First of all we note that, up to possible extraction of a subsequence $\varepsilon = (\varepsilon_k)_{k\in\mathbb{N}}$, f_ε converges almost everywhere to f as $\varepsilon \to 0$. To prove this, it is sufficient to show that $g_\varepsilon(x) := f(x/(1-\varepsilon))$ converges almost everywhere to $f(x)$ as $\varepsilon \to 0$. Clearly, g_ε is bounded in $\dot{W}^{1,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$, and it also converges to f in the sense of distributions, since for all compactly supported test-functions φ one can write

$$\int g_\varepsilon \varphi = (1-\varepsilon)^n \int f(x)\varphi((1-\varepsilon)x)\, dx \to \int f\varphi.$$

So g_ε converges weakly to f in $\dot{W}^{1,p}$, and therefore locally strongly in L^r for any $r \in (1, p^\star)$. It follows that (up to extraction of a subsequence) $g_\varepsilon \to f$ almost everywhere. As a consequence, f_ε converges to f almost everywhere. Since $f_\varepsilon \leqslant f \in L^{p^\star}$, by dominated convergence theorem $f_\varepsilon \to f$ in $L^{p^\star}$. Similarly (or as a consequence of the L^p convergence of f_ε to f) ∇f_ε converges to ∇f in distributional sense on $\mathbb{R}^n$, and is also bounded in L^p, so ∇f_ε converges weakly in L^p to ∇f. On the other hand, again because $f_\varepsilon \leqslant f$, we know that $(f_\varepsilon)^{p^\star/q}||\nabla\varphi|| \in L^q$. So, by dominated convergence, $(f_\varepsilon)^{p^\star/q}\nabla\varphi$ converges (strongly) in L^q to $f^{p^\star/q}\nabla\varphi$. Thus we can pass to the limit as $\varepsilon \to 0$ in the right-hand side of (47), and by Fatou's

lemma we obtain

$$\int f^{p^\star(1-1/n)}\Delta\varphi \leqslant -c_{n,p}\lim_{\varepsilon\to 0}\int (f_\varepsilon)^{p^\star/q}\,\nabla f_\varepsilon\cdot\nabla\varphi = -c_{n,p}\int f^{p^\star/q}\,\nabla f\cdot\nabla\varphi.$$

This concludes the proof of (44). □

Proof of Proposition 6. With the notations of Theorem 2, let us fix nonnegative functions f and g for which there is equality in (16). We will trace back the equality cases in the proof of (16). Recall that $\nabla\varphi$ denotes the Brenier map pushing $f^{p^\star}(x)dx$ forward to $g^{p^\star}(y)\,dy$. Our goal is to prove that $\nabla\varphi$ is a dilation–translation map. As before, we denote by Ω the interior of the convex set where $\varphi<+\infty$; we recall that $\bar{\Omega}$ contains $\mathrm{supp}(f)$, and that $\partial\Omega$ is of zero measure.

The proof will be done in three steps:

Step 1: The function f is positive on Ω;
Step 2: $D^2_{\mathscr{D}'}\varphi$ has no singular part on Ω;
Step 3: $\nabla\varphi$ is a dilation–translation map.

Let us first show that f is positive everywhere, or more rigorously that *for every compact subset K of Ω, there exists a positive constant α_K such that*

$$\forall x\in K,\quad f(x)\geqslant\alpha_K>0. \tag{48}$$

Here, of course "$\forall x$" should be understood as "for almost all x". A proof was suggested to us by Almut Burchard; we reproduce her argument almost verbatim below.

For equality to hold in Hölder's inequality (11) it is necessary that, for some positive constant $k>0$,

$$||X||_*^p = k||Y||^q \quad \text{almost everywhere} \tag{49}$$

(a short proof is recalled at the end of the paper).

Therefore, equality in (26) implies

$$||\nabla f(x)||_*^p = kf^{p^\star}(x)||\nabla\varphi(x)||^q \tag{50}$$

for almost every $x\in\Omega$.

Let us introduce $f_m(x)=\max(f(x),1/m)$. We know that $\nabla f_m\in L^p$ and that in fact $\nabla f_m=\nabla f 1_{f>1/m}$. It follows that

$$||\nabla f_m(x)||_*^p \leqslant ||\nabla f(x)||_*^p = kf^{p^\star}(x)||\nabla\varphi(x)||^q \leqslant kf_m^{p^\star}(x)||\nabla\varphi(x)||^q.$$

As a consequence,

$$||\nabla(f_m^{-p/(n-p)})||_* \leqslant k^{1/p}\left(\frac{p}{n-p}\right)||\nabla\varphi||^{1/(p-1)}. \tag{51}$$

Since $||\nabla\varphi||$ is locally bounded on Ω, it follows from (51) that the functions $f_m^{-p/(n-p)}$ are uniformly (in m) locally Lipschitz on Ω. Taking m to infinity shows that $f^{-p/(n-p)}$ is locally Lipschitz, and therefore locally bounded, on Ω. From this we deduce that f is positive, locally bounded away from 0 on Ω, in the sense of (48). This implies in particular that the support of f is $\bar{\Omega}$.

We now prove that $D^2_{\mathscr{D}'}\varphi$ has no singular part. Since this is a nonnegative matrix-valued measure, it is enough to prove that its trace $\Delta_{\mathscr{D}'}\varphi$ is itself absolutely continuous in Ω. Let $\Delta_s\varphi$ be the singular part of $\Delta_{\mathscr{D}'}\varphi$; recall that $\Delta_s\varphi$ is a nonnegative measure and that $\Delta_{\mathscr{D}'}\varphi = \Delta\varphi + \Delta_s\varphi$. Since there should be equality in (44), we deduce from the proof of Lemma 7 that

$$\lim_{\varepsilon\to 0}\ \liminf_{\delta\to 0}\ \langle (f_\varepsilon^\delta)^{\frac{p(n-1)}{n-p}}, \Delta_s\varphi\rangle_{\mathscr{D}'} = 0. \tag{52}$$

Without loss of generality, we assume that $0\in\Omega$. Let K be an arbitrary convex compact subset of Ω containing 0 in its interior. For $d_K := d(K,\Omega^c)$, let $K' = \{x\in\Omega;\ d(x,K)\leqslant d_K/2\}$. From its definition K' is a convex compact subset of Ω whose interior is a neighborhood of K. By (48) we know that there exists $\alpha = \alpha_{K'} > 0$ such that $f \geqslant \alpha\, 1_{K'}$, where $1_{K'}$ stands for the indicator function of K'. If ε is small enough, we can make sure that $K/(1-\varepsilon)^2 \subset K'$; then, with the notation of Lemma 7 we have $f_\varepsilon(x) \geqslant \alpha\, 1_{K/(1-\varepsilon)}(x)$. If δ is small enough, this implies

$$f_\varepsilon^\delta \geqslant \alpha\, 1_K.$$

As a consequence, when both ε and δ are small enough we see that

$$\langle (f_\varepsilon^\delta)^{p^\star(1-1/n)}, \Delta_s\varphi\rangle_{\mathscr{D}'} \geqslant \alpha^{p^\star(1-1/n)}\Delta_s\varphi[K]. \tag{53}$$

Combining this with (52) and the positivity of α, we find that $\Delta_s\varphi[K] = 0$. Since K is arbitrary, we conclude that $\Delta_s\varphi$ vanishes. As announced above, this means that $D^2_{\mathscr{D}'}\varphi$ is absolutely continuous.

We can now conclude the proof of Proposition 6. Since we have equality in the arithmetic–geometric inequality (23) for $f^{p^\star(1-1/n)}(x)\,dx$ almost every $x\in\Omega$, and therefore for almost every $x\in\Omega$, we conclude that $D^2\varphi$, which can be identified with $D^2_{\mathscr{D}'}\varphi$, is proportional to the identity matrix at almost every $x\in\Omega$. Let κ be a smooth regularizing kernel with support included in a small ball of radius ε. Since $D^2(\varphi * \kappa) = D^2\varphi * \kappa$, we deduce that the smooth function $\varphi * \kappa$ is such that its Hessian is also pointwise proportional to the identity matrix on $\Omega_\varepsilon := \{x\in\Omega;\ d(x,\partial\Omega) > \varepsilon\}$. From this one easily shows that $D^2(\varphi * \kappa)$ is a constant multiple of the identity. By making κ tend to a Dirac mass, we see that $D^2_{\mathscr{D}'}\varphi$ is also a

constant multiple of the identity on the whole of Ω, and therefore $\nabla\varphi$ is a dilation–translation map on Ω. This concludes the proof of Proposition 6. $\square$

Remark. (1) No strict convexity of the norm is required for (49), as shown by the following short argument. Let $\lambda>0$ satisfy

$$\left(\int ||X||_*^p\right)^{1/p}\left(\int ||Y||^q\right)^{1/q}=\frac{\lambda^{-p}}{p}\left(\int ||X||_*^p\right)+\frac{\lambda^q}{q}\left(\int ||Y||^q\right).$$

Then, equality in Hölder's inequality (11) implies a pointwise (almost everywhere) equality in Young's inequality (10). When there is equality in Young's inequality, the function $\alpha(t):=(X\cdot Y)t-(\lambda^{-p}||X||_*^p/p)t^p$ achieves its maximum at $t=1$, and therefore $\lambda^{-p}||X||_*^p=\lambda^q||Y||^q$. This implies (49) with $k=\lambda^{p+q}$.

(2) In the case $g=h_p>0$, once the strict positivity of f has been proven, it is possible to appeal to Caffarelli's interior regularity results [10] for solutions of the Monge–Ampère equation, in order to conclude directly that $\varphi\in W^{2,\alpha}_{\mathrm{loc}}$ $(\alpha>1)$. This argument also implies that $D^2_{\mathscr{D}'}\varphi$ has no singular part; it has however the drawback to rely on very sophisticated results.

(3) If we look for extremal g's in (17), we can set $f=h_p$ in (16) and check for equality cases there. From Proposition 6 we know that g has to be a dilation–translation image of h_p, and that $\nabla\varphi$ is a dilation–translation map. But, as in the proof of Proposition 6, equality in Hölder's inequality with $f=h_p$ implies $||\nabla\varphi(x)||=\lambda||x||$ almost everywhere, for some $\lambda>0$ (see (50)). Therefore $\nabla\varphi(x)=\pm\lambda x$, and the only cases of equality are dilations of h_p. Of course, in (29) with $f=h_p$, the only equality cases will again be the dilation–translation images of h_p, as in Theorem 5.

(4) Replacing Hölder's inequality by Young's inequality—in fact, we eventually used the cases of equality in Young's inequality!—in the proof of Proposition 6, we can conclude that for equality to hold in (30), it is necessary that f be a translation–dilation image of g.

(5) It was pointed out to us by Maggi [24] that the technicalities encountered above can be greatly simplified if one restricts to radially symmetric functions. Indeed, in this case we have to deal with a one-dimensional transportation problem, which is completely elementary. The interest of this remark lies in the fact that it is often possible, for many variational problems, to show a priori that optimal functions have to be radially symmetric around some point, by sharp rearrangement inequalities (in this case, the Brothers–Ziemer theorem). Once this reduction has been performed, the classical procedure for the identification of extremals is still somewhat subtle, and even in this context the mass transportation argument leads to substantial simplifications. On the other hand, these sharp rearrangement inequalities are in general nontrivial. A proof of the Brothers–Ziemer theorem for general norms has been recently announced by Ferone and Volpicelli (after a similar result for strictly convex norms, by Esposito and Trombetti); by combining this with Maggi's remark, one can devise an alternative proof of Theorem 5.

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