2.10.44. The conclusion of the preceding theorem may fail if either $\mathbf{R}^m$ or $\mathbf{R}^n$ is remetrized by some norm not induced by an inner product. For example if

$$S=\{(1,-1),(-1,1),(1,1)\}\subset\mathbf{R}^2,\ f\colon S\to\mathbf{R}^2,$$

$$f(1,-1)=(1,0),\quad f(-1,1)=(-1,0),\quad f(1,1)=(0,\sqrt{3}),$$

$$\mu(x)=\sup\{|x_1|,|x_2|\},\ \nu(x)=[(x_1)^2+(x_2)^2]^{\frac{1}{2}}\ \text{for}\ x\in\mathbf{R}^2,$$

then $\mu(u-v)=2=\nu[f(u)-f(v)]$ for $u,v\in S$, and $\mu(u)=1$ for $u\in S$, but there exists no $\eta\in\mathbf{R}^2$ with $\nu[\eta-f(u)]\le 1$ for $u\in S$; thus f has no extension to $S\cup\{(0,0)\}$ with μ, ν Lipschitz constant 1.

On the other hand, if S is a subset of an arbitrary metric space X, then every Lipschitzian map $f\colon S\to\mathbf{R}$ has a Lipschitzian extension $g\colon X\to\mathbf{R}$ with $\operatorname{Lip}(g)=\operatorname{Lip}(f)$; in fact one may define

$$g(\xi)=\inf\{f(x)+\operatorname{Lip}(f)\cdot\operatorname{dist}(x,\xi)\colon x\in S\}\ \text{for}\ \xi\in X.$$

2.10.45. Theorem. *Suppose Z is a metric space and $\mathbf{R}^m\times Z$ is metrized so that*

$$\operatorname{dist}[(y,z),(u,v)]^2=\operatorname{dist}(y,u)^2+\operatorname{dist}(z,v)^2$$

whenever $y,u\in\mathbf{R}^m$ and $z,v\in Z$. For each $V\subset Z$ with $\mathscr{H}^k(V)<\infty$ there exists a number c such that

$$\alpha(m)^{-1}\le c\,\alpha(k)\,\alpha(k+m)^{-1}\le 2^{-m}(m+1)^{(k+m)/2},$$

$$\mathscr{H}^{m+k}(U\times V)=c\,\mathscr{L}^m(U)\,\mathscr{H}^k(V)\ \textit{for every}\ \mathscr{L}^m\ \textit{measurable set}\ U.$$

Proof. We let $\gamma(U)=\mathscr{H}^{k+m}(U\times V)$ for $U\subset\mathbf{R}^m$, and observe that γ measures $\mathbf{R}^m$, all closed subsets of $\mathbf{R}^m$ are γ measurable by 2.3.2(9), γ is invariant under translations, and

$$\mathscr{L}^m(U)\,\mathscr{H}^k(V)\,\alpha(k+m)\,\alpha(k)^{-1}\alpha(m)^{-1}\le\gamma(U)\ \text{for}\ U\subset\mathbf{R}^m$$

by 2.10.27, 2.10.35. Letting $\xi=\alpha(k+m)\,\alpha(k)^{-1}2^{-m}(m+1)^{(k+m)/2}$ we will prove next that

$$\gamma(U)\le\mathscr{L}^m(U)\,\xi\,\mathscr{H}^k(V)$$

whenever U is an m dimensional cube in $\mathbf{R}^m$. Suppose u is the side length of U, $\delta>0$, and G is any countable covering of V such that $0<\operatorname{diam}T\le\delta$ for $T\in G$; with each $T\in G$ we associate the positive integer $j(T)$ such that

$$[j(T)-1]\operatorname{diam}T<u\le j(T)\operatorname{diam}T,$$

Proof. The assertion is trivial in case $n=1$. Applying induction with respect to n we assume $n>1$ and $\operatorname{diam}S>0$, choose

$$a_1,b_1\in S\ \text{with}\ |a_1-b_1|=\operatorname{diam}S,$$

$$p\in\mathbf{O}^*(n,n-1),\quad a_1-b_1\in\ker p,$$

then select $\alpha_2,\beta_2,\ldots,\alpha_n,\beta_n\in p(S)$ and a rectangular box W with side lengths $c_2\ge c_3\ge\cdots\ge c_n$ such that $p(S)\subset W\subset\mathbf{R}^{n-1}$ and

$$|(\alpha_2-\beta_2)\wedge\cdots\wedge(\alpha_m-\beta_m)|=c_2\cdot\cdots\cdot c_m\ \text{for}\ m=2,\ldots,n.$$

Finally we take $c_1=|a_1-b_1|$,

$$Q=\mathbf{R}^n\cap\{x\colon 0\le(x-a_1)\bullet(b_1-a_1)\le|a_1-b_1|^2,\ p(x)\in W\},$$

$$a_i,b_i\in S\ \text{with}\ p(a_i)=\alpha_i,\ p(b_i)=\beta_i\ \text{for}\ i=2,\ldots,n,$$

and compute

$$\Big|\bigwedge_{i=1}^{m}(a_i-b_i)\Big|=\Big|(a_1-b_1)\wedge\bigwedge_{i=2}^{m}[(p^*\circ p)(a_i-b_i)]\Big|$$

$$=|a_1-b_1|\cdot\Big|\bigwedge_{i=2}^{m}(\alpha_i-\beta_i)\Big|=c_1\cdot c_2\cdot\cdots\cdot c_m.$$

2.10.39. Corollary. $\mathscr{S}^m(A)\le n^{m/2}2^{m-1}\mathscr{T}^m(A)$ *for* $A\subset\mathbf{R}^n$.

Proof. For each rectangular box Q with side lengths $c_1\ge c_2\ge\cdots\ge c_n>0$ we can choose positive integers $k_1,\ldots,k_{m-1}$ such that

$$k_i/2<c_i/c_m\le k_i\ \text{whenever}\ i=1,\ldots,m-1,$$

and express Q as the union of $k_1\cdot\cdots\cdot k_{m-1}$ rectangular boxes with side lengths

$$c_1/k_1,\ldots,c_{m-1}/k_{m-1},\ c_m,\ldots,c_n.$$

Each of these new boxes is contained in a cube with side length c_m, hence in a closed ball with diameter $n^{\frac{1}{2}}c_m$. The sum of the m'th powers of the diameters of the balls so obtained equals

$$k_1\cdot\cdots\cdot k_{m-1}\cdot n^{m/2}\cdot(c_m)^m=n^{m/2}(k_1c_m)\cdot\cdots\cdot(k_{m-1}c_m)\cdot c_m$$

$$<n^{m/2}(2c_1)\cdot\cdots\cdot(2c_{m-1})\cdot c_m=n^{m/2}2^{m-1}c_1\cdot\cdots\cdot c_m.$$

2.10.40. Lemma. *If $\varnothing\ne P\subset\mathbf{R}^n\times\{r\colon 0<r<\infty\}$, P is compact, and*

$$Y_t=\{y\colon|y-a|\le rt\ \textit{whenever}\ (a,r)\in P\}$$

for $0\le t<\infty$, then $c=\inf\{t\colon Y_t\ne\varnothing\}<\infty$, Y_c consists of a single point b, and b belongs to the convex hull of

$$A=\{a\colon \text{for some}\ r,\ (a,r)\in P\ \text{and}\ |b-a|=rc\}.$$

Proof. Since each Y_t is compact, and

$$0 \in Y_t \text{ for } t \geq \sup\{|a|/r: (a,r) \in P\},$$

we see that $Y_c = \bigcap \{Y_t: c < t < \infty\} \neq \varnothing$. We define

$$\mu = \sup\{r: (a,r) \in P \text{ for some } a\}.$$

If $y, z \in Y_c$, then $(a,r) \in P$ implies

$$|(y+z)/2 - a|^2 = |y+z|^2/4 + |a|^2 - (y+z) \bullet a$$
$$= |y|^2/2 + |z|^2/2 - |y-z|^2/4 + |a|^2 - y \bullet a - z \bullet a$$
$$= (|y-a|^2 + |z-a|^2)/2 - |y-z|^2/4 \leq r^2 c^2 - r^2 |y-z|^2/(4\mu^2),$$

thus $(y+z)/2 \in Y_t$ with $t = [c^2 - |y-z|^2/(4\mu^2)]^{\frac{1}{2}}$, hence $y = z$. Subjecting $\mathbf{R}^n$ to a translation, we henceforth assume $Y_c = \{0\}$.

If $u \in \mathbf{R}^n$ and $|u| = 1$, then $\varepsilon > 0$ implies $\varepsilon u \notin Y_c$, hence there exists $(a,r) \in P$ with

$$|a|^2 \leq r^2 c^2 < |u\varepsilon - a|^2 = \varepsilon^2 + |a|^2 - 2\varepsilon u \bullet a, \quad u \bullet a \leq \varepsilon/2;$$

since P is compact it follows that

$$P \cap \{(a,r): |a| = rc \text{ and } u \bullet a \leq 0\} \neq \varnothing, \quad A \cap \{a: u \bullet a \leq 0\} \neq \varnothing.$$

Thus no $n-1$ dimensional plane separates 0 from the compact set A.

2.10.41. Jung's theorem. *If $S \subset \mathbf{R}^n$ and $0 < \operatorname{diam} S < \infty$, then S is contained in a unique closed ball with minimal diameter, which does not exceed*

$$[2n/(n+1)]^{\frac{1}{2}} \operatorname{diam} S.$$

Proof. We suppose S is compact and apply 2.10.40 with $P = S \times \{1\}$. Assuming $b = 0$, hence $S \subset \mathbf{B}(0,c)$, we choose

$$a_1, \ldots, a_{n+1} \in A = S \cap \{a: |a| = c\}$$

and nonnegative numbers $\lambda_1, \ldots, \lambda_{n+1}$ with

$$0 = \sum_{i=1}^{n+1} \lambda_i a_i \text{ and } 1 = \sum_{i=1}^{n+1} \lambda_i.$$

We infer for each $j \in \{1, \ldots, n+1\}$ that

$$2c^2 = \sum_{i=1}^{n+1} \lambda_i (2c^2 - 2a_i \bullet a_j) = \sum_{i=1}^{n+1} \lambda_i |a_i - a_j|^2$$
$$\leq \sum_{i \neq j} \lambda_i (\operatorname{diam} S)^2 = (1 - \lambda_j)(\operatorname{diam} S)^2,$$

and conclude by summation with respect to j that

$$(n+1) 2c^2 \leq n (\operatorname{diam} S)^2.$$

2.10.42. Corollary. $\mathscr{S}^m(A) \leq [2n/(n+1)]^{m/2} \mathscr{H}^m(A)$ *for* $A \subset \mathbf{R}^n$.

2.10.43. Kirszbraun's theorem. *If $S \subset \mathbf{R}^m$ and $f: S \to \mathbf{R}^n$ is Lipschitzian, then f has a Lipschitzian extension $g: \mathbf{R}^m \to \mathbf{R}^n$ with* $\operatorname{Lip}(g) = \operatorname{Lip}(f)$.

Proof. We suppose $\operatorname{Lip}(f) = 1$ and consider the class Ω of all those Lipschitzian extensions of f which map some subset of $\mathbf{R}^m$ into $\mathbf{R}^n$ and have the Lipschitz constant 1. By Hausdorff's maximal principle Ω has a maximal (with respect to inclusion) element $g: T \to \mathbf{R}^n$, where $T \subset \mathbf{R}^m$.

We will show that if $\xi \in \mathbf{R}^m \sim T$ there would exist $\eta \in \mathbf{R}^n$ with

$$|\eta - g(x)| \leq |\xi - x| \text{ whenever } x \in T,$$

hence $g \cup \{(\xi, \eta)\} \in \Omega$, and g would not be maximal in Ω. Thus we must prove that

$$\bigcap \{\mathbf{B}[g(x), |x - \xi|]: x \in T\} \neq \varnothing;$$

since these balls are compact it will suffice to verify that

$$\bigcap \{\mathbf{B}[g(x), |x - \xi|]: x \in F\} \neq \varnothing$$

for every finite subset F of T. For this purpose we apply 2.10.40 with

$$P = \{(g(x), |x - \xi|): x \in F\},$$

choose distinct points $x_1, \ldots, x_k \in F$ and positive numbers $\lambda_1, \ldots, \lambda_k$ with

$$g(x_i) \in A, \text{ hence } |b - g(x_i)| = |x_i - \xi| c \text{ for } i = 1, \ldots, k,$$

$$b = \sum_{i=1}^{k} \lambda_i g(x_i), \quad 1 = \sum_{i=1}^{k} \lambda_i,$$

and use the identity $2u \bullet v = |u|^2 + |v|^2 - |u-v|^2$ to obtain

$$0 = 2|\sum_i \lambda_i [g(x_i) - b]|^2 = 2 \sum_{i,j} \lambda_i \lambda_j [g(x_i) - b] \bullet [g(x_j) - b]$$
$$= \sum_{i,j} \lambda_i \lambda_j [|g(x_i) - b|^2 + |g(x_j) - b|^2 - |g(x_i) - g(x_j)|^2]$$
$$\geq \sum_{i,j} \lambda_i \lambda_j [c^2 |x_i - \xi|^2 + c^2 |x_j - \xi|^2 - |x_i - x_j|^2]$$
$$= \sum_{i,j} \lambda_i \lambda_j [2c(x_i - \xi) \bullet c(x_j - \xi) + (c^2 - 1)|x_i - x_j|^2]$$
$$= 2|c \sum_i \lambda_i (x_i - \xi)|^2 + (c^2 - 1) \sum_{i,j} \lambda_i \lambda_j |x_i - x_j|^2;$$

hence either $k = 1$ and $c = 0$ (because $\xi \neq x_1 \in T$), or $k > 1$ and $c \leq 1$; we conclude that $c \leq 1$, $b \in Y_1$.