AN INEQUALITY FOR REARRANGEMENTS

G. G. LORENTZ, University of Toronto

Let $f_1(x), f_2(x), \cdots$ denote positive measurable functions on $(0, 1)$ and $f_1^*(x), f_2^*(x), \cdots$ their equimeasurable decreasing rearrangements (see [1], [3]). For the work dealing with rearrangements, the following simple inequality is basic:

$$\int_0^1 f_1(x)f_2(x)dx \leqq \int_0^1 f_1^*(x)f_2^*(x)dx. \tag{1}$$

There are, however, also other combinations of $f_1, f_2, \cdots$ for which relations similar to (1) hold. One of these was given by Ruderman [2, Theorem II]. In this note we propose to determine, quite generally, necessary and sufficient conditions on a continuous function $\Phi(x, u_1, \cdots, u_n)$ defined for $0<x<1$, $u_k \geqq 0$, $k=1, 2, \cdots, n$, under which

$$\int_0^1 \Phi(x, f_1(x), \cdots, f_n(x))dx \leqq \int_0^1 \Phi(x, f_1^*(x), \cdots, f_n^*(x))dx \tag{2}$$

is satisfied for each set $f_k(x)$, $k=1, \cdots, n$, of positive bounded measurable functions on $(0, 1)$. (We assume the $f_k(x)$ bounded in order to insure the existence of both integrals in (2).)

In inequalities containing values of the function Φ at different points, we shall omit those of the arguments $x, u_1, \cdots, u_n$ which take the same but arbitrary values. For a set I of indices i, $1 \leqq i \leqq n$, we put $U_I = \{u_i\}_{i \epsilon I}$. We also put $U_I + U_I' = \{u_i + u_i'\}_{i\epsilon I}$, if $U_I' = \{u_i'\}$.

THEOREM. *In order that Φ satisfy (2) it is necessary and sufficient that Φ have the properties*

$$\Phi(u_i + h, u_j + h) - \Phi(u_i + h, u_j) - \Phi(u_i, u_j + h) + \Phi(u_i, u_j) \geqq 0, \tag{3}$$

$$\int_0^\delta \{\Phi(x - t, u_i + h) + \Phi(x + t, u_i) - \Phi(x + t, u_i + h) - \Phi(x - t, u_i)\}dt \geqq 0 \tag{4}$$

for all $0<x<1$, $u_k \geqq 0$, $k=1, \cdots, n$, $h>0$, $0<\delta<x$, $\delta<1-x$, and $i \neq j$. If Φ has continuous second partial derivatives with respect to all variables, conditions (3), (4) *are equivalent to*

$$\frac{\partial^2\Phi}{\partial u_i \partial u_j} \geqq 0, \tag{3a}$$

$$\frac{\partial^2\Phi}{\partial x \partial u_i} \leqq 0. \tag{4a}$$

Proof. Suppose $0<a<1$, $0<\delta<a$, $\delta<1-a$, $i \neq j$. Define $f_i(x) = u_i + h_i$ for

$x \leqq a-\delta$ and $a<x\leqq a+\delta$ and $f_i(x)=u_i$ for other x, $f_j(x)=u_j+h_j$ for $x\leqq a$, $f_j(x)=u_j$ for $x>a$, further $f_k(x)=u_k$, $0<x<1$ for k different from i and j. Then the inequality (2) reduces to

$$\int_0^{\delta}\{\Phi(a-t, u_i+h_i, u_j+h_j)-\Phi(a+t, u_i+h_i, u_j)-\Phi(a-t, u_j, u_j+h_j) + \Phi(a+t, u_i, u_j)\}dt \geqq 0.$$

Putting here $h_j=0$, we obtain (4). Dividing through by δ and making $\delta\to 0$, we obtain (3).

To prove that the conditions are sufficient, we first deduce from (3) that for any two disjoint groups of indices I, J and $h_i, h_j\geqq 0$,

$$(5)\quad \Phi(U_I+H_I, U_J+H_J)-\Phi(U_I+H_I, U_J)-\Phi(U_I, U_J+H_J)+\Phi(U_I, U_J)\geqq 0.$$

From (3) we have

$$\Phi(u_i+sh, u_j+h)-\Phi(u_i+sh, u_j)-\Phi(u_i+(s-1)h, u_j+h) + \Phi(u_i+(s-1)h, u_j)\geqq 0.$$

Adding these relations for $s=1, 2, \cdots, p$ we deduce

$$(6)\quad \Phi(u_i+ph, u_j+h)-\Phi(u_i+ph, u_j)-\Phi(u_i, u_j+h)+\Phi(u_i, u_j)\geqq 0.$$

Treating now the second argument in (6) in the same way we obtain, for positive integers p, q and $h_i=ph$, $h_j=qh$,

$$(7)\quad \Phi(u_i+h_i, u_j+h_j)-\Phi(u_i+h_i, u_j)-\Phi(u_i, u_j+h_j)+\Phi(u_i, u_j)\geqq 0.$$

An appeal to the continuity of Φ establishes (7) for arbitrary $h_i, h_j\geqq 0$.

To prove (5), let I' be the group consisting of I and the index k, which belongs neither to I nor to J. Then

$$(8)\quad \begin{aligned} &\Phi(U_{I'}+H_{I'}, U_J+H_J)-\Phi(U_{I'}+H_{I'}, U_J)-\Phi(U_{I'}, U_J+H_J)+\Phi(U_{I'}, U_J)\\ &\quad=\{\Phi(U_I+H_I, u_k+h_k, U_J+H_J)-\Phi(U_I+H_I, u_k+h_k, U_J)\\ &\qquad-\Phi(U_I, u_k+h_k, U_J+H_J)+\Phi(U_I, u_k+h_k, U_J)\}\\ &\qquad+\{\Phi(U_I, u_k+h_k, U_J+H_J)-\Phi(U_I, u_k+h_k, U_J)\\ &\qquad-\Phi(U_I, u_k, U_J+H_J)+\Phi(U_I, u_k, U_J)\}.\end{aligned}$$

Applying this relation we can, beginning with (7), prove (5) by induction with respect to the number of elements of I and J.

In the same way, we can generalize (4) to

$$(9)\quad \int_0^{\delta}\{\Phi(x-t, U_I+H_I)+\Phi(x+t, U_I)-\Phi(x+t, U_I+H_I) - \Phi(x-t, U_I)\}dt\geqq 0.$$

Replacing in identity (8) u_k by $x-t$, u_k+h_k by $x+t$, and combining (5) and (9), we obtain finally

$$(10)\quad \int_0^\delta \{\Phi(x-t, U_I+H_I, U_J+H_J) - \Phi(x-t, U_I, U_J+H_J) - \Phi(x+t, U_I+H_I, U_J) + \Phi(x+t, U_I, U_J)\}dt \geqq 0.$$

We can now prove (2) under the assumption that each of the functions $f_k(x)$ is a step-function, constant on each of the intervals $((s-1)/p, s/p)$, $s=1, \cdots, p$. For $1 \leqq s < p$ we consider the following *elementary operation* which gives a new set of functions $\bar{f}_k(x)$. We put $\bar{f}_k(x) = f_k(x)$ outside of $((s-1)/p, (s+1)/p)$; on $((s-1)/p, (s+1)/p)$, $\bar{f}_k(x)$ is the decreasing rearrangement of $f_k(x)$ on this interval. If I consists of the indices k for which $f_k(x)$ increases on $((s-1)/p, (s+1)/p)$, J of the indices for which $f_k(x)$ decreases, u_k is the smaller, u_k+h_k the larger of the two values of $f_k(x)$, then (10) with $x=s/p$, $\delta=1/p$ is exactly the inequality

$$\int_0^1 \Phi(x, f_1, \cdots, f_n)dx \leqq \int_0^1 \Phi(x, \bar{f}_1, \cdots, \bar{f}_n)dx.$$

By a finite number of elementary operations we can transform $f_1, \cdots, f_n$ into $f_1^*, \cdots, f_n^*$. This proves (2) in our particular case. In the general case we consider sequences $f_1^{(p)}, \cdots, f_1^{(p)}$, $p=1, 2, \cdots$ of uniformly bounded step-functions of our type such that $f_k^{(p)}(x) \to f_k(x)$ almost everywhere and pass to the limit $p \to \infty$ in the relation (2) for the $f_k^{(p)}$. This gives (2) in full generality.

It remains to show that (3) is equivalent to (3a) and (4) to (4a), if Φ has continuous second derivatives. If (4) holds, then for any i, $0<x<1$, $u_k \geqq 0$, there are arbitrary small $t>0$ with

$$\Phi(x+t, u_i+t) - \Phi(x-t, u_i+t) - \Phi(x+t, u_i) + \Phi(x-t, u_i) \leqq 0.$$

Dividing by $2t^2$ and making $t \to 0$, we obtain (4a). Conversely, from (4a) we deduce a relation stronger than (4), namely

$$(4b)\quad \Delta^2\Phi = \Phi(x+t, u_i+h) - \Phi(x+t, u_i) - \Phi(x, u_i+h) + \Phi(x, u_i) \leqq 0.$$

For if (4b) does not hold, there is a $c>0$ and a rectangle $R=(x, x+t; u_i, u_i+h)$ with side lengths t, h for which $\Delta^2\Phi \geqq cht$. Subdividing R, we obtain a sequence of rectangles with the same property which converge to a point (x^0, u_i^0). Then

$$\frac{\partial^2\Phi}{\partial x^0 \partial u_i^0} = \lim \frac{\Delta^2\Phi}{ht} \geqq c > 0,$$

which contradicts (4a). In the same way we treat the pair of relations (3), (3a).

Examples. The inequality (2) holds if $\Phi(u_1, \cdots, u_n) = u_1 \cdots u_n$. It holds for $\Phi = F(u_1+u_2+\cdots u_n)$ if and only if $F(u)$ is convex, that is $F(u+2h) - 2F(u+h) + F(u) \geqq 0$. For example, $F(u) = -\log u$ has this property. Writing (2) in this case for sums instead of integrals, we obtain Ruderman's inequality [2, Theorem II]

$$\prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk} \geqq \prod_{s=1}^{p}\sum_{k=1}^{n} a_{sk}^{*}, \tag{11}$$

where $a_{sk}\geqq 0$ and the a_{sk}^*, $s=1, \cdots, p$ are the a_{sk}, $s=1, \cdots, p$ arranged in order of decreasing magnitude.

References

1. G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1934.
2. H. D. Ruderman, Two new inequalities, this MONTHLY, 59, 1952, pp. 29–32.
3. A. Zygmund, Trigonometrical series, Warsaw-Lwow, 1935.

ON SUMS INVOLVING BINOMIAL COEFFICIENTS*

EMIL GROSSWALD, Institute for Advanced Study

In some problems of algebra† we are led to consider sums of the form $\sum_{\nu\geqq 0} k_\nu A(n,r,\nu)\, B(n,r,\nu)\, C(n,r,\nu,)$ where $A, B, C, \cdots$, are binomial coefficients, depending on ν and also on one or two other integral parameters, and where the summation proceeds up to the first value of ν, for which one of the factors vanishes. A certain number of such sums can be found in [1] and [3]. However, the sums computed in (4), do not seem to appear in the literature. They do not follow readily by the methods of [3], and a direct proof, or a proof by induction, seems rather difficult. In what follows, we give a simple proof of (4), using well-known properties of Legendre's polynomials $P_n(x)$ and of the hypergeometric function $F(a, b; c; x)$.

Let $P_n(x) = \sum_{r=0}^{n} a_r^{(n)} x^r$ be the nth Legendre polynomial, and let $P_n^{(r)}(x)$ be its rth derivative. Then, by Maclaurin's formula, $P_n(x) = \sum_{r=0}^{n}\{x^r P_n^{(r)}(0)/r!\}$ so that

$$P_n^{(r)}(0)/r! = a_r^{(n)}. \tag{1}$$

Here the values of $a_r^{(n)}$ are (see [2], p. 11)

$$\begin{aligned} a_r^{(n)} &= (-1)^{(n-r)/2}2^{-n}\binom{n+r}{n}\binom{n}{(n+r)/2} \quad \text{if} \quad n \equiv r \pmod 2 \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{2}$$

It also is known (see [4], pp. 61–62) that‡ the hypergeometric function

* This paper was prepared under contract with the Office of Naval Research.

† *E.g.* the study of the algebraic irreducibility of Legendre's polynomials in the field of rational numbers.

‡ The idea of this proof is due to Professor E. D. Rainville, who kindly suggested it to me in a letter.