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VISCOUS FINGERING: AN OPTIMAL BOUND ON THE GROWTH RATE OF THE MIXING ZONE*

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Abstract. We consider the flow of two immiscible fluids of different mobility in a porous medium. If the more mobile fluid displaces the other, a macroscopically sharp interface is unstable. By growing a network of fingers on a mesoscopic scale, the two phases mix on a macroscopic scale. We are interested in the evolution of this mixing zone. We show that the effect of a large but *finite* mobility ratio λ is strong enough to limit the growth rate of the mixing zone. This is done by rigorously deriving an a priori estimate for the Saffman–Taylor model. In this geometry of an infinite channel, the estimate essentially states that the mobility ratio λ itself (in the nondimensionalized setting with unit velocity imposed at infinity) is the optimal bound on the velocity by which the penetrating phase progresses in direction of the channel.

Since the introduction of diffusion-limited aggregation, various stochastic algorithms simulating this two-phase flow have been developed. The generated clusters, which correspond to the distribution of the highly mobile displacing phase, are fractal in the limiting case of $\lambda = \infty$ and "compact" for $\lambda = 1$. With support of numerical experiments and renormalization-group arguments, it had been conjectured that they eventually cross over from fractal to compact for all finite $\lambda \in (1, \infty)$. Our result may be interpreted as another confirmation of this conjecture.

Key words. magnetic fluids, pattern formation, evolution of microstructure

AMS subject classifications. 28A80, 76S05, 76T05, 82C27

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Introduction. We are interested in the flow of two immiscible fluids of different mobility in a porous medium. If the more mobile phase displaces the less mobile one, a macroscopically sharp interface is observed to be unstable. By forming a network of fingers on a mesoscopic scale, the two phases effectively mix on a macroscopic scale and the mixing zone grows in time (see for instance [31, pp. 261–267; 8]). The two-phase flow between closely spaced parallel sheets of glass (the Hele–Shaw cell) can to some extent be considered an experimental simulation of a two-dimensional two-phase flow in a porous medium [26].

A free boundary model. It seems natural to model the flow of two immiscible fluids in a Hele–Shaw cell by a two-dimensional free boundary problem for the interface — as long as the length scales of the patterns in the phase distribution are large compared to the spacing of the sheets of glass. We will follow this approach of Saffman and Taylor [26]. We assume that the two-dimensional flow domain Ω is an infinite strip

$$\Omega \;:=\; \{\, (x, y) \in \mathbb{R}^2 \,|\, y \in (0, \pi) \,\} \,.$$

Let s be the phase distribution; i.e.,

$$s(t, x, y) \in \{0, 1\}$$

indicates whether at time $t \in [0, \infty)$ and location $(x, y) \in \Omega$ we are in the more mobile

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phase ($s(t,x,y)=1$) or the less mobile phase ($s(t,x,y)=0$). Let

$$v(t,x,y)\in\mathbb{R}^2 \quad \text{and} \quad p(t,x,y)\in\mathbb{R}$$

denote, respectively, the fluid velocity and the pressure at time t and location (x,y). We assume that the flow is incompressible and that there is no flux across the lateral walls $\partial\Omega$. This is expressed in the fact that v has zero spatial divergence and that its normal component vanishes on the boundary

$$\text{div}\, v \;=\; 0 \;\text{ in } \Omega \quad \text{and} \quad v\cdot\nu \;=\; 0 \;\text{ on } \partial\Omega\,. \tag{1}$$

Since we expect v to be discontinuous at the interface, we have to interpret (1) in the distributional sense. The driving force is an imposed pressure gradient far downstream and upstream, which we prefer to state in terms of the velocity, namely,

$$v\to e \quad \text{for }\; x\to\pm\infty\,,$$

where $e:=(1,0)$ is the downstream-pointing unit vector. To be more precise: We actually suppose that at each time t, $v(t)-e$ is square integrable on the unbounded domain Ω:

$$v(t)-e \;\in\; L^2(\Omega) \quad \text{for all } t\in(0,\infty)\,. \tag{2}$$

We further assume that the phase distribution is advected by the flow, which is expressed by the transport equation

$$\partial_t s \;+\; \nabla s\cdot v \;=\; 0 \;\text{ in } \Omega\,.$$

Using (1), this equation can be written in conservation form,

$$\partial_t s \;+\; \text{div}\,[s\,v] \;=\; 0 \;\text{ in } \Omega, \tag{3}$$

and interpreted in the distributional sense. Observe that (1) implies that the normal component of v is continuous across the interface; (3) then states that the normal velocity of the interface is given by this normal component. Since we are interested in phase distributions which are such that the more mobile fluid displaces the less mobile, we always think of $s=1$ far upstream and $s=0$ far downstream. Last not least we assume that Darcy's law holds. Darcy's law (nondimensionalized) states that

$$\text{velocity} \;=\; -\,\text{mobility}\times\text{pressure gradient}\,.$$

In our case the mobility depends on the phase so that Darcy's law assumes the form

$$v \;=\; -\,(1+\lambda\,(1-s))\,\nabla p\,, \tag{4}$$

where $\lambda\in[1,\infty)$ is the mobility ratio. Observe that given the phase distribution $s(t)$ at a time t, the pressure $p(t)$ and thus the velocity $v(t)$ are determined as a solution of the elliptic problem (1), (2), (4). Thus (1)–(4) formally define an evolution of the phase distribution $s(t)$ in time or equivalently: an evolution of the interface.

Linear stability analysis. Consider the moving planar interface

$$s(t,x,y) \;=\; \left\{\begin{array}{ll} 1 & \text{for} \quad x<t \\ 0 & \text{for} \quad x>t \end{array}\right\}.$$

This s is a solution of (1)–(4). Reflecting the fact that this evolution of the phase distribution will not be observed in experiments, the linear stability analysis of s shows that it is mathematically highly unstable: Disturbances of this planar interface grow in time with a rate proportional to the inverse of their wavelength [26, 3]. More precisely, the growth rate of a small amplitude Fourier mode with wavenumber n is given by

$$\mu\, n, \quad \text{where} \quad \mu := \frac{\lambda - 1}{\lambda + 1} > 0. \tag{5}$$

Thus the linearization of our problem essentially behaves like the heat equation backward—with a hyperbolic instead of a parabolic scaling. In particular there is no bound on the growth rate of small-amplitude disturbances of the planar interface. Only the modeling of additional physical effects, such as a line tension for the interface, can provide such a bound. We are interested in the later stages of the evolution of a perturbed moving planar interface. Is there a bound on the growth rate of the mixing zone; more precisely: is there a bound on the velocity by which the penetrating phase $\{s = 1\}$ progresses in the x-direction? *We would like to convince the reader with an estimate that it is the effect of the finite viscosity ratio which limits the growth rate of the mixing zone.* This must be a nonlinear effect, as the linear stability analysis shows no qualitative difference between $\lambda < \infty$ and $\lambda = \infty$.

Numerical simulations with noise. In the case of the infinite mobility ratio $\lambda = +\infty$, the diffusion-limited aggregation (DLA) algorithm (introduced by [32]), which generates a random family of clusters $\{K(t)\}_{t\in\Lambda}$ on a grid which grow in discrete time $t \in \Lambda$, can be interpreted as a discretization of the above evolution problem for $\{s(t)\}_{t\in(0,\infty)}$ [23, 17]. The cluster $K(t)$ (which corresponds to the set $\{s(t) = 1\}$) scales like a fractal; it has a specific dimension [33, 7, 30] and specific dynamic properties [15, 16]. Stimulated by the success of DLA in qualitatively simulating the formation and dynamics of ramified pattern, more general stochastic algorithms have been developed (first in the context of dielectric breakdown [18]) and used as a discretization of our flow problem in the case of finite λ (a simple modification of DLA [27], a more sophisticated [5, 25, 24, 12, 13] and a deterministic algorithm, where noise is introduced by simulating random pore sizes [4, 11, 2, 6]). Supported by numerical results, it had been conjectured [11, 2, 6] that in the case of finite λ, the generated cluster eventually crosses over from fractal to "compact" in the limit $t \uparrow \infty$ or, equivalently, of vanishing grid size a and length of time step. We prefer to think in terms of the limit $a \downarrow 0$, since only in this case do the lateral boundary conditions scale appropriately. Recently, renormalization-group arguments in favor of a limiting compact behavior have been given [14, 19].

Statement of the a priori estimate. We will prove the following estimate for fixed $\lambda \in [1, \infty)$: For any $C > \lambda$, there exist an $\alpha > 0$ such that

$$\partial_t \left[\int_\Omega s(t,x,y)\, \exp(\alpha\,(x - C\,t))\, dx\, dy \right] \leq 0 \tag{6}$$

for all solutions s of (1)–(4).

Interpretation of the estimate. Let us draw a conclusion from this result: For any $C > \lambda$, the penetrating phase $\{s = 1\}$ progresses—apart from a volume which can be chosen arbitrarily small—with a velocity in the x-direction not bigger than C. More precisely: For a given initial phase distribution and any $\rho > 0$, there exists an $M < \infty$ such that the volume of the invading phase in the moving half channel $\{x > C\,t + M\}$ is less than ρ:

$$\int_{Ct+M}^{\infty}\int_0^{\pi} s(t,x,y)\,dx\,dy \;<\; \rho \quad \text{for all } t \in (0,\infty)\,.$$

Hence the mobility ratio λ itself is an upper bound (in our nondimensionalized setting with unit velocity imposed at infinity; see (2)) on the velocity by which the penetrating phase progresses in the x-direction. This result confirms the conjecture of "compact" behavior in the case of the finite mobility ratio.

Optimality of the estimate. The condition $C > \lambda$ in our a priori estimate is *optimal* in the following sense: If $\alpha > 0$ and $C < \infty$ are such that

$$\partial_t \left[\int_\Omega s(t,x,y)\,\exp(\alpha\,(x - C\,t))\,dx\,dy\right] \;\le\; 0$$

holds for *all* solutions of (1)–(4), then necessarily $C \ge \lambda$. This is a consequence of the existence of the Saffman–Taylor fingers [26]. Those form a one-parameter family of exact traveling-wave solutions of the free boundary problem with tip velocity (depending on their relative width) ranging in $(1,\lambda)$. Hence the mobility ratio λ is the optimal upper bound on the velocity by which the penetrating phase $\{s = 1\}$ progresses in the x-direction. We would like to interpret this result in the following way: *It is the effect of the finite mobility ratio which limits the growth rate of the mixing zone.* We conjecture that λ is also the *generic* bound on the velocity by which the cluster $\{K(t)\}_{t\in\Lambda}$ grows in the x-direction in the stochastic algorithms described above. This question is related to the issue whether some (mean) statistic properties of $\{K(t)\}_{t\in\Lambda}$ can be described by solutions of partial differential equations. In the case of the original DLA algorithm, it has been discovered numerically [1] that a specific level set of the mean occupancy agrees with the Saffman–Taylor finger of width $\frac{1}{2}$ (and hence tip velocity 2), provided that the time scale is chosen appropriately. (Time is measured in units decreasing with the grid size.) In [21], we will propose a mean-field approach for the finite mobility ratio in the physical time scale, that is, the time scale determined by the fixed velocity imposed at infinity (see (2)).

Extension. It is well known that the free boundary problem (1)–(4) is mathematically ill posed: the free boundary may develop a singularity in finite time [9, 10]. Hence it might be objected that we prove a result for solutions of an ill-posed problem—after all, there may be no solution for generic initial data. This is why we also investigated a singular perturbation of (1)–(4) which leads to a well-posed evolution problem and proved (6) with an α not depending on the perturbation parameter. The interested reader will find this rigorous analysis in [22].

Derivation of the estimate. For details and mathematical rigor, we refer the reader to [22] and give only a sketch here. The main ingredient for (6) is the L^2-estimate (12) with exponential weight

$$\omega_\alpha(x,y) \;:=\; \exp(\alpha\,x)$$

for the elliptic problem (1), (2), (4). (A related study of an elliptic equation in a semi-infinite strip can be found in [29].) For convenience, we eliminate the somewhat underdetermined pressure and state (1), (2), (4) in terms of v alone: For given $\{0,1\}$-valued s find a velocity field v such that $v-e\in L^2(\Omega)$ and

$$\left.\begin{aligned} \operatorname{div} v \;=\; 0 \;\text{ in }\Omega \quad\text{and}\quad v\cdot\nu \;=\; 0 \;\text{ on }\partial\Omega\,,\\ \operatorname{curl}\frac{v}{1+(\lambda-1)s} \;=\; 0 \;\text{ in }\Omega\,. \end{aligned}\right\} \tag{7}$$

Let us first observe that v can be expressed in terms of s with help of the Helmholtz projection Γ for the infinite strip Ω:

$$v-e \;=\; (\lambda-1)\Big(\operatorname{id}-\Gamma\Big)\Big(\operatorname{id}-\mu\,(1-2\,s)\,\Gamma\Big)^{-1} s\,e\,, \tag{8}$$

where the Atwood mobility ratio $\mu\in[0,1)$ is given in (5). We recall that Γ is the orthogonal projection on the curl-free vector fields with respect to $L^2(\Omega)$, the Hilbert space of square integrable vector fields on Ω. Hence (7) is equivalent to

$$\Gamma\,(v-e) \;=\; 0 \quad\text{and}\quad \Big(\operatorname{id}-\Gamma\Big)\left(\frac{v}{1+(\lambda-1)\,s}-e\right) \;=\; 0$$

and thus to

$$\Big(\operatorname{id}+(\lambda-1)\,\Gamma\,s\Big)\left(\frac{v}{1+(\lambda-1)\,s}-e\right) \;=\; -(\lambda-1)\,\Gamma\,s\,e\,,$$

which can be restated as

$$\begin{aligned} v-e &= (\lambda-1)\Big\{\,s\,e-(1+(\lambda-1)\,s)\Big(\operatorname{id}+(\lambda-1)\,\Gamma\,s\Big)^{-1}\Gamma\,s\,e\,\Big\}\\ &= (\lambda-1)\Big\{\,s\,e-(1+(\lambda-1)\,s)\,\Gamma\Big(\operatorname{id}+(\lambda-1)\,s\,\Gamma\Big)^{-1} s\,e\,\Big\}\\ &= (\lambda-1)\Big(\operatorname{id}-\Gamma\Big)\Big(\operatorname{id}+(\lambda-1)\,s\,\Gamma\Big)^{-1} s\,e\\ &= (\lambda-1)\Big(\operatorname{id}-\Gamma\Big)\Big(\operatorname{id}-\mu\,(1-2\,s)\,\Gamma\Big)^{-1} s\,e. \end{aligned}$$

Thanks to the above algebra, we may use the following estimate for Γ:

$$\left(\int_\Omega |\,\Gamma f\,|^2\,\omega_\alpha\right)^{1/2} \;\le\; \frac{1+(\frac{\alpha}{2})^2}{1-(\frac{\alpha}{2})^2}\left(\int_\Omega |f|^2\,\omega_\alpha\right)^{1/2} \quad \text{for } |\alpha|<2\,. \tag{9}$$

Inequality (9) can be shown as follows: It is equivalent to the estimate without weight

$$\left(\int_\Omega |\,\Gamma^{(\alpha)} f\,|^2\right)^{1/2} \;\le\; \frac{1+\alpha^2}{1-\alpha^2}\left(\int_\Omega |f|^2\right)^{1/2} \quad \text{for } |\alpha|<1 \tag{10}$$

for the conjugated operator $\Gamma^{(\alpha)}$ given by

$$\Gamma^{(\alpha)} f \;=\; \omega_\alpha\,\Gamma\,(\omega_{-\alpha}\,f)\,.$$

$\Gamma^{(\alpha)}$ has a simple representation in terms of the Fourier transform in the x-variable and the Fourier series in the y-variable:

$$\begin{aligned} &(\Gamma^{(\alpha)} f)(\xi,n)\\ &= \frac{1}{(\xi-i\alpha)^2+n^2}\; f(\xi,n)\cdot\begin{pmatrix} i\xi-\alpha\\ n\end{pmatrix}\begin{pmatrix} i\xi-\alpha\\ n\end{pmatrix}. \end{aligned}$$

Here ξ is the dual variable of x, the Fourier coefficients are numbered by n, and $\cdot$ denotes the scalar product on the two-dimensional vector space over the complex numbers. Equation (10) follows immediately from this representation.

Let us now infer from (8) and (9) the existence of an $\alpha_0 > 0$ and a continuous $C: [0, \alpha_0] \to [\lambda, \infty)$, depending only on $\lambda \in [1, \infty)$, such that

$$C(0) \;=\; \lambda \tag{11}$$

and with the following property: for any v and s satisfying (7) and $\alpha \in (0, \alpha_0]$ the estimate

$$\int_\Omega |v - e|^2 \, \omega_\alpha \;\leq\; (C(\alpha) - 1)^2 \int_\Omega s \, \omega_\alpha \tag{12}$$

holds. Indeed, from (9) we immediately deduce that there exist $\alpha_0 > 0$ and $C_0 < \infty$ such that

$$\int_\Omega \left| \Big(\mathrm{id} - \Gamma \Big) \Big(\mathrm{id} - \mu \, (1 - 2\, s) \, \Gamma \Big)^{-1} g \right|^2 \omega_{\alpha_0} \;\leq\; C_0^2 \int_\Omega |g|^2 \, \omega_{\alpha_0} \, . \tag{13}$$

But in order to obtain the optimal bound $C(0) = \lambda$, we need the second estimate

$$\int_\Omega \left| \Big(\mathrm{id} - \Gamma \Big) \Big(\mathrm{id} - \mu \, (1 - 2\, s) \, \Gamma \Big)^{-1} s \, f \right|^2 \;\leq\; \int_\Omega |f|^2 \, , \tag{14}$$

which is shown as follows: Define g by

$$\Big(\mathrm{id} - \mu \, (1 - 2\, s) \, \Gamma \Big) \, g \;=\; s \, f \, ,$$

multiply this identity by $g + \mu \, (1 - 2\, s) \, \Gamma \, g$ and integrate over Ω

$$\int_\Omega (g - \mu \, (1 - 2\, s) \, \Gamma \, g) \cdot (g + \mu \, (1 - 2\, s) \, \Gamma \, g) \;=\; \int_\Omega s \, f \cdot (g - \mu \, \Gamma \, g) \, .$$

The left-hand side can be written as

$$\begin{aligned}
&\int_\Omega (g - \mu \, (1 - 2\, s) \, \Gamma \, g) \cdot (g + \mu \, (1 - 2\, s) \, \Gamma \, g) \\
&= \int_\Omega |g|^2 \;-\; \mu^2 \int_\Omega |\, \Gamma \, g \,|^2 \\
&= \int_\Omega |\, (\mathrm{id} - \Gamma) \, g \,|^2 \;+\; (1 - \mu^2) \int_\Omega |\, \Gamma \, g \,|^2 \, ,
\end{aligned}$$

whereas the right-hand side is estimated as follows:

$$\begin{aligned}
&\int_\Omega s \, f \cdot (g - \mu \, \Gamma \, g) \\
&\leq \left(\int_\Omega |f|^2 \right)^{1/2} \left(\int_\Omega |\, g - \mu \, \Gamma \, g \,|^2 \right)^{1/2} \\
&= \left(\int_\Omega |f|^2 \right)^{1/2} \left(\int_\Omega |\, (\mathrm{id} - \Gamma) \, g \,|^2 \;+\; (1 - \mu)^2 \int_\Omega |\, \Gamma \, g \,|^2 \right)^{1/2} .
\end{aligned}$$

Because of $1-\mu^2 \geq (1-\mu)^2$, we obtain in particular

$$\left(\int_\Omega |\,(\mathrm{id}-\Gamma)\,g\,|^2\right)^{1/2} \;\leq\; \left(\int_\Omega |f|^2\right)^{1/2},$$

which establishes (14). By complex interpolation of (13) (for $g = s\,f$) and (14) (for complex interpolation, see, for instance, [28, Chap. V]) we obtain for all $\alpha \in [0, \alpha_0]$ the estimate

$$\int_\Omega \left|(\lambda-1)\Big(\mathrm{id}-\Gamma\Big)\Big(\mathrm{id}-\mu\,(1-2\,s)\,\Gamma\Big)^{-1} s\,f\right|^2 \omega_\alpha \;\leq\; (C(\alpha)-1)^2 \int_\Omega |f|^2\,\omega_\alpha \tag{15}$$

with

$$C(\alpha) \;:=\; 1 \,+\, (\lambda-1)\,C_0^{\frac{\alpha}{\alpha_0}}\,.$$

We apply (15) to $f = s\,e$ and so infer (12) from (8).

Now we are in the situation to prove (6). We assume that all integrals exist (and refer the reader once again to [22] for mathematical rigor). We multiply (3) by

$$\exp(\alpha\,(x - C(\alpha)\,t))$$

and obtain after partial integration

$$\begin{aligned}
&\partial_t\left[\int_\Omega s(t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right]\\
&= \alpha\,(1-C(\alpha))\int_\Omega s(t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\\
&\quad + \alpha\int_\Omega [s\,(v-e)\cdot e](t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\,.
\end{aligned}$$

Let us apply the Cauchy–Schwarz inequality to the last integral of the right-hand side:

$$\begin{aligned}
&\left|\int_\Omega [s\,(v-e)\cdot e](t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right|\\
&\leq \left(\int_\Omega s(t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right)^{1/2}\\
&\quad \times \left(\int_\Omega |v(t,x,y)-e|^2\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right)^{1/2}.
\end{aligned}$$

Consider the second factor of the right-hand side. According to (1), (2), (4), $v = v(t)$ solves the elliptic problem (7) with $s = s(t)$; we thus obtain from (12)

$$\begin{aligned}
&\left(\int_\Omega |v(t,x,y)-e|^2\,\exp(\alpha\,(x-C(\alpha)\,t)\,dx\,dy\right)^{1/2}\\
&\leq (C(\alpha)-1)\left(\int_\Omega s(t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right)^{1/2}.
\end{aligned}$$

Hence we have

$$\partial_t\left[\int_\Omega s(t,x,y)\,\exp(\alpha\,(x-C(\alpha)\,t))\,dx\,dy\right] \;\leq\; 0.$$

Together with (11), this proves our a priori estimate (6).

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