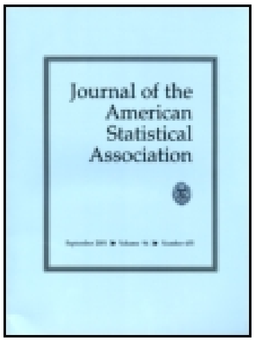

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RECTANGULAR CONFIDENCE REGIONS FOR THE MEANS OF MULTIVARIATE NORMAL DISTRIBUTIONS*

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For rectangular confidence regions for the mean values of multivariate normal distributions the following conjecture of O. J. Dunn [3], [4] is proved: Such a confidence region constructed for the case of independent coordinates is, at the same time, a conservative confidence region for any case of dependent coordinates. This result is based on an inequality for the probabilities of rectangles in normal distributions, which permits one to factor out the probability for any single coordinate.

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1. INTRODUCTION

IN TESTING the vector of mean values of a multivariate normal distribution, the common procedure is to apply Hotelling's T^2-statistic. The corresponding confidence regions for the vector of mean values, obtained by this procedure, have the shape of ellipsoids. (See e.g. T. W. Anderson [2, Section 5.3].) For an experimenter, however, the mathematical expression and the shape of ellipsoids is too complicated and difficult to imagine. Moreover, this procedure gives only a simultaneous confidence region for all mean values, but it does not yield any reasonable clear-cut confidence statements for the individual mean values separately.

Thus we are led to the problem of finding rectangular confidence regions, which are free of the mentioned difficulties and have rather attractive properties from the practical point of view.

A survey of several procedures for finding rectangular confidence regions has been given by O. J. Dunn [3], [4]. However, the "best" of these procedures, yielding the shortest confidence intervals and being based on certain inequalities for normal probabilities, and on the confidence intervals for independent variables, has been established only for some special cases: for the case of two-dimensional or three-dimensional variables, and for the case where the correlation coefficients ρ_{ij} have the special structure $\rho_{ij}=b_ib_j$.

The aim of the present paper is to show the validity of this "best" procedure generally.

2. AN INEQUALITY FOR PROBABILITIES OF RECTANGLES

Theorem 1. Let $X=(X_1, X_2, \cdots, X_k)$ be the vector of random variables having the k-dimensional normal distribution with zero means, arbitrary variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$, and an arbitrary correlation matrix $R=\{\rho_{ij}\}$. Then, for any positive numbers $c_1, c_2, \cdots, c_k$,

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$$P(|X_1| \leqq c_1, |X_2| \leqq c_2, \cdots, |X_k| \leqq c_k) \geqq P(|X_1| \leqq c_1)\cdot P(|X_2| \leqq c_2, \cdots, |X_k| \leqq c_k). \tag{1}$$

Proof. First, let the variables $X_1, X_2, \cdots, X_k$ have a non-singular distribution, and let $f(x_1, x_2, \cdots, x_k)$ be their density. Similarly, let $f(x_1)$ and $f(x_2, \cdots, x_k)$ denote the corresponding marginal densities, and $f(x_2, \cdots, x_k | x_1)$ the conditional density for $X_1 = x_1$. Put

$$F(c_1) = \int_{-c_1}^{c_1}\int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k} f(x_1, x_2, \cdots, x_k)dx_1dx_2\cdots dx_k - \int_{-c_1}^{c_1}\int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k} f(x_1)f(x_2, \cdots, x_k)dx_1dx_2\cdots dx_k.$$

It is easy to see that

$$\begin{aligned} F(c_1) &= 2\int_0^{c_1}\int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k}[f(x_1, x_2, \cdots, x_k) \\ &\qquad - f(x_1)f(x_2, \cdots, x_k)]dx_1dx_2\cdots dx_k \\ &= 2\int_0^{c_1} f(x_1)\left\{\int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k}[f(x_2, \cdots, x_k | x_1)\right. \\ &\qquad \left. - f(x_2, \cdots, x_k)]dx_2\cdots dx_k\right\} dx_1. \end{aligned}$$

Therefore

$$\frac{\partial F(c_1)}{\partial c_1} = 2f(c_1)\int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k}[f(x_2, \cdots, x_k | c_1) - f(x_2, \cdots, x_k)]dx_2\cdots dx_k.$$

Now, we shall investigate the behavior of the function

$$G(c_1) = \int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k}[f(x_2, \cdots, x_k | c_1) - f(x_2, \cdots, x_k)]dx_2\cdots dx_k,$$

applying T. W. Anderson's Corollary 2 in [1] which asserts the following: If X is a random vector with density $g(x)$ such that $g(x) = g(-x)$ and the set $\{x;\ g(x) \geqq u\}$ is convex for every non-negative u, and if E is a convex set, symmetric about the origin, y is a vector and k a number, $0 \leqq k \leqq 1$, then $P\{X + ky \in E\} \geqq P\{X + y \in E\}$.

Continuing our proof, put finally

$$H(c_1) = \int_{-c_2}^{c_2}\cdots\int_{-c_k}^{c_k} f(x_2, \cdots, x_k | c_1)dx_2\cdots dx_k, \tag{2}$$

and note that $f(x_2, \cdots, x_k | c_1)$ is the density of a normal distribution with mean values $\rho_{12}\sigma_2\sigma_1^{-1}c_1, \cdots, \rho_{1k}\sigma_k\sigma_1^{-1}c_1$, and with some variances and correlation coefficients not depending on c_1. Thus the density $f(x_2, \cdots, x_k | c_1)$ is obtained by "shifting" the density $f(x_2, \cdots, x_k | 0)$. Obviously, $f(x_2, \cdots, x_k | 0)$ and the

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integration region in (2) satisfy the assumptions of Anderson's Corollary 2, which shows that $H(c_1)$ is a decreasing function of c_1 (except the case $\rho_{12} = \cdots = \rho_{1k} = 0$, in which $H(c_1)$ is constant; this case may be omitted in the sequel, since the assertion of Theorem 1 is here obvious),

Thus $G(c_1)$ is also decreasing, and, taking into account $\lim_{c_1 \to \infty} H(c_1) = 0$, that is $\lim_{c_1 \to \infty} G(c_1) < 0$, we see that the following two cases may occur: either $G(c_1) \leqq 0$ for all c_1, $0 \leqq c_1 < \infty$, or there exists a c^* such that $G(c_1) > 0$ for $0 \leqq c_1 < c^*$, $G(c_1) < 0$ for $c^* < c_1 < \infty$. The same inequalities are then true for the derivative $\partial F(c_1)/\partial c_1$.

Thus, in the second case, $F(c_1)$ is increasing for $0 \leqq c_1 < c^*$ and decreasing for $c^* < c_1 < \infty$. Moreover,

$$F(0) = 0, \qquad \lim_{c_1 \to \infty} F(c_1) = 0, \tag{3}$$

so that $F(c_1) \geqq 0$ for all c_1, $0 \leqq c_1 < \infty$.

In the first case, $F(c_1)$ would be decreasing, but this is clearly impossible in view of (3).

The inequality (1) is thus proved provided the distribution of $X_1, X_2, \cdots, X_k$ is non-singular. If their distribution is a singular one, it may be approximated by a sequence of non-singular distributions; hence, by an obvious passage to the limit, the validity of (1) can be established in general.

Remark 1. The following conjecture seems to be very plausible, and I hope to publish its proof later. Let $p(\lambda) = P(|X_1| \leqq c_1, \cdots, |X_k| \leqq c_k)$, where $(X_1, \cdots, X_k)$ is the vector described in Theorem 1, except that the correlation coefficients ρ_{1j}, $2 \leqq j \leqq k$, are replaced by $\lambda\rho_{1j}$. Then $p(\lambda)$ is an increasing function of λ, $0 \leqq \lambda \leqq 1$.

By induction we can immediately prove the following

Corollary 1. Under the assumptions of Theorem 1 we have

$$P(|X_1| \leqq c_1, \cdots, |X_k| \leqq c_k) \geqq \prod_{i=1}^{k} P(|X_i| \leqq c_i). \tag{4}$$

This result was obtained by O. J. Dunn [3] for the following special cases: for $k = 2$ or 3, and for $\rho_{ij} = b_i b_j$, $1 \leqq i, j \leqq k$, $i \neq j$, with $0 < b_i < 1$, $1 \leqq i \leqq k$.

It might also be mentioned here that the following, in some sense analogous, one-sided result was found by D. Slepian [11] (see also [7], p. 805): Let the vector X have, under P_R, the normal distribution with zero means, unity variances and correlation matrix $R = \{\rho_{ij}\}$; and let it have, under P_K, the same distribution except that its correlation matrix is now $K = \{\kappa_{ij}\}$. If $\rho_{ij} \geqq \kappa_{ij}$ for all $i \neq j$, then

$$P_R\{X_1 \leqq c_1, \cdots, X_k \leqq c_k\} \geqq P_K\{X_1 \leqq c_1, \cdots, X_k \leqq c_k\}.$$

3. CONFIDENCE RECTANGLES IN THE CASE OF KNOWN VARIANCES

Let us consider a random sample of n vectors $Y_\nu = (Y_{1\nu}, \cdots, Y_{k\nu})$, $\nu = 1, \cdots, n$, where each Y_ν has the same normal distribution with unknown mean values $\mu_1, \cdots, \mu_k$ and known variances $\sigma_1^2, \cdots, \sigma_k^2$. Then the variables $X_i = n^{1/2}(\bar{Y}_i - \mu_i)/\sigma_i$, $i = 1, \cdots, k$, with

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$$\overline{Y}_i = n^{-1} \sum_{\nu=1}^{n} Y_{i\nu},$$

satisfy the assumptions of Theorem 1 and Corollary 1 with $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = 1$.

Wishing to find a confidence rectangle for $\mu_1, \cdots, \mu_k$ with the confidence level $1-\alpha$, we may determine $c_1, \cdots, c_k$ such that the right-hand side of (4) equals $1-\alpha$, and the desired confidence rectangle is then

$$\overline{Y}_i - c_i\sigma_i n^{-1/2} \leqq \mu_i \leqq \overline{Y}_i + c_i\sigma_i n^{-1/2}, \qquad i = 1, \cdots, k. \tag{5}$$

In other words, we may always act as if all coordinates of the vectors Y_ν were independent. For any case of dependent coordinates (4) shows that the confidence level of (5) can never be less than $1-\alpha$.

Usually we shall put $c_1 = c_2 = \cdots = c_k = c_\alpha$ (say), so that c_α will be chosen to satisfy $\Phi(c_\alpha) = \frac{1}{2}[1+(1-\alpha)^{1/k}]$, where Φ is the standardized normal distribution function.

4. AN EXTENSION OF THE INEQUALITY FOR PROBABILITIES OF RECTANGLES

Theorem 2. Let us consider the following two probability distributions P, P_1 of random variables $Z_1, \cdots, Z_k, s$. Under P, let the vector $Z = (Z_1, \cdots, Z_k)$ have the k-dimensional normal distribution with zero means, arbitrary variances $\sigma_1^2, \cdots, \sigma_k^2$, and an arbitrary correlation matrix $R = \{\rho_{ij}\}$. Under P_1, let $Z = (Z_1, \cdots, Z_k)$ have the same distribution with the only exception that Z_1 is now independent of $Z_2, \cdots, Z_k$ (i.e. $\rho_{12}, \rho_{13}, \cdots, \rho_{1k}$ are replaced with 0). Finally suppose that s is a positive random variable, which is independent of $Z_1, \cdots, Z_k$ and has the same distribution both under P and P_1. Then, for any positive constants $c_1, \cdots, c_k$,

$$\begin{aligned} (6) \quad & P\left(\frac{|Z_1|}{s} \leqq c_1, \frac{|Z_2|}{s} \leqq c_2, \cdots, \frac{|Z_k|}{s} \leqq c_k\right) \\ & \geqq P_1\left(\frac{|Z_1|}{s} \leqq c_1, \frac{|Z_2|}{s} \leqq c_2, \cdots, \frac{|Z_k|}{s} \leqq c_k\right) \\ & \geqq P\left(\frac{|Z_1|}{s} \leqq c_1\right)\cdot P\left(\frac{|Z_2|}{s} \leqq c_2, \cdots, \frac{|Z_k|}{s} \leqq c_k\right). \end{aligned}$$

Proof. By Theorem 1 we obtain for conditional probabilities

$$\begin{aligned} (7) \quad P(|Z_1| \leqq c_1 s, |Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s \mid s) \geqq P_1(&|Z_1| \leqq c_1 s, \\ &|Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s \mid s). \end{aligned}$$

Since the distribution of s is the same both under P and P_1, we may take expectations with respect to this distribution on both sides of (7), and the first inequality in (6) follows.

For proving the second inequality in (6), we make use of an inequality given by A. W. Kimball [9], namely $E\{F(s)\cdot G(s)\} \geqq E\{F(s)\}\cdot E\{G(s)\}$, which is valid for non-negative increasing functions F, G of a random variable s. Applying it in our case, we have

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$$\begin{aligned} P_1(|Z_1| \leqq c_1 s, |Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s) \\ = E\{P_1(|Z_1| \leqq c_1 s, |Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s \mid s)\} \\ = E\{P(|Z_1| \leqq c_1 s \mid s) \cdot P(|Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s \mid s)\} \\ \geqq E\{P(|Z_1| \leqq c_1 s \mid s)\} \cdot E\{P(|Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s \mid s)\} \\ = P(|Z_1| \leqq c_1 s) \cdot P(|Z_2| \leqq c_2 s, \cdots, |Z_k| \leqq c_k s). \end{aligned}$$

Corollary 2. Let the distribution P of the vector $Z=(Z_1, \cdots, Z_k)$ be the same as given in Theorem 2. Further, under the distribution P_k, let $Z=(Z_1, \cdots, Z_k)$ have the k-dimensional normal distribution with zero means, the same variances $\sigma_1^2, \cdots, \sigma_k^2$, and all coordinates $Z_1, \cdots, Z_k$ independent (i.e. all ρ_{ij}, $i \neq j$, are replaced with 0). Finally, let s be a positive random variable, which is independent of $Z_1, \cdots, Z_k$ and has the same distribution both under P and P_k. Then

$$(8) \quad P\left(\frac{|Z_1|}{s} \leqq c_1, \cdots, \frac{|Z_k|}{s} \leqq c_k\right) \geqq P_k\left(\frac{|Z_1|}{s} \leqq c_1, \cdots, \frac{|Z_k|}{s} \leqq c_k\right) \geqq \prod_{i=1}^{k} P\left(\frac{|Z_i|}{s} \leqq c_i\right).$$

Proof. The first inequality in (8) is proved similarly to the corresponding inequality in (6), replacing P_1 in (7) by P_k. The second inequality in (8) follows by induction from (6), putting there P_k in place of P.

Let us mention that O. J. Dunn [3] proved the first inequality in (8) for the special case $k=2$ or 3, and the inequality between the first and the last terms in (8) for the special case of correlation coefficients $\rho_{ij}=b_i b_j$, $1 \leqq i, j \leqq k$, $i \neq j$, with $0<b_i<1$, $1 \leqq i \leqq k$. A one-sided analogue of the latter inequality was obtained, also for the same case $\rho_{ij}=b_i b_j$, by C. W. Dunnett and M. Sobel [6].

5. CONFIDENCE RECTANGLES IN THE CASE OF UNKNOWN BUT EQUAL VARIANCES

Theorem 2 and Corollary 2 are useful for the following experimental situation: We observe n vectors $Y_\nu=(Y_{1\nu}, \cdots, Y_{k\nu})$, $\nu=1, \cdots, n$, as in Section 3, and we suppose in addition $\sigma_1^2=\sigma_2^2= \cdots =\sigma_k^2=\sigma^2$ (say), where this common variance σ^2 is now unknown. We may estimate σ^2 by the sample variance

$$s_g^2 = (n-1)^{-1} \sum_{\nu=1}^{n} (Y_{g\nu} - \overline{Y}_g)^2,$$

where g is some fixed index chosen from $1, \cdots, k$, and then the variables $Z_i=n^{1/2}(\overline{Y}_i-\mu_i)$, $i=1, \cdots, k$, and $s=s_g$ satisfy the assumptions of Theorem 2 and Corollary 2 (see [2, Theorem 3.3.2]). The confidence rectangle for $\mu_1, \cdots, \mu_k$ is now

$$(9) \qquad \overline{Y}_i - c_i s_g n^{-1/2} \leqq \mu_i \leqq \overline{Y}_i + c_i s_g n^{-1/2}, \qquad i=1, \cdots, k,$$

and we can make statements concerning its confidence level analogous to those in Section 3.

In the last-mentioned situation, the middle probability in (8) is given by the k-dimensional Student distribution with zero correlations and $n-1$ degrees of

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freedom (see, e.g., [7], p. 806). Critical values $c_1 = c_2 = \cdots = c_k = c_\alpha$ (say) for this distribution may be found in K. C. S. Pillai and K. V. Ramachandran [10], reproduced in O. J. Dunn [4], for $\alpha = 0.05$, $1 \leqq k \leqq 8$, and selected degrees of freedom. More comprehensive tables were recently prepared by O. J. Dunn and F. J. Massey [5] for $\alpha = 0.50, 0.40, 0.30, 0.20, 0.10, 0.05, 0.025, 0.01$, $k = 2, 6, 10, 20$, and $n-1 = 4, 10, 30, \infty$. (This paper [5] contains also a survey of related topics and tables.)

Somewhat worse, but more easily available, method is to find c_i's such that the last term in (8) equals $1-\alpha$, which can be done by applying the common Student distribution with $n-1$ degrees of freedom. The paper [5] contains also the values $c_1 = c_2 = \cdots = c_k = c_\alpha$ (say) found by this procedure, for the same cases as mentioned above.

Remark 2. O. J. Dunn [3], [4] suggested also the use of a "pooled" estimate

$$s^2 = k^{-1} \sum_{g=1}^{k} s_g^2$$

in place of s_g^2, keeping, however, $n-1$ degrees of freedom. Now, it is known that the distributions of this s^2 under P, P_1, and P_k are different (see [2, Theorem 3.3.2]), and even $(n-1)s^2/\sigma^2$ may not have a χ^2 distribution. Thus our Theorem 2 and Corollary 2 can not be applied; more precisely, the inequalities between the first and the middle terms in (6) and in (8) can not be established by our method. Still, it is easy to show, by a proof similar to that given above, that the inequalities between the first and the last terms in (6) and in (8) remain true for this s^2.

Remark 3. The described procedure is applicable only in the case of equal variances, $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$. Therefore it would be useful to prove the analogues of Theorem 2 and Corollary 2 in which the variable s in the i-th coordinate would be replaced by s_i. I hope to publish this result soon also.

6. NUMERICAL COMPARISON OF CRITICAL VALUES

The following Table 1 may give the reader some idea of the closeness of the two procedures obtained from (8) in Section 4. In this table we present several critical values $c_\alpha = c_1 = \cdots = c_k$ for $\alpha = 0.05$; for each k, in each double-column, the first columns contain those c_α (reproduced from [10] or [4]) which make the middle probability in (8) equal to 0.95, and analogously for the second columns

TABLE 1. COMPARISON OF TWO KINDS OF CRITICAL VALUES FOR $\alpha = 0.05$

$n-1$ \ k	2		5		8	
5	3.09	3.15	3.78	4.01	4.14	4.50
10	2.61	2.63	3.10	3.16	3.35	3.43
15	2.47	2.48	2.91	2.94	3.12	3.17
20	2.41	2.42	2.82	2.84	3.02	3.05
∞	2.23	2.24	2.57	2.57	2.73	2.73

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and the last probability in (8) (computed from [12]). It is seen that the critical values of the two procedures are very close, unless k is large and the number of degrees of freedom small. Naturally, the two values for each k in the last row should be equal (thus it seems that the reproduced value 2.23 for $k=2$ is erroneous).

These conclusions of ours are also confirmed by more extended tables of the both kinds of c_α's given in [5].

7. DISCUSSION

As was pointed out by O. J. Dunn [3], [4], if the variances are known but we know nothing about the correlations, the present procedure is the "best" one and yields the shortest confidence intervals. As a matter of fact, the confidence rectangle must be chosen to be valid in particular for independent variables, but then, by our results, it is valid for any dependent variables.

Let us compare briefly the procedure described here and the procedure derived from the "classical" confidence ellipsoids. Having such an ellipsoid of the confidence level $1-\alpha$, one may circumscribe a rectangular region around it. Though the ellipsoids for different correlation matrices are different, it is easy to see that the circumscribed rectangles coincide; they have always a confidence level $>1-\alpha$. Moreover, if the rectangle has for independent variables the confidence level (say) $1-\beta>1-\alpha$, then, by our results, it has a confidence level $\geqq 1-\beta$ for any kind of dependent variables. Thus this rectangular confidence region is unnecessarily large, and it may be made smaller.

The validity of the "best" procedure described in this paper is now proved only for the variances known, or unknown but equal. For the variances unknown and unequal we may apply the procedure based on confidence ellipsoids, or, better, the procedure based on the Bonferroni inequality (for details see [3], [4]; also [8] is related to this topic); the latter procedure is in most cases very close to the "best" one.

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