MAT 477HF — Fall Semester Syllabus 2017-18
PRIMARY SOURCES

Weeks

- W1 Introduction and overview of optimal transport [9][10] [11] [12] [13]. Decide who presents what.

- W2? L. Kantorovich. On the translocation of masses. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **37**, 199–201 (1942).

  T.C. Koopmans. Optimum utilization of the transportation system. *Econometrica (Supplement)*, 17:136–146, 1949.

  Kantorovich and Koopmans shared the 1975 Nobel Prize in Economics for these. The mathematics is in the Kantorovich paper, the economics in Koopmans. Both were trained as mathematicians, though Koopmans also had a degree in physics. A short modern proof appears in 1.6.3 of [11]. An unexpected application to stable matching was found in 3.1 of Shapley and Shubik A5. Any background in real and functional analysis or linear programming would be a plus.

- W3? R.T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math.* **17**, 497–502 (1966).

  Along with Theorem 1 (see also §6 of [10] for additional intuition), it would be good if the presenter could cover some basic background on convex functions and their Legendre transforms [1, Chapter 8.3] and/or Alexandrov's theorem on second differentiability of convex functions (statement from 6.4 of [4]; proof from Theorem 3.2 sketched in [9]).

- W4 Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C.R. Acad. Sci. Paris Sér. I Math.* **305**, 805–808 (1987).

  This short though difficult French paper proves a nice theorem with a lot of nice connections, to e.g. Hodge theory. The basic argument got expanded in English [2] but it is probably easier to follow in one of [5, 8] or the following paper by Caffarelli. I recommend these two presenters coordinate to divide up the proofs and the applications.

- W5 L. Caffarelli. Allocation maps with general cost functions. In P. Marcellini et al, editors, *Partial Differential Equations and Applications*, pages 29–35. Dekker, New York, 1996.

W6 A.D. Aleksandrov. Existence and uniqueness of a convex surface with a given integral curvature. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **35**, 131–134 (1942).

Spectacular, but not easy. Some background in geometry (the Gauss map, curvature) is a plus. The uniqueness argument is spelled out in more detail in:

W7 R.J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80**, 309–323 (1995).

Brenier's theorem from Aleksandrov and Rockafellar's perspective.

W8 S. Alesker, S. Dar and V. Milman. A remarkable measure preserving diffeomorphism between two convex bodies in $\mathbf{R}^n$. *Geom. Dedicata* **74** (1999) 201–212.

More than you ever wanted to know about inequalities governing vector sums of convex bodies. The precursor is Brunn-Minkowski (Remark 2.4 of [6]); domination the geometric by the arithemetic mean gone bananas.

W9 D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. **182** (2004) 307–332.

Combines entropy and transportation to recover this fundamental inequality in analysis. Although the paper is long, the key theorem and its proof are more succinctly covered on pages 200-203 of [12].

W10 A. Aleksandrov. Smoothness of the convex surface of bounded Gaussian curvature. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **36** (1942) 195–199.

Regularity (smoothness) theorems are hard, but rewarding. This paper was ahead of its time; on of the more challenging works we cover this semester. Uses a geometrical argument to show the surface constructed in W6 above is actually differentiable.

W11 R. Myerson. Optimal Auction Design. *Math. Oper. Res.* **6** (1981) 58-73.

A seller plans to auction off an object. Potential buyers have different private valuations for the object; the seller knows only a statistical (probablistic) description of their likely valuations. What rules should the auction follow to maximize the seller's expected revenue?

W12 F. Otto. Viscous fingering: an optimal bound on the growth rate of the mixing zone. *SIAM J. Appl. Math.* **57** (1997) 982–990.

Derives an inequality which limits the rate of mixing between two immiscible fluids at the unstable interface as the more mobile one displaces the other under pressure but forms fingers as the displacement occurs in a porous medium (or a channel between two glass plates). The presenter should have a basic comfort level with partial differential equations, ideally those which arise in the study of fluids.

W13 R. Latala and D. Matlak. Royen's proof of the Gaussian correlation inequality. In *Geometric Aspects of Functional Analysis* (Cham: Springer, 2017) 265–275.

2014 proof of a long-standing and well-known conjecture, stating that the Gaussian mass of the intersection of any pair of convex symmetric sets A and B dominates the product of their Gaussian masses. In other words, convex symmetric events are positively correlated under Gaussian measure. This presentation may take place on Dec 7 or 8.

Alternates (To be presented if time allows)

A2 D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage, *Amer. Math. Monthly* **69** (1962) 9–15.

Introduced the stable marriage problem and solved it using the deferred acceptance algorithm assuming no transfers between spouses.

A4 L.S. Shapley and M. Shubik. The assignment game I: The core. *Internat. J. Game Theory* **1** (1972) 111–130.

Showed that when utility is transferable between husbands and wives, the stable marriage problem becomes an optimal transport problem.

A5 G.G. Lorentz. An inequality for rearrangements, *Amer. Math. Monthly* **60** (1953) 176–179. (Also Appendix B of [7].)

Assortativity of 1D matching (Becker / Spence / Mirrlees Nobel prizes).

A6 H. Hotelling. Stability in competition. *Econom. J.* **39** (1929) 41–57.

Why different political parties end up sharing the same policies.

A7 H. Sonnenschein. Price dynamics based on the adjustment of firms. *Amer. Econom. Rev.* **72** (1982) 1088–1096. (c.f. Ch 8 of [11])

Unexpectedly anticipates steepest descent wrt transportation metric.

MAT 477H — Fall Semester Supplementary

References

[1] R Adams and J Fournier "Sobolev Spaces. 2ed" Academic Press (2003).

[2] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44**, 375–417 (1991).

[3] L.C. Evans. Partial differential equations and Monge-Kantorovich mass transfer. In R. Bott et al., editors, *Current Developments in Mathematics*, pages 26–78. International Press, Cambridge, 1997.

[4] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press (1992) Boca Raton.

[5] W. Gangbo and R.J. McCann. Optimal maps in Monge's mass transport problem. *C.R. Acad. Sci. Paris Sér. I Math.* **321**, 1653–1658 (1995).

[6] R.J. McCann A convexity principle for interacting gases. *Adv. Math.* **128** (1997) 153–179.

[7] R.J. McCann. Exact Solutions to the Transportation Problem on the Line. Proc. Roy. Soc. London Ser. A **455** (1999) 1341–1380.

[8] R.J. McCann. Polar Factorization of Maps on Riemannian Manifolds. *Geom. Funct. Anal.* **11** (2001) 589–608.

[9] R.J. McCann and N. Guillen. Five lectures on optimal transportation: geometry, regularity and applications. In *Analysis and Geometry of Metric Measure Spaces*. G. Dafni et al, eds. Providence: Amer. Math. Soc. (2013) 145-180.

[10] R.J. McCann. A glimpse into the differential topology and geometry of optimal transport. Discrete Contin. Dyn. Syst. **34** (2014) 1605–1621.

[11] F. Santambrogio. *Optimal Transport for Applied Mathematicians.* Birkhauser (2015) Cham.

[12] C. Villani. *Topics in Optimal Transportation.* American Mathematical Society (2003) Providence.

[13] C. Villani. *Optimal Transport, Old and New.* Spring (2009) Berlin. `http://cedricvillani.org/wp-content/uploads/2012/08/preprint-1.pdf`