Topological Properties of Causal Spaces

by

Osama Farooqui

A thesis submitted in conformity with the requirements
for the degree of Master's of Mathematics

Department of Mathematics
University of Toronto

Topological Properties of Causal Spaces

Osama Farooqui
Master's of Mathematics

Department of Mathematics
University of Toronto
2022

Abstract

In this paper, we study causal spaces, i.e. spaces equipped with a time separation function. We introduce some natural topologies on this space and study their properties. We study how causal path-connectedness simplifies the topologies, and induces desirable properties such as first countability. We also study the compatibility of these topologies with subspaces and products. Finally we introduce an analog of the Hausdorff distance for metric spaces, called the Hausdorff time separation.

Contents

Chapter 1

Introduction

1.1 History and Motivation

This body of work is contained in a field which may be called non-smooth geometry. A $\mathcal{C}^2$-smooth pseudo-Riemannian manifold (M, g) is a $\mathcal{C}^2$ manifold M equipped with a metric g, ie a symmetric non-degenerate 2-form. The central objects of study in non-smooth geometry are spaces which have (in some abstract sense) the same properties as pseudo-Riemannian manifolds, but which are not necessarily $\mathcal{C}^2$-smooth, or even smooth at all.

In principle we are concerned with Lorentzian manifolds, where the metric has one negative eigenvalue and all others positive. However there is a wealth of knowledge for Riemannian manifolds. Like many problems in Lorentzian geometry, one might like to appeal to well-established ideas in the Riemannian world, and transplant them into the Lorentzian setting. In non-smooth Riemannian geometry, the program is as follows. Let (M, g) be a Riemannian manifold, and let $\gamma : [0, 1] \to M$ be an arclength parametrized $\mathcal{C}^1$ curve on M. The length of the curve γ as measured by g is defined to be

$$L_g(\gamma) \coloneqq \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt$$

Then one can define a distance function d_g, via

$$d_g(x, y) \coloneqq \inf\{L_g(\gamma) : \gamma \text{ a } \mathcal{C}^1 \text{ curve joining } x \text{ to } y\}$$

which makes (M, d_g) a metric space. The idea is to "forget" the manifold structure of M, and view it purely as a metric space with distance function d_g. It is a theorem of Toponogov [Top59] that a lower bound on the sectional curvature on M can be described by d_g using so called "comparison triangles". That is, in every small neighbourhood of M, one embeds a triangle. The side lengths of the triangle are measured, and an equivalent triangle is embedded in a manifold $M(k)$ of constant sectional curvature k. One then compares the median of the triangles. The median is the distance from a vertex to the midline on the opposite edge. If the median of the triangle in M is larger than that in $M(k)$, for every embedding in the neighbourhood, then the curvature must be higher than k. If this holds in every small neighbourhood of M, then M has curvature at least k. In a similar way, triangle comparison characterizes upper sectional curvature bounds as well. Using comparison triangles, one can define a synthetic notion of sectional curvature bounds for any metric space in which triangles can be embedded. Such metric spaces are called *length spaces*. This was the work of A.D. Alexandrov in [Ale51]. These spaces with sectional curvature bounded below are called Alexandrov spaces.

Remark 1.1.1. There is some disagreement on the original contributors to triangle comparison geometry. See [PZ11].

Among the main contributors to the modern theory of Alexandrov spaces are [GLP81], [Per95] and [BGP92]. A detailed course on non-smooth Riemannian geometry can be found in [BBI01]. Crucially in [GLP81], Gromov introduced two distance functions, the Lipshitz and Gromov-Hausdorff distances (although in the text this latter function is called a Hausdorff distance), on the collection of all length spaces. In particular these are distances on the collection of Riemannian manifolds, and the collection of Alexandrov spaces. These tools give us a notion of convergence for these spaces, which in turn give rise to stability properties, i.e those properties of length spaces which are stable under limits. This gives a language to speak of spaces which are 'approximately' Riemannian manifolds.

Kunzinger and Sämann introduced a Lorentzian analog to these ideas in [KS18]. Here they introduce the notion of a *Lorentzian pre-length space* which, broadly speaking, is a space $(X, \leq, \ll, \tau, d)$ consisting of two preorders $\leq$ and $\ll$, a metric d, and a function τ known as a *time separation* which is compatible with the preorders. Under some additional topological and causal properties, these pre-length spaces become what Kunzinger and Sämann call *Lorentzian length spaces*, which serve as an analog to Gromov's length spaces; the time separation serves as a Lorentzian analog to the metric in a length space. However in the Lorentzian setting, a metric d is still necessary to generate a useful and well-understood topology. In [BS22], Sämann and Beran develop a Lorentzian analog of comparison triangles, and to curvature bounds. However, there are elements which are as of yet missing. One major element is the lack of a 'Lorentzian Gromov-Hausdorff distance', i.e a lack of a meaningful notion of convergence for these spaces.

This is one instance of borrowing ideas from the Riemannian world. Another is the following. Much like the metric on a Riemannian manifold induces a distance function, the Lorentzian metric η on a time orientable Lorentzian manifold (M, η) induces a number of preorders ($\leq$, $\ll$, and $\to$). They are defined as follows. M is said to be time orientable if, broadly speaking, there is a choice of orientation, called the future, for every tangent plane of M, and furthermore that this choice can be made "smoothly" across the manifold. A tangent is called *future directed* if it is oriented towards the future. Given points $x, y \in M$, we then say that:

- $x \leq y$ if there is a future directed curve γ joining x to y such that $\eta(\gamma'(t), \gamma'(t)) \leq 0 \; \forall t$
- $x < y$ if there is a future directed curve γ joining x to y such that $\eta(\gamma'(t), \gamma'(t)) < 0 \; \forall t$
- $x \to y$ if there is a future directed curve γ joining x to y such that $\eta(\gamma'(t), \gamma'(t)) = 0 \; \forall t$

$\leq$ is called *causality*, $\ll$ is called *chronology* and $\to$ is called *horismos*. Kronheimer and Penrose showed in [KP67] that much of the causal structure of time orientable Lorenztian manifolds can be recovered from these three orders, in the same way that the distance d_g recovers geometric properties. Hence in some sense it is sufficient to study the space $(X, \leq, \ll, \to)$, under some additional axioms. These are called *causal spaces*. The depth of research in such spaces is broad. See [GS05] for a general synopsis.

1.2 Present Work and Results

In this work we consider a pair (X, τ) consisting of a set and a time separation, in the same vein as [KS18]. We drop the assumption of a metric compatible with the time separation, and instead consider topologies arising from τ itself. In section 2.2, we identify several such topologies, and especially consider a class of three topologies $\mathcal{H}_i$ for $i = 0, 1, 2$, which are suitable to generate interesting results.

In section 3.2, we show that under additional assumptions, the finest of our topologies, $\mathcal{H}_2$, imposes desirable topological properties on the causal space. In section 3.3 we consider the analog of path metric spaces of [GLP81], which first appear in [KS18].

In chapter 4, we compare our $\mathcal{H}_i$ topologies to the standard subspace, product and quotient topologies, and show that $\mathcal{H}_2$ is in general finer, while $\mathcal{H}_0, \mathcal{H}_1$ are in general coarser.

Finally, in chapter 5, we introduce a causal space analog of the Hausdorff metric for metric spaces. The conclusion of our work introduces three main questions for further study:

- Under what conditions are these causal spaces metrizable, and therefore become Lorentzian pre-length spaces, in the sense of [KS18]?
- What can one say about the collection of maps between causal spaces? Is there a notion of precompactness, or the Arzela-Ascoli theorem?
- Can the Hausdorff time separation induce a "Gromov-Hausdorff time separation", in the spirit of the metric case in [GLP81]?

Chapter 2

Causal Spaces

2.1 Definitions, Conventions, and Notations

Definition 2.1.1. A *causal space* is a pair (X, τ) consisting of a set X and a function

$$\tau : X \times X \to \{-\infty\} \sqcup [0, \infty]$$

called a *time separation*, which satisfies the following axioms of causality:

1. $\tau(x, x) \geq 0 \quad \forall x \in X$
2. $\tau(x, y) \geq 0 \implies \tau(y, x) = -\infty \quad \forall x \neq y \in X$
3. $\tau(x, z) \geq \tau(x, y) + \tau(y, z) \quad \text{if} \quad \tau(x, y) + \tau(y, z) \geq 0$

with the convention that $-\infty + a = a + (-\infty) = -\infty$ for all $a \in \{-\infty\} \sqcup [0, \infty]$.

The time separation induces a relation $\leq$, given by $x \leq y \iff \tau(x, y) \geq 0$. This relation is reflexive by (a) and transitive by (c), and so is a preorder. We say $x < y \iff \tau(x, y) > 0$. $\leq$ is called a *causality*, and $<$ is called a *chronology*.

Remark 2.1.1. By axioms 1 and 3, $\tau(x, x) = 0$ or $\tau(x, x) = \infty$ for all $x \in X$.

Using causality and chronology, we distinguish points using the same language as [KP67].

Definition 2.1.2. Given $x, y \in X$, we say:

- x is *causal* with y if $x \leq y$ or $y \leq x$

- x is *timelike* with y if $x < y$ or $y < x$
- x is *horismal* with y if $\tau(x, y) = 0$ or $\tau(y, x) = 0$.
- x is *acausal* with y if x is not causal with y.

We also define the following sets, which will be the central sets of study in our topologies:

Definition 2.1.3. Let $x \in X$ and $r \geq 0$. We say

- $I_r^+(x) \coloneqq \{y \in X : \tau(x, y) > r\}$ is the *r-chronological future* of x
- $I_r^-(x) \coloneqq \{y \in X : \tau(y, x) > r\}$ is the *r-chronological past* of x
- $J_r^+(x) \coloneqq \{y \in X : \tau(x, y) \geq r\}$ is the *r-causal future* of x
- $J_r^-(x) \coloneqq \{y \in X : \tau(y, x) \geq r\}$ is the *r-causal past* of x

For convenience we denote $I^\pm(x) \coloneqq I_0^\pm(x)$, and $J^\pm(x) \coloneqq J_0^\pm(x)$.

Given $a, b \in X$, we say

- $I(a, b) \coloneqq I^+(a) \cap I^-(b)$ is the *chronological diamond* of a and b
- $J(a, b) \coloneqq J^+(a) \cap J^-(b)$ is the *causal diamond* of a and b.

Next, we distinguish four classes of points in a causal space.

Definition 2.1.4. Let $x \in X$. We call $x\ldots$

- *inclusive* if $I^\pm(x) \neq \varnothing$
- a *(resp. true) root* if $I^-(x) = \varnothing$ (*resp.* and $I^+(x) \neq \varnothing$)
- a *(resp. true) tip* if $I^+(x) = \varnothing$ (*resp.* and $I^-(x) \neq \varnothing$)
- *exclusive* if $I^\pm(x) = \varnothing$.

For some clarity later in the paper we define some notations. Given $A \subset X$, we define

$$\partial_\tau^+(A) = \{x \in X : I^-(x) \cap A = \varnothing\}$$
$$\partial_\tau^-(A) = \{x \in X : I^+(x) \cap A = \varnothing\}$$

Therefore $\partial_\tau^\pm(X)$ denotes the set of roots/tips of X respectively.

Lastly, we define the useful notion of the *reduced time separation*, $\hat{\tau}$ which is given by

$$\hat{\tau}(x,y) := \begin{cases} \tau(x,y) & x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.1.2. In [KS18], $(X, \leq, <)$ is a causal space, following [KP67], and $(X, \leq, <, \hat{\tau}, d)$ is a Lorentzian pre-length space.

We now present some examples of causal spaces. These examples serve to illustrate two points: that causal spaces are in some sense ubiquitous, and that they are in some sense non-trivial.

Example 2.1.1. $(\mathbb{R}, \tau)$ is a causal space, with

$$\tau(x,y) = \begin{cases} y - x & x \leq y \\ -\infty & \text{otherwise} \end{cases}$$

Similarly, for all intervals $I \subset \mathbb{R}$, (I, τ) is a causal space.

Example 2.1.2. Let (M, η) be a time orientable Lorentzian manifold with no closed causal curves. For all future causal piecewise C^1 curves $\gamma : [0,1] \to M$, define

$$L_\eta(\gamma) := \int_0^1 \sqrt{-\eta(\gamma'(t), \gamma'(t))}dt$$

Then

$$\tau_\eta(x,y) = \sup\{L_\eta(\gamma) : \gamma \text{ future causal piecewise } \mathcal{C}^1 \text{ joining } x \text{ to } y\}$$

is a time separation, when $\sup \varnothing := -\infty$. Thus (M, τ_η) is a causal space.

Example 2.1.3. Let $G = (V, E)$ be a directed graph with no directed loops. For $x, y \in V$, define

$$\tau(x,y) = \sup\{n : \exists \{a_i\}_{i=1}^n \subset V \text{ such that } (x, a_1), (a_i, a_{i+1}), (a_n, y) \in E \text{ for all } i = 1, \ldots, n-1\}$$

Again if we define $\sup \varnothing = -\infty$, then τ is a time separation. Then (V, τ) is a causal space. Conceptually, τ measures the maximal number of vertices on any directed path from x to y. If there is no path from x to y, it returns $-\infty$. This is simply a discrete version of the example above.

2.2 Topologies on Causal Spaces

Here is a list of topologies on (X, τ), which are defined as the coarsest topologies on which the following sets are open.

1. The *Alexandrov topology* $\mathcal{A}$ generated by $\{I(a,b) : a, b \in X\}$
2. The *quasi-uniform topology* $\mathcal{Q}$ generated by $\{J^+(a) : a \in X\}$
3. The *chronological topology* $\mathcal{H}_0$ generated by $\{I^\pm(a) : a \in X\}$
4. The *fine chronological topology* $\mathcal{H}_1$ generated by $\{I_r^\pm(a) : a \in X, r \geq 0\}$
5. The *superfine chronological topology* $\mathcal{H}_2$ generated by $\{I_r^\pm(a) : a \in X, r \geq 0\} \cup \{J^\pm(a) : a \in \partial_\tau^\pm(X)\}$

Remark 2.2.1. The term "Alexandrov topology" is attached to many of these. $\mathcal{Q}$ is an Alexandrov topology in the sense that every point has a minimal neighbourhood. This is the convention of [Spe07], and has nothing to do with A.D Alexandrov. In [KS18], $\mathcal{A}$ is the Alexandrov topology. Others, including [GS05] denote $\mathcal{H}_0$ as the Alexandrov topology. We prefer to follow the conventions of [KS18] whenever possible, so $\mathcal{A}$ is called the Alexandrov topology. We prefer to call $\mathcal{Q}$ quasi-uniform in the sense of uniform spaces, see [Kel75].

It is easy to see that $\mathcal{A} \subset \mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2$. Note that $\mathcal{H}_1$ is the coarsest topology such that the reduced time separation $\hat{\tau}$ is lower-semicontinuous in each argument, i.e for each $a \in X$:

$$\liminf_{x \to y} \hat{\tau}(x,a) \geq \hat{\tau}(y,a)$$

$$\liminf_{x \to y} \hat{\tau}(a,x) \geq \hat{\tau}(a,y)$$

Lower semicontinuity is a desirable property. In a smooth spacetime (M, η), the standard time separation is lower-semicontinuous with respect to the topology induced by the background Riemannian metric. This may be why lower-semicontinuity is axiomatic in the construction of Lorentzian length spaces in [KS18]. In our case, it is essential to a large part of our proofs. Thus $\mathcal{H}_1$ is the coarsest topology necessary for our purposes. We will see in the following chapter that $\mathcal{H}_0$ is sufficent under suitable hypotheses. This omits $\mathcal{A}$ and $\mathcal{Q}$ from our discussion, but $\mathcal{A}$ may be reinserted under additional hypothesis; this is the content of corollary 3.2. One might wonder what the use of $\mathcal{H}_2$ is. In some sense, roots and tips are 'poorly covered' by $\mathcal{H}_1$. Indeed, the only neighbourhood of any exclusive point is the space itself, and the only neighbourhoods covering roots/tips are chronological pasts/futures respectively. This is insufficient for our needs. $\mathcal{H}_2$ is the simplest remedy for this situation.

Remark 2.2.2. One can choose to work in the quasi-uniform topology $\mathcal{Q}$. This is the topology which is compatible with the preorder $\leq$. There are many useful properties under this topology: every causal space is locally compact and locally path connected, and maps between causal spaces are continuous iff they are monotone. However, there are some undesirable properties. Namely, the image of a continuous curve may be finite and the space is metrizable only if it is discrete. See [Spe07].

Considering this discussion, we single out the $\mathcal{H}_i$ topologies as most useful, in particular $\mathcal{H}_1$, and $\mathcal{H}_2$ when it is necessary.

Example 2.2.1. $(1+1)$-Minkowski space $\mathbb{R}^{1,1}$ is a causal space with time separation

$$\tau((t,x),(s,y)) = \begin{cases} \sqrt{(t-s)^2-(x-y)^2} & (t-s)^2 \geq (x-y)^2 \\ -\infty & \text{otherwise} \end{cases}$$

The standard topology on $\mathbb{R}^{1,1}$ is the Euclidean topology $\mathcal{E}$ of $\mathbb{R}^2$. Notice that $\mathcal{A}$ coincides with this topology, since within every point in a chronological diamond, one can inscribe a ball around that point within the diamond, and for every point within a ball, one can inscribe a diamond around that point within the ball. $\mathcal{H}_0$ coincides with $\mathcal{A}$ since every chronological past/future is a union of diamonds; this is a consequence of the fact that $\mathbb{R}^{1,1}$ has no roots/ tips. $\mathcal{H}_1$ coincides with $\mathcal{H}_0$ since every r-chronological past/future can be written as a union of chronological pasts/futures. This is the contents of corollary 3.2. Finally $\mathcal{H}_2$ coincides with $\mathcal{H}_1$ since $\mathbb{R}^{1,1}$ has no roots or tips. In summary,

$$\mathcal{E} = \mathcal{A} = \mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2$$

Example 2.2.2. Consider a subspace of $D \subset \mathbb{R}^{1,1}$, $D := J((-1,0),(1,0))$, the causal diamond of $(-1,0)$ and $(1,0)$. This is a causal space as it inherits the time separation from Minkowski space. For the same reason as in the previous example, $\mathcal{H}_0 = \mathcal{H}_1$. However, $\mathcal{A}$ is strictly coarser than $\mathcal{H}_0$. For instance, $(1,0)$ is a tip in D, and therefore there is no open set in $\mathcal{A}$ which contains it. However, $I^+((-1,0))$ is an open set in $\mathcal{H}_0$ which contains $(1,0)$. Every open set in $\mathcal{H}_0$ is open in the Euclidean subspace topology, since the $\mathcal{H}_0$ topology on $\mathbb{R}^{1,1}$ coincides with the Euclidean topology on $\mathbb{R}^{1,1}$, and elements of $\mathcal{H}_0$ in D are simply intersections of elements of $\mathcal{H}_0$ in $\mathbb{R}^{1,1}$ with D.

Therefore they are open in the Euclidean subspace topology on D. However $\mathcal{H}_0$ is strictly coarser than the Euclidean subspace topology. This is because $(0,1)$ is a exclusive point in D, so there is no open set in $\mathcal{H}_0$ which contains it. But there clearly is an open set in the Euclidean subspace topology which contains it: a ball of small radius centred at that point, for instance. Finally, the Euclidean subspace topology is coarser than the $\mathcal{H}_2$ topology. This is a consequence of the fact that the Euclidean topology coincides with the $\mathcal{H}_1$ topology on $\mathbb{R}^{1,1}$, and corollary 4.1. However it is strictly coarser. This is again because $(0,1)$ is exclusive. Therefore $J((0,1),(0,1)) = \{(0,1)\}$ is open in $\mathcal{H}_2$, while it is not open in the Euclidean subspace topology. Therefore we have the chain

$$\mathcal{A} \subsetneq \mathcal{H}_0 = \mathcal{H}_1 \subsetneq \mathcal{E} \subsetneq \mathcal{H}_2$$

Example 2.2.2 illustrates how the introduction of roots, tips and exclusive points can distinguish the various topologies, even away from a standard topology like $\mathcal{E}$.

Chapter 3

Path spaces and timelike curves

3.1 Curves

Definition 3.1.1. A *curve* is a continuous map $\gamma : [a, b] \to X$. It is *causal* (*resp. timelike*) if it is (*resp.* strictly) monotone with respect to the preorder on X. It is *future causal* (*resp. future timelike*) if it is (*resp.* strictly) monotone increasing, and *past causal* (*resp. past timelike*) if it is (*resp.* strictly) monotone decreasing.

When we require γ to be continuous, we typically refer to $\mathcal{H}_1$-continuity. This is necessary in order for the composition of $\hat{\tau}(a, \cdot)$ with a curve and $\hat{\tau}(\cdot, a)$ with a curve to be lower-semicontinuous, which is a fact we use often. Of course we could just as easily use $\mathcal{H}_2$ to achieve this, so we leave the definition of continuity ambiguous.

Definition 3.1.2. The *length* of a causal curve γ is defined as

$$L_\tau(\gamma) = \inf \left\{ \sum_{i=1}^{n} \tau(\gamma(t_{i-1}), \gamma(t_i)) : n \in \mathbb{N}, a = t_0 \leq \cdots \leq t_n = b \right\}$$

whenever γ is future causal, and similarly defined when γ is past causal.

Theorem 3.1.1. *(Properties of Length) Let $\gamma : I = [a, b] \to X$ be a causal curve. Then:*

1. *$L_\tau(\gamma \circ \phi) = L_\tau(\gamma)$ for any homeomorphism $\phi : I' \to I$*

2. *If $\sigma : I' = [c, d] \to X$ is another causal curve of the same causality, and $\gamma(b) = \sigma(c)$, then $L_\tau(\gamma \oplus \sigma) = L_\tau(\gamma) + L_\tau(\sigma)$, where $\gamma \oplus \sigma$ is the concatenation of γ and σ.*

3. *If γ is $\mathcal{H}_1$-continuous, $f_\gamma(t) \coloneqq L_\tau(\gamma|_{[a,t]})$ is lower-semicontinuous. If in addition $L(\gamma)$ is finite, then $g_\gamma(t) \coloneqq L_\tau(\gamma|_{[t,b]})$ is upper-semicontinuous.*

Proof. WLOG assume γ is future causal.

(a) First assume ϕ is order preserving. Let $\{t_i\}_{i=0}^n$ be a partition of I'. Then $\{s_i\} = \{\phi(t_i)\}$ is a partition of I, and

$$\sum_{i=1}^n \tau(\gamma \circ \phi(t_{i-1}), \gamma \circ \phi(t_i)) = \sum_{i=1}^n \tau(\gamma(s_{i-1}), \gamma(s_i))$$

Which implies $L_\tau(\gamma \circ \phi) \geq L_\tau(\gamma)$. On the other hand, if $\{t_i\}_{i=1}^n$ is a partition of I, then $\{s_i'\} = \{\phi^{-1}(t_i)\}$ is a partition of I', and

$$\sum_{i=1}^n \tau(\gamma(t_{i-1}), \gamma(t_i)) = \sum_{i=1}^n \tau(\gamma \circ \phi(s_{i-1}'), \gamma \circ \phi(s_i'))$$

So $L_\tau(\gamma \circ \phi) \leq L_\tau(\gamma)$. Hence $L_\tau(\gamma \circ \phi) = L_\tau(\gamma)$.

If ϕ is order reversing, then $\{s_{n-i}\}$ is a partition of I, $\{s_{n-i}'\}$ is a partition of I', and

$$\sum_{i=1}^n \tau(\gamma \circ \phi(t_i), \gamma \circ \phi(t_{i-1})) = \sum_{i=1}^n \tau(\gamma(s_{i-1}), \gamma(s_i))$$

$$\sum_{i=1}^n \tau(\gamma(t_{i-1}), \gamma(t_i)) = \sum_{i=1}^n \tau(\gamma \circ \phi(s_i'), \gamma \circ \phi(s_{i-1}'))$$

Which again implies $L_\tau(\gamma \circ \phi) = L_\tau(\gamma)$.

(b) Let $\sigma : I' \to X$ be a future causal curve, and WLOG assume $I' = [b, c]$ for some $c \geq b$, so that $\gamma \oplus \sigma : [a, c] \to X$. Otherwise, $\gamma \oplus \sigma$ can always be transformed to such a curve through a homeomorphism, and by (a), the length is invariant under such transformations. Now, any pair of partitions $\{t_i\}_{i=0}^n$ and $\{t_i'\}_{i=0}^m$ of I and I' respectively can be concatenated into a partition $\{s_i\}_{i=0}^{n+m+1}$ of $[a, c]$ in an obvious way. Hence $L_\tau(\gamma \oplus \sigma) \leq L_\tau(\gamma) + L_\tau(\sigma)$. On the other hand, let $\{t_i\}$ be a partition of $[a, c]$ which does not decompose into a pair of partitions of I and I', ie there is k such that $t_k \leq b \leq t_{k+1}$. Then

$$\tau(\gamma \oplus \sigma(t_k), \gamma \oplus \sigma(t_{k+1})) \geq \tau(\gamma \oplus \sigma(t_k), \gamma \oplus \sigma(b)) + \tau(\gamma \oplus \sigma(b), \gamma \oplus \sigma(t_{k+1}))$$

And hence

$$\sum_{i=1}^n \tau(\gamma \oplus \sigma(t_{i-1}), \gamma \oplus \sigma(t_i)) \geq \sum_{i=1}^{n+1} \tau(\gamma \oplus \sigma(s_{i-1}), \gamma \oplus \sigma(s_i))$$

where $\{s_i\}$ is the partition given by $s_i = t_i$ for $i \leq k$, $s_{k+1} = b$, and $s_i = t_{i-1}$ for $i > k$. Therefore

$L_\tau(\gamma \oplus \sigma) \geq L_\tau(\gamma) + L_\tau(\sigma)$. Hence $L_\tau(\gamma \oplus \sigma) = L_\tau(\gamma) + L_\tau(\sigma)$.

(c) To show that $f_\gamma(t)$ is lower semicontinuous, we must show that the preimage of every open ray $(r, \infty]$ is open. First note that $f_\gamma(t)$ is increasing. It follows that $f_\gamma^{-1}((r,\infty]) = \varnothing$ for $r \geq f_\gamma(b)$, and $f_\gamma^{-1}((r,\infty]) \supset (s,b]$ otherwise, where $s = \inf\{t : f_\gamma(t) > r\}$. If $s = a$, then $f_\gamma^{-1}((r,\infty])$ must be open, since it is either equal to $(a,b]$ which is open, or $[a,b]$, which is also open. So assume $s > a$. We want to show that $s \notin f_\gamma^{-1}((r,\infty])$, and therefore that $f_\gamma^{-1}((r,\infty]) = (s,b]$.

Since $a < s$, $(a,s) \neq \varnothing$. For every $t_* \in (a,s)$, consider the map $[t_*, s] \ni t \mapsto \hat{\tau}(\gamma(t_*), \gamma(t))$. Since γ is $\mathcal{H}_1$-continuous, and $\hat{\tau}$ is lower-semicontinuous in $\mathcal{H}_1$, $\hat{\tau}(\gamma(t_*), \gamma(t))$ is lower-semicontinuous at $t = s$. Since γ is causal, $\tau(\gamma(t_*), \gamma(t)) \geq 0$, and therefore $\hat{\tau}(\gamma(t_*), \gamma(t)) = \tau(\gamma(t_*), \gamma(t))$, so $\tau(\gamma(t_*), \gamma(t))$ is lower-semicontinuous. Note that it is also increasing. Hence

$$\liminf_{t \to s} \tau(\gamma(t_*), \gamma(t)) = \tau(\gamma(t_*), \gamma(s))$$

Therefore for every $\varepsilon > 0$, there is $t_* < t_\varepsilon < s$ such that

$$\tau(\gamma(t_*), \gamma(t_\varepsilon)) > \tau(\gamma(t_*), \gamma(s)) - \frac{\varepsilon}{2}$$

On the other hand, for each $\varepsilon' > 0$ and each interval $[a,t]$, there is a partition $\{t_i\}$ such that

$$f_\gamma(t) \leq \sum_{i=1}^{n} \tau(\gamma(t_{i-1}), \gamma(t_i)) \leq f_\gamma(t) + \frac{\varepsilon'}{2}$$

Choose $\varepsilon' = \varepsilon$ and interval $[a, t_\varepsilon]$. Then there is a partition $\{t_i\}$ of this interval, so that $t_n = t_\varepsilon$ WLOG we can assume $t_{n-1} = t_*$, since if not we may refine the partition to include this point and the sum may only decrease. Then:

$$\sum_{i=1}^{n} \tau(\gamma(t_{i-1}), \gamma(t_i)) \leq f_\gamma(t_\varepsilon) + \frac{\varepsilon}{2} \leq r + \frac{\varepsilon}{2}$$

Then

$$f_\gamma(s) \leq \sum_{i=1}^{n-1} \tau(\gamma(t_{i-1}), \gamma(t_i)) + \tau(\gamma(t_*), \gamma(s)) < \sum_{i=1}^{n-1} \tau(\gamma(t_{i-1}), \gamma(t_i)) + \tau(\gamma(t_*), \gamma(t_\varepsilon)) + \frac{\varepsilon}{2} \leq r + \varepsilon$$

From which it follows that $f_\gamma(s) \leq r$. So $s \notin f_\gamma^{-1}((r,\infty])$. Hence $f_\gamma(t)$ is lower semicontinuous.

By (b), $L_\tau(\gamma) = f_\gamma(t) + g_\gamma(t)$. Since $f_\gamma(t) \leq L(\gamma) < \infty$, $g_\gamma(t) = L_\tau(\gamma) - f_\gamma(t)$ is well-defined.

Since $f_\gamma(t)$ is lower-semicontinuous, $g_\gamma(t)$ is upper-semicontinuous. $\square$

3.2 Causal Path Connection

We would like to embed triangles in our causal spaces, in the same sense as [Ale51] and [KS18]. To begin, we should want to embed curves, i.e we should look at spaces which have a path-connectedness like property. However, we should embed in a way which respects the causal structure of our spaces. A first guess it to have causal curves between causally related points. Unfortunately, the existence of only *causal* curves is insufficient for our purposes. In fact, we would like these causal curves to be timelike whenever possible.

Definition 3.2.1. Let (X, τ) be a causal space. We say X is *causally path connected* if for every $a, b \in X$:

- If $a \leq b$, then there is a causal curve joining a and b
- If $a < b$ then there is a timelike curve joining a and b

Once we impose this condition, the structure of our spaces is both enriched and simplified. We begin by simplifying their topology.

Lemma 3.2.1. *Let (X, τ) be a $\mathcal{H}_1$-causally path connected space. For any $x \in X$ and $r \geq 0$,*

$$I_r^\pm(x) = \bigcup_{y \in I_r^\pm(x)} I^\pm(y)$$

Proof. It is clear that $I_r^\pm(x) \supset \bigcup_{y \in I_r^\pm(x)} I^\pm(y)$. We will show that $I_r^+(x) \subset \bigcup_{y \in I_r^+(x)} I^+(y)$, as the other case is similar. Let $y \in I_r^+(x)$. We want to show that there is a $z \in I_r^+(x)$ such that $y \in I^+(z)$. Since $x < y$, there is a future timelike curve joining $\gamma : [a, b] \to X$ joining x and y. From the lower-semicontinuity of $\hat{\tau}(x, \cdot)$,

$$\liminf_{t \to b} \hat{\tau}(x, \gamma(t)) \geq \hat{\tau}(x, y) > r$$

Therefore there is $t < b$ such that $\hat{\tau}(x, \gamma(t)) > r \implies \hat{\tau}(x, \gamma(t)) = \tau(x, \gamma(t))$. Let $z = \gamma(t)$. Then $z \in I_r^+(x)$, and since γ is timelike and $t < b$, $z < y$, so $y \in I^+(z)$. $\square$

Corollary. *If X is a causally path connected space, then $\mathcal{H}_0 = \mathcal{H}_1$*

Proof. We know already that $\mathcal{H}_0 \subset \mathcal{H}_1$. Lemma 3.2.1 establishes that the subbasis elements of $\mathcal{H}_1$ topology are open in $\mathcal{H}_0$, and hence $\mathcal{H}_1 \subset \mathcal{H}_0$. So $\mathcal{H}_0 = \mathcal{H}_1$. $\square$

The remainder of this section deals with $\mathcal{H}_2$-causally path connected spaces, and some results about their structure. First we show that roots and tips are restricted in such spaces.

Lemma 3.2.2. *Let X be $\mathcal{H}_2$-causally path connected. Then every point in the future horismos of a root is a root and every point in the past horismos of a tip is a tip.*

Proof. We will prove this for roots only, as the proof for tips is similar. Let a be a root, and let x be in the future horismos of a, i.e. $\tau(a, x) = 0$. Suppose x is not a root, so there is $y < x$. Then there is a $\mathcal{H}_2$-continuous timelike curve $\gamma : [0, 1] \to X$ joining y to x. Note that $a \not\leq y$, since if it were, $a \leq y < x \implies a < x$, which contradicts the fact that x is future horismal with a. Then by continuity, $\gamma^{-1}(J^+(a))$ is open, contains 1 and does not contain 0. So $\gamma^{-1}(J^+(a)) = (t, 1]$ for some $t < 1$. Therefore for any $t < s \leq 1$, $\gamma(s) \in J^+(a)$. But γ is timelike, so for $t < s < 1$, $a \leq \gamma(s) < \gamma(b) = x$ and again $a < x$. Hence x must be a root. $\square$

Corollary. *Let X be $\mathcal{H}_2$-causally path connected. Let $x \in X$. Then x is a root/tip iff it is horismal with a root/tip.*

Lemma 3.2.2 and its corollary are instrumental for the remaining results in this section. Next we show that intersection of chronological futures also contain chronological futures, which gives a simpler description of the topology.

Lemma 3.2.3. *Let X be $\mathcal{H}_1$-causally path connected. If $z \in I^{\pm}(x) \cap I^{\pm}(y)$, then there is w such that $z \in I^{\pm}(w) \subset I^{\pm}(x) \cap I^{\pm}(y)$.*

Proof. We will only prove one case, as the other is similar. Let $z \in I^+(x) \cap I^+(y)$. Then $x < z$, so there is timelike curve $\gamma : [a, b] \to X$ joining x to z. By the lower semicontinuity of $\tau(y, \gamma(t))$, there is $t < b$ such that $\tau(y, \gamma(t)) > 0$. Let $w = \gamma(t) < \gamma(b) = z$. Then

$$z \in I^+(w) \subset I^+(x) \cap I^+(y)$$

as desired. $\square$

Theorem 3.2.1. *Let X be $\mathcal{H}_2$-causally path connected. Let $x \in X$.*

1. *If x is inclusive, every neighbourhood about x contains an element of the form $I(a, b)$, where $a < x < b$*

2. *If x is a true root, every neighbourhood about x contains an element of the form $J^+(x) \cap I^-(a)$ for some $x < a$.*

3. *If x is a true tip, every neighbourhood about x contains an element of the form $I^+(a) \cap J^-(x)$ for some $a < x$.*

Proof. Let x be inclusive, and let $x \in U$ be a basis element containing x. A basis of $\mathcal{H}_2$ topology is given by a finite intersection of subbasis elements, i.e a finite intersection of the sets $I^{\pm}(a)$ for inclusive a, and $J^{\pm}(b)$ for true roots/tips b. In general we have

$$x \in \bigcap_{i=1}^{n} I^+(a_i) \cap \bigcap_{j=1}^{m} I^-(b_j) \cap \bigcap_{k=1}^{n'} J^+(c_k) \cap \bigcap_{\ell=1}^{m'} J^-(d_\ell) =: U$$

where a_i, b_j are inclusive, c_k are roots and d_ℓ are tips. Firstly, by lemma 3.2.2, x must be timelike related to any such c_k, d_ℓ, and therefore there is inclusive c'_k, d'_ℓ such that $c_k < c'_k < x$ and $x < d'_\ell < d_\ell$. Hence after relabelling we have

$$x \in \underbrace{\bigcap_{i=1}^{n+n'} I^+(a_i) \cap \bigcap_{j=1}^{m+m'} I^-(b_j)}_{=:V} \subset U$$

Then by lemma 3.2.3, there is w, z such that $a_i < w < x < z < b_j$, and hence

$$x \in I(w, z) \subset V \subset U$$

The case with x being a true root or true tip is similar. □

It is not hard to see that Lemma 3.2.3 combined with theorem 3.2.1 imply that the collection of the above sets, together with the collection of all singleton sets of all exclusive points, form a basis for the $\mathcal{H}_2$ topology. Notably, these basis elements are not far from chronological diamonds.

Corollary. *Let X be a timelike path connected causal space with no roots or tips. Then $\mathcal{A} = \mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2$, and the basis for any of these topologies is $\{I(a, b) : a, b \in X\}$.*

Finally we present some results about limit points for timelike path connected spaces.

Theorem 3.2.2. *Let X be a $\mathcal{H}_2$-causally path connected causal space. Then X is first countable.*

Proof. Let $x \in X$. We want to show that there is a countable collection of open neighbourhoods of x such that any open neighbourhood of x contains an element of this collection. Such a collection is called a neighbourhood basis of x. If x is exclusive, then $\{x\}$ suffices. Now suppose x is inclusive. Let $a < x < b$, and let $\gamma, \sigma : [0,1] \to X$ be timelike curves joining a to x and x to b respectively. Let $a_n = \gamma\left(1 - \frac{1}{n}\right)$ and let $b_n = \sigma\left(\frac{1}{n}\right)$. We claim that $\{I(a_n, b_n) : n \in \mathbb{N}\}$ is a neighbourhood basis.

By theorem 3.2.1 any basis element containing x is of the form $x \in I(c,d)$ for some $c, d \in X$. By lower semicontinuity of $\hat{\tau}(c, \gamma(t))$, we have that there is $t_* < 1$ such that for all $t > t_*$, $\hat{\tau}(c, \gamma(t)) > 0$, and therefore $c < \gamma(t)$. Therefore there is $N = \frac{1}{1-t_*}$ such that for all $n > N$, $c < a_n$. By a similar argument, there is M such that for all $m > M$, $b_m < d$. Then take $n > \max\{N, M\}$, so that $c < a_n < x < b_n < d$. Then $x \in I(a_n, b_n) \subset I(c,d)$. By a similar argument, one can show that $\{J^+(x) \cap I^-(b_n)\}$ is a neighbourhood basis of x whenever x is a true root, and $\{I^+(a_n) \cap J^-(x)\}$ is a neighbourhood basis whenever it is a true tip. □

Remark 3.2.1. A similar version of theorem 3.2.1 can be proven using the $\mathcal{H}_1$ topology, and therefore theorem 3.2.2 holds in the $\mathcal{H}_1$ topology as well. We omit these proofs as they are practically identical to the $\mathcal{H}_2$ case.

First countable spaces are known to be *sequential*, meaning the limit point of any set is a limit point of a convergent sequence contained in that set. This is the first result on limit points we would like to present. The second is on the closure of future and past chronologies. In Minkowski space, $\overline{I^\pm(x)} = J^\pm(x)$. One might wonder how close this equality is in a general causal space. If τ is continuous in the $\mathcal{H}_0$ product topology, then $J^\pm(x)$ is closed, so the closure of $I^\pm(x)$ is no larger than $J^\pm(x)$. We now present a converse for $\mathcal{H}_2$-causally path connected spaces. In these spaces, the closure of $I^\pm(x)$ is no smaller than $J^\pm(x)$. The proof relies primarily on the following result and lemma 3.2.2.

Lemma 3.2.4. *Let X be $\mathcal{H}_2$-causally path connected. Let $x \in X$. If $I^\pm(x) \neq \varnothing$, then x is a limit point of $I^\pm(x)$.*

Proof. We will only prove this for $I^+(x) \neq \varnothing$. A basis element about x is of the form $I(a,b)$ or $J^+(x) \cap I^-(b)$. In either case, the element contains the set $I(x,b)$. Since $x < b$, and X is causally path connected, $I(x,b) \neq \varnothing$. Hence every open set containing x must intersect $I^+(x)$, so x is a limit point of $I^+(x)$. □

Theorem 3.2.3. *Let X be $\mathcal{H}_2$-causally path connected and let $x \in X$. Then $J^\pm(x) \subset \overline{I^\pm(x)}$, whenever $I^\pm(x)$ is non-empty.*

Proof. As usual, we will only prove the claim for $I^+(x) \neq \varnothing$. Let $y \in J^+(x)$. If $y \in I^+(x)$, then $y \in \overline{I^+(x)}$, so suppose $\tau(x,y) = 0$. By lemma 3.2.2, $I^+(y) \neq \varnothing$. If not, then x would be past horismal with a tip, and therefore must be a tip, which contradicts $I^+(x) \neq \varnothing$. Then by the previous lemma

$$y \in \overline{I^+(y)} \subset \overline{I^+(x)}$$

Therefore $J^+(x) \subset \overline{I^+(x)}$. $\square$

Remark 3.2.2. Although there are variations of theorems 3.2.1 and 3.2.2 for $\mathcal{H}_1$-causally path connected spaces, the above theorem cannot be reproduced in this setting. The reason is that, given $x \in X$, there may be a tip $z \in X$ which is in the horismal future of x. Furthermore, there could be $y \in I^-(z)$ such that $I^+(y) \cap I^+(x) = \varnothing$, and therefore z fails to be a limit point of $I^+(x)$, despite the fact that $z \in J^+(x)$. Lemma 3.2.2 omits this possibility for $\mathcal{H}_2$-causally path connected spaces.

Lastly, we can fully characterize the closure of $I^{\pm}(x)$:

Theorem 3.2.4. *Let X be $\mathcal{H}_2$-causally path connected and let $x \in X$. Then*

$$\overline{I^{\pm}(x)} = I^{\pm}(x) \cup \{y \in X : \varnothing \neq I^{\pm}(y) \subset I^{\pm}(x)\}$$

whenever $I^{\pm}(x)$ is non-empty.

Proof. We will only prove this for $I^+(x)$. Let y be a limit point of $I^+(x) \neq \varnothing$. If y is a tip, then $J^-(y) \cap I^+(x) \neq \varnothing$, which implies $x \leq y$. By lemma 3.2.2, $\tau(x, y) \neq 0$, so $x < y$, so $y \in I^+(x)$. If y is not a tip, then for all $y < a$, $I^-(a) \cap I^+(x) \neq \varnothing$, and therefore $x < a$. Hence $I^+(y) \subset I^+(x)$. Therefore the LHS set is contained in the RHS set in the above expression.

Now let y be such that $\varnothing \neq I^+(y) \subset I^+(x)$. Let y be inclusive. Then a basis element containing y is of the form $y \in I(a, b)$, with $b \in I^+(x)$. Note that $b \in I^+(x) \cap I^+(y)$. Then by lemma 3.2.3, there is c such that $x < c < b$, and $a < y < c < b$. Therefore $c \in I(a, b)$, so $I(a, b) \cap I^+(x) \neq \varnothing$. Hence y is a limit point of $I^+(x)$. If y is a root, then a basis element is of the form $J^+(y) \cap I^-(b)$, with $y < b$. Then by the same principle, there is c such that $x < c$ and $y \leq c < b$. Hence $c \in J^+(y) \cap I^-(b)$, so y is a limit point of $I^+(x)$. Hence the RHS is contained in the LHS. $\square$

3.3 Path Spaces

Definition 3.3.1. The *path time separation* is defined as:

$$\tau^*(x, y) = \sup \{L_\tau(\gamma) : \gamma : [a, b] \to X \text{ future causal}, \gamma(a) = x, \gamma(b) = y\}$$

Proposition 1. *The path time separation is a time separation.*

Proof. First note that $L_\tau(\cdot)$ is a non-negative function. Hence if there exists future timelike curve from x to y, then $\tau^*(x,y) \geq 0$. Otherwise, $\tau^*(x,y) = \sup\{\varnothing\} = -\infty$. Hence the image of τ^* is $\{-\infty\} \cup [0,\infty]$.

Next, for any $x \in X$, the constant curve $\gamma(t) = x$ is causal, and so $\tau^*(x,x) \geq L_\tau(\gamma) \geq 0$. Now suppose $\tau^*(x,y) \geq 0$. Then $x \leq y$. If $x \neq y$, then $y \not\leq x$, so there cannot be a future causal curve from y to x. Hence $\tau^*(y,x) = -\infty$.

Finally, suppose $\tau^*(x,y)$ and $\tau^*(y,z)$ are non-negative. Let γ, σ be causal curves from x to y and y to z respectively. Then $\gamma \oplus \sigma$ is a causal curve from x to z, and

$$L_\tau(\gamma \oplus \sigma) = L_\tau(\gamma) + L_\tau(\sigma)$$

Then we have:

$$\begin{aligned}
\tau^*(x,z) &= \sup\{L_\tau(\gamma) : \gamma \text{ future casual from } x \text{ to } z\} \\
&\geq \sup\{L_\tau(\gamma \oplus \sigma) : \gamma \text{ future casual from } x \text{ to } y, \sigma \text{ future causal from } y \text{ to } z\} \\
&= \sup\{L_\tau(\gamma) + L_\tau(\sigma) : \gamma \text{ future casual from } x \text{ to } y, \sigma \text{ future causal from } y \text{ to } z\} \\
&= \sup\{L_\tau(\gamma) : \gamma \text{ future casual from } x \text{ to } y\} + \sup\{L_\tau(\sigma) : \sigma \text{ future causal from } y \text{ to } z\} \\
&= \tau^*(x,y) + \tau^*(y,z)
\end{aligned}$$

□

We call (X, τ^*) the *path* causal space of X and denote it X^*.

Remark 3.3.1. Of course, the definition of τ^* depends on causal curves, and continuity of those curves depends on the topology on X. Therefore there are two useful path spaces: one for $(X, \mathcal{H}_1)$ and one for $(X, \mathcal{H}_2)$. Again, by default we use $\mathcal{H}_1$.

Given that $\tau^*(x,y) \leq \tau(x,y)$, one might wonder about the converse case. When is $\tau(x,y) \leq \tau^*(x,y)$?

Definition 3.3.2. A causal space (X,τ) is *intrinsic* if $\tau^*(x,y) = \tau(x,y)$ for any $x,y \in X$. It is *strongly intrinsic* if there is a curve γ from x to y such that $\tau(x,y) = L_\tau(\gamma)$.

One would hope that X^* is intrinsic, but to prove this requires some technical steps. The content of the proof is showing that $X^* = X^{**} := (X^*)^*$, which is equivalent to showing that $\tau^* = \tau^{**}$.

$\tau^*(x,y)$ is the maximal τ-length of causal curves in X which join x to y, while $\tau^{**}(x,y)$ is the maximal τ^*-length of causal curves in X^* which join x to y. To prove the above claim, we should relate these ideas: how are causal curves in X related to causal curves in X^*, and how are their respective lengths related? Lemma 3.3.2 below shows that $L_\tau = L_{\tau^*}$, so there is no difference between τ-length and τ^*-length. Similarly lemma 3.3.1 shows that there is no difference between τ-causality of curves and τ^*-causality. It is left to deal with the 'curve' part. Specifically, how does continuity differ in X and X^*? One would like to relate the topology of these spaces via their respective time separations. That is, endow X with the $\mathcal{H}_1$ topology, and X^* with the $\mathcal{H}_1^*$ topology. The issue is that, a priori, there is no relation between the two. A possible solution is to fix the same topology on all three spaces X, X^*, and X^{**}. It needn't be $\mathcal{H}_1$, even, since we do not need the lower-semicontinuity of $\hat{\tau}$ at all. Once fixing a topology on X, the result follows.

Lemma 3.3.1. *Fix a topology on X. A curve γ on X is τ-causal iff it is τ^*-causal.*

Proof. Let γ be a curve, and WLOG let γ be future causal (either τ or τ^*). Let $0 \le s \le t \le 1$. If γ is τ^*-causal, then $\gamma(s) \le^* \gamma(t)$. Then since $\tau^* \le \tau$, $\gamma(s) \le \gamma(t)$. Hence γ is τ-causal. If γ is τ-causal, then $\gamma(s) \le \gamma(t)$. Furthermore, $\gamma|_{[s,t]}$ is a future τ-causal curve from $\gamma(s)$ to $\gamma(t)$. Therefore $\tau^*(\gamma(s),\gamma(t)) \ge L_\tau(\gamma|_{[s,t]}) \ge 0$. Hence $\gamma(s) \le^* \gamma(t)$, so γ is τ^*-causal. □

Lemma 3.3.2. *Fix a topology on X. For all causal curves γ on X, $L_\tau(\gamma) = L_{\tau^*}(\gamma)$.*

Proof. By lemma 3.3.1, we need not distinguish between τ-causal curves and τ^*-causal curves. Let $\gamma : [a,b] \to X$ be a future causal curve on X, and let $\{t_i\}_{i=0}^n$ be a partition of $[a,b]$. Then since $\tau^* \le \tau$,

$$\sum_{i=1}^n \tau^*(\gamma(t_{i-1}),\gamma(t_i)) \le \sum_{i=1}^n \tau(\gamma(t_{i-1}),\gamma(t_i))$$

From which it follows that $L_{\tau^*}(\gamma) \le L_\tau(\gamma)$. On the other hand, let partition $\{t_i\}_{i=1}^n$ be a partition of $[a,b]$. For each $i = 1,\dots,n$, and each $\varepsilon > 0$, there is a partition $\{s_{i,j}\}_{j=0}^{n_i}$ of $[t_{i-1},t_i]$, such that

$$\tau^*(\gamma(t_{i-1}),\gamma(t_i)) \ge L_\tau(\gamma|_{[t_{i-1},t_i]}) > \sum_{j=1}^{n_i} \tau(\gamma(s_{i,j-1}),\gamma(s_{i,j})) - \varepsilon$$

It follows that

$$\sum_{i=1}^n \tau^*(\gamma(t_{i-1}),\gamma(t_i)) > \sum_{i=1}^n \sum_{j=1}^{n_i} \tau(\gamma(s_{i,j-1}),\gamma(s_{i,j})) - \varepsilon \ge L_\tau(\gamma) - \varepsilon \tag{3.1}$$

Since this holds for all partitions $\{t_i\}$ of $[a,b]$, it follows that

$$L_{\tau^*}(\gamma) \geq L_\tau(\gamma)$$

Hence $L_{\tau^*}(\gamma) = L_\tau(\gamma)$ $\square$

Theorem 3.3.1. *Fix a topology on X. Then X^* is intrinsic.*

Proof. Once we fix a topology, the definition of a causal curve in X and X^* is unambiguous, and equivalent. Then for all $x, y \in X$:

$$\tau^{**}(x,y) = \{L_{\tau^*}(\gamma) : \gamma \text{ future causal curve from } x \text{ to } y\} = \{L_\tau(\gamma) : \gamma \text{ future causal curve from } x \text{ to } y\} = \tau^*(x,y)$$

$\square$

Following this discussion, one might wonder how to relate the $\mathcal{H}_i$ topologies with the $\mathcal{H}_i^*$. In the metric case this is immediate. Passing from a metric d to the path metric d^* always refines the topology. In the causal case the situation is more delicate. One issue is that passing from X to X^* ensures there is always a causal curve between causally related points. However it does not guarantee that there is a timelike curve between timelike related points. The second issue is that passing to the path causal space can produce more roots and tips, which are not well-covered by $\mathcal{H}_1^*$. Therefore, one should work with path spaces X^* which are $\mathcal{H}_1$-causally path connected, and endow these spaces with the $\mathcal{H}_2^*$-topology. In the following lemmas, we use $I^\pm(x)$ to denote the chronological future/past of x with respect to the τ time separation, and $(I^*)^\pm$ to denote the chronological future/past with respect to the τ^* time separation. $J^\pm(x)$ and $(J^*)^\pm(x)$ are similarly defined.

Lemma 3.3.3. *Suppose X^* is $\mathcal{H}_1$-causally path connected. For every $x \in X^*$ and $r \geq 0$:*

$$I_r^\pm(x) = \bigcup_{\substack{y \in I_r^\pm(x) \\ y \notin \partial^\pm_{\tau^*}(X^*)}} (I^*)^\pm(y) \cup \bigcup_{\substack{y \in I_r^\pm(x) \\ y \in \partial^\pm_{\tau^*}(X^*)}} (J^*)^\pm(y)$$

whenever $I_r^\pm(x) \neq \varnothing$.

Proof. We only prove this for $I_r^+(x)$, as the other case is similar. It is clear that the RHS is contained in the LHS, since $\tau^* \leq \tau$. Let $y \in I_r^+(x)$. If $y \in \partial^+_{\tau^*}(X^*)$ then $y \in (J^*)^+(y)$. Suppose $y \notin \partial^+_{\tau^*}(X^*)$. Then there is $z <^* y$. Then there is a $\mathcal{H}_1$-continuous future timelike curve $\gamma : [a,b] \to X^*$ from z to

y. Then by the lower semicontinuity of $\hat{\tau}(x,\cdot)$:

$$\liminf_{t\to b} \hat{\tau}(x,\gamma(t)) \geq \hat{\tau}(x,y) > r$$

Therefore there is $t < b$ such that $\hat{\tau}(x,\gamma(t)) > r$. Let $z' = \gamma(t)$. Then $z' \in I_r^+(x)$, and $y \in I^+(z')$. $\square$

Corollary. *If X^* is $\mathcal{H}_1$-causally path connected, then $\mathcal{H}_1 \subset \mathcal{H}_2^*$.*

Lemma 3.3.4. *Suppose X^* is $\mathcal{H}_2$-causally path connected. For every $x \in \partial_\tau^\pm(X)$:*

$$J^\pm(x) = \bigcup_{\substack{y\in J^\pm(x)\\ y\notin\partial^\pm_{\tau^*}(X^*)}} (I^*)^\pm(y) \cup \bigcup_{\substack{y\in J^\pm(x)\\ y\in\partial^\pm_{\tau^*}(X^*)}} (J^*)^\pm(y)$$

whenever $J^\pm(x) \neq \varnothing$.

Proof. We only prove this for $x \in \partial_\tau^+(X)$, as the other case is similar. It is clear that the RHS is contained in the LHS, since $\tau^* \leq \tau$. Let $y \in J^+(x)$. If $y \in I^+(x)$, then the LHS is contained in the RHS by lemma 3.3.3. So suppose $\tau(x,y) = 0$. If $y \in \partial^+_{\tau^*}(X^*)$, then $y \in (J^*)^+(y)$.

Now suppose further that $y \notin \partial^+_{\tau^*}(X^*)$. Then there is $z <^* y$. Then there is a $\mathcal{H}_2$-continuous future timelike curve $\gamma : [a,b] \to X^*$ from z to y. Then $\gamma^{-1}(J^+(x))$ is open since γ is $\mathcal{H}_2$ continuous and x is a roots, and does not contain z. Therefore $\gamma^{-1}(J^+(x)) = (s,b]$ for some $a < s$ or equals $[a,b]$. Either way, there is $t < b$ such that $\gamma(t) \in J^+(x)$. Then $x \leq \gamma(t) < \gamma(b) = y$, which contradicts $\tau(x,y) = 0$. Hence $y \in \partial^+_{\tau^*}(X^*)$. $\square$

Corollary. *If X^* is $\mathcal{H}_2$-causally path connected, then $\mathcal{H}_2 \subset \mathcal{H}_2^*$.*

Chapter 4

Constructing Causal Spaces

There are three standard ways of constructing new spaces from old ones: restrictions, products, and quotients. In this chapter, we investigate all three methods, and the correlation between their respective topologies and the $\mathcal{H}_i$ topologies induced by their time separations.

4.1 Subspaces

Proposition 2. *Let (X, τ) be a causal space, and let $A \subset X$. Then (A, τ_A) is a causal space, where $\tau_A = \tau|_{A\times A}$.*

The fact that τ_A is a time separation is obvious. Once X is endowed with a topology, A is given the corresponding subspace topology. On the other hand, the time separation τ_A induces its own topology $\mathcal{H}_i^A$. In general $\mathcal{H}_1^A$ is coarser than the subspace topology. The issue is that roots/tip of A are not necessarily roots/tips of X. Therefore they may be poorly covered in $\mathcal{H}_1^A$, while they are well covered in the subspace topology. The issue is resolved by passing to $\mathcal{H}_2^A$, which covers roots and tips well.

Lemma 4.1.1. *Let (X, τ_X) be a causal space with the $\mathcal{H}_1$ topology, and $A \subset X$. Suppose (A, τ_A) is causally path connected in the subspace topology. Then for all $x \in X$*

$$I_r^{\pm}(x) \cap A = \bigcup_{\substack{y\in I_r^{\pm}(x)\cap A \\ y\notin\partial^{\pm}_{\tau_A}(A)}} I^{\pm}(y) \cap A \cup \bigcup_{\substack{y\in I_r^{\pm}(x)\cap A \\ \tau\in\partial^{\pm}_{\tau_A}(A)}} J^{\pm}(y) \cap A$$

whenever $I_r^{\pm}(x) \cap A \neq \varnothing$.

Proof. We will only prove the case for $I^+(x)$, as the other case is similar. If $y \in I_r^+(x) \cap A$, then

$$I^+(y) \subset J^+(y) \subset I_r^+(x) \implies I^+(x) \cap A \subset J^+(y) \cap A \subset I_r^+(x) \cap A$$

Which shows that the RHS is contained in the LHS. On the other hand, let $y \in I_r^+(x) \cap A$. If $y \in \partial_{\tau_A}^+(A)$, then $y \in J^+(y) \cap A$. Suppose $y \notin \partial_{\tau_A}^+(A)$, so there is $z \in A$, with $z < y$. Then since A is causally path connected, there is $\gamma : [0,1] \to A$ future timelike joining z and y, which is continuous in the $\mathcal{H}_1$ subspace topology. Then $\hat{\tau}(x, \gamma(t))$ is lower-semicontinuous at $t = 1$, so there is $s < 1$ such that $\hat{\tau}(x, \gamma(s)) > r$. Let $w = \gamma(s)$. Then $w \in I_r^+(x) \cap A$, $w \notin \partial_{\tau_A}^+(A)$, and $y \in I^+(w) \cap A$. Therefore the LHS is contained in the RHS, so they must be equal. □

Corollary. *Let $A \subset X$ be causally path connected in the $\mathcal{H}_1$-subspace topology. Then $\mathcal{H}_2^A$ is finer than the subspace topology.*

We can find a similar result for the subspace topology induced by $\mathcal{H}_2$, as long as we assume some stronger path connected properties.

Lemma 4.1.2. *Let (X, τ_X) be a causal space, and $A \subset X$. Suppose (A, τ_A) is causally path connected in the $\mathcal{H}_2$ subspace topology. Let x be a root/tip. Then*

$$J^\pm(x) \cap A = \bigcup_{\substack{y \in J^\pm(x) \cap A \\ y \notin \partial_{\tau_A}^\pm(A)}} I^\pm(y) \cap A \cup \bigcup_{\substack{y \in J^\pm(x) \cap A \\ y \in \partial_{\tau_A}^\pm(A)}} J^\pm(y) \cap A$$

Proof. Again, we will only prove the claim for $J^+(x) \cap A$. It is clear that the RHS is contained in the LHS. Let $y \in J^+(x) \cap A$. If $x < y$, then y is contained in the RHS, by lemma 4.1.1. Now suppose $\tau(x, y) = 0$. If y is τ_A-inclusive, i.e there is $w < y$, then there is future timelike curve $\gamma : [0,1] \to X$ from w to y. Then since x is a root, $\tau(x, \gamma(t))$ is lower semi-continuous, which implies that there is $s < 1$ such that $\tau(x, \gamma(t)) = 0$. Hence $x \leq \gamma(t) < y$, which contradicts that $\tau(x, y) = 0$. Hence y must be a root of A. Then $y \in J^+(y) \cap A$. Hence the RHS contains the LHS, so they are equal. □

Corollary. *Let $A \subset X$ be causally path connected in the $\mathcal{H}_2$-subspace topology. Then $\mathcal{H}_2^A$ is finer than the subspace topology.*

Proof. The combination of lemmas 4.1.1 and 4.1.2 mean that $I_r^\pm(x) \cap A$, which is open in the subspace topology, is also open in the $\mathcal{H}_2$ topology. Similarly $J^\pm(x) \cap A$ is open in $\mathcal{H}_2$ whenever x is a root/tip. It follows that the subspace topology is coarser than the $\mathcal{H}_2^A$ topology. □

It may be difficult to show that A which are causally path connected in a subspace topology. However, the matter can be simplified by asking for path connectedness of X, and 'convexity' of A.

Definition 4.1.1. Let (X, τ) be a causal space. A subset $A \subset X$ is called *causally convex* if for all $a, b \in A$ with $a \leq b$, $J(a, b) \subset A$. The *causally convex hull* of a set is

$$\operatorname{conv}_\tau(A) = \bigcup_{\substack{a,b \in A \\ a \leq b}} J(a, b)$$

It is the smallest causally convex set containing A. We denote the set of all causally convex subsets of X by $\mathcal{J}(X)$.

Corollary. *Let (X, τ) be a $\mathcal{H}_i$-causally path connected space, and $A \subset X$ be causally convex. Then $\mathcal{H}_2^A$ is finer than the $\mathcal{H}_i$-subspace topology, for $i = 1, 2$.*

Proof. Fix $i = 1, 2$. Let $a, b \in A$ with $a < b$. Since X is $\mathcal{H}_i$-causally path connected, there is a $\mathcal{H}_i$-continuous, future timelike path γ joining a to b. Since A is causally convex, $\gamma \subset J(a, b) \subset A$. It follows that γ is continuous in the $\mathcal{H}_i$ subspace topology. Hence A is causally path connected in the $\mathcal{H}_i$ subspace topology. Then by corollary 4.1 or 4.1, the $\mathcal{H}_2^A$ is finer than the $\mathcal{H}_i$ subspace topology. □

4.2 Products

Proposition 3. *(Product) Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a collection of causal spaces, where Λ is some index set. Then (X, τ) is a causal space, where $X = \prod_\alpha X_\alpha$, and*

$$\tau((x_\alpha), (y_\alpha)) = \inf_\alpha \{\tau_\alpha(x_\alpha, y_\alpha)\}$$

Proof. Let $(x_\alpha)_{\alpha \in \Lambda} \in X$. Then

$$\tau((x_\alpha), (x_\alpha)) = \inf_\alpha \{\underbrace{\tau_\alpha(x_\alpha, x_\alpha)}_{\geq 0}\} \geq 0$$

Which verifies the first axiom of causality.

Next let $(y_\alpha) \in X$, with $(x_\alpha) \neq (y_\alpha)$, and $\tau((x_\alpha), (y_\alpha)) \geq 0$. Then by definition, $\tau_\alpha(x_\alpha, y_\alpha) \geq 0$ for all α. Therefore since $(x_\alpha) \neq (y_\alpha)$, there is $\alpha \in \Lambda$ such that $x_\alpha \neq y_\alpha$, and thus $\tau_\alpha(y_\alpha, x_\alpha) = -\infty$. It follows that $\tau((y_\alpha), (x_\alpha)) = -\infty$. This verifies the second axiom of causality.

Finally, let $(z_\alpha) \in X$, and suppose $\tau((x_\alpha),(y_\alpha)) \geq 0$ and $\tau((y_\alpha),(z_\alpha)) \geq 0$. Then for all α, $\tau_\alpha(x_\alpha, y_\alpha) \geq 0$ and $\tau_\alpha(y_\alpha, z_\alpha) \geq 0$. Hence $\tau_\alpha(x_\alpha, z_\alpha) \geq \tau_\alpha(x_\alpha, y_\alpha) + \tau_\alpha(y_\alpha, z_\alpha)$. Then

$$\begin{aligned}
\tau((x_\alpha),(z_\alpha)) &= \inf_\alpha\{\tau_\alpha(x_\alpha, z_\alpha)\} \\
&\geq \inf_\alpha\{\tau_\alpha(x_\alpha, y_\alpha) + \tau_\alpha(y_\alpha, z_\alpha)\} \\
&\geq \inf_\alpha\{\tau_\alpha(x_\alpha, y_\alpha)\} + \inf_\alpha\{\tau_\alpha(y_\alpha, z_\alpha)\} = \tau((x_\alpha),(y_\alpha)) + \tau((y_\alpha),(z_\alpha))
\end{aligned}$$

Which proves the third axiom of causality. □

Similarly to the case of subsets of X, we want to compare the product topology (once a topology is fixed on X_i), and the $\mathcal{H}_i$ topology induced by τ. For this we must restrict to finite products. We will see that, like for subspaces, $\mathcal{H}_1$ is coarse, and $\mathcal{H}_2$ is fine.

Lemma 4.2.1. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of causal spaces, and let (X, τ) be the product space. Then for all $x = (x_i) \in X$, and all $r \geq 0$:*

$$I_r^\pm(x) = \prod_{i=1}^n I_r^\pm(x_i)$$

Proof. The result is a consequence of the fact that $\tau(x,y) > r \iff \tau_i(x_i, y_i) > r$ for all i. This is evident from the definition of τ and the fact that the collection is finite. □

Corollary. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of causal spaces with the $\mathcal{H}_1$ topology, and let (X, τ) be the product space. Then the $\mathcal{H}_1$ topology on X is coarser than the product topology.*

Lemma 4.2.2. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of causal spaces, and let (X, τ) be the product space. Then for all $x = (x_i) \in X$:*

$$J^\pm(x) = \prod_{i=1}^n J^\pm(x_i)$$

Proof. The result is a consequence of the fact that $\tau(x,y) \geq 0 \iff \tau_i(x_i, y_i) \geq 0$ for all i. Again, this is evident from the definition of τ and the fact that the collection is finite. □

Lemma 4.2.3. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of causal spaces, and let (X, τ) be the product space. Then $x = (x_i) \in X$ is a root/tip iff at least one x_i is a root/tip.*

Proof. This is a consequence of lemma 4.2.1. Since $I^\pm(x) = \prod I^\pm(x_i)$, $I^\pm(x) = \varnothing \iff I^\pm(x_i) = \varnothing$ for some x_i. □

Theorem 4.2.1. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite collection of $\mathcal{H}_2$-causally path connected spaces. Let (X, τ) be the product. Then the $\mathcal{H}_2$ topology on X is finer than the product topology.*

Proof. We need only show that the preimage of a subbasis element in X_i under the i-th projection map π_i is open in $\mathcal{H}_2$, for every i. Since each X_i is $\mathcal{H}_2$-causally path connected, the subbasis elements are $I^{\pm}(x)$ for inclusive x, and $J^{\pm}(x)$ for roots/tips x.

First we show that $U = \pi_i^{-1}(I^+(x_i))$ is open for every inclusive x_i. Let $y = (y_i) \in U$. If y is a root, then $y \in J^+(y)$ which is open in $\mathcal{H}_2$. Furthermore, for every $z = (z_i) \in J^+(y)$, $0 \le \tau(y, z) \le \tau_i(y_i, z_i)$. Therefore $y_i \le z_i$. Then $x_i < y_i \le z_i$ hence $z \in U$. Hence

$$y \in J^+(y) \subset \pi_i^{-1}(I^+(x_i))$$

Now suppose y is not a root. Then by lemma 4.2.3, y_j is not a root for every j. Then there is $z_j < y_j$ for each j. In particular, there is $z_i < y_i$. Then since X_i is causally path connected and lemma 3.2.3, there is $z_i < w_i < y_i$ such that $x_i < w_i$. Define $z = (z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_n)$. Then $z \in U$, since $x_i < w_i$, and $y \in I^+(z)$, since $z_j < y_j$ for all j, and $w_i < y_i$. Therefore

$$y \in I^+(z) \subset \pi_i^{-1}(I^+(x_i))$$

Hence $\pi_i^{-1}(I^+(x_i))$ is open in $\mathcal{H}_2$. Similarly, $\pi_i^{-1}(I^-(x_i))$ is open.

Now we consider $U = \pi_i^{-1}(J^+(x_i))$, where x_i is a root. Let $y = (y_i) \in U$. If y is a root, then $y \in J^+(y)$, and $J^+(y) \in U$. Suppose y is not a root. Then y_i is not a root, and therefore by lemma 3.2.2, $x_i < y_i$. By the process as above, we have $z_j < y_j$ for all j, and $x_i < w_i < y_i$. Defining $z = (z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_n)$, we have

$$y \in I^+(z) \in \pi_i^{-1}(J^+(x_i))$$

So $\pi_i^{-1}(J^+(x_i))$ is open in $\mathcal{H}_2$ for all roots x, and similarly $\pi_i^{-1}(J^-(x_i))$ is open for all tips x. $\square$

4.3 Quotients

In this section we ask the question of when a space can be made into a causal space. The axioms of causality imply the following properties: the first axiom asserts that every point be causally related to itself, the second asserts that there are no closed causal loops, and the third asserts that time separation between points is maximal. It is relatively easy to construct functions which relate points to themselves and are maximal. It is harder to ensure that no causal loops are present. A possible solution is to identify causal loops as equivalence classes and pass to the quotient. In some sense, we take causal loops and 'shrink them to a point'.

Definition 4.3.1. A set (X, τ) is a *precausal space* if τ satisfies the first and third axioms of causality, ie

1. $\tau(x, x) \geq 0 \quad \forall x \in X$

2. $\tau(x, z) \geq \tau(x, y) + \tau(y, z) \quad \text{if} \quad \tau(x, y) + \tau(y, z) \geq 0$

τ is called a *pre-time separation*.

A pre-time separation defines a reflexive and transitive order $\leq$ in a similar manner to a standard time separation. However, this order is not necessarily asymmetric. Still, we can endow a precausal space (X, τ) with the same topologies as a causal space.

Given a precausal space (X, τ), we define a relation $x \sim y \iff y \leq x \leq y$. This relation is reflexive since $\leq$ is reflexive, and it is both symmetric and transitive since $\leq$ is transitive. Therefore it is an equivalence relation.

Lemma 4.3.1. *Let $x \sim y \in X$, and let $z \in X$. Then $\tau(x, z) = \tau(y, z)$ and $\tau(z, x) = \tau(z, y)$.*

Proof. We will only prove $\tau(x, z) = \tau(y, z)$, as the other claim is symmetric. First observe that $x \leq z \iff y \leq z$. Hence if $x \nleq z$, then $y \nleq z$, and $\tau(x, z) = -\infty = \tau(y, z)$. Now assume $x \leq z$ and $y \leq z$. Then $y \leq x \leq z$ and $x \leq y \leq z$, so

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z) \geq \tau(y, z)$$
$$\tau(y, z) \geq \tau(y, x) + \tau(x, z) \geq \tau(x, z)$$

Which implies $\tau(x, z) = \tau(y, z)$. □

By lemma 4.3.1, τ respects the equivalence relation $\sim$, and therefore there is a well-defined map $\tilde{\tau}$ on the quotient $\tilde{X} = X/\sim$, given by:

$$\tilde{\tau} : \tilde{X} \times \tilde{X} \to \{-\infty\} \sqcup [0, \infty] \quad ([x], [y]) \mapsto \tau(x, y)$$

Theorem 4.3.1. *$(\tilde{X}, \tilde{\tau})$ is a causal space.*

Proof. $\tilde{\tau}$ inherits the first and third axioms of causality from τ. It is left to show that it satisfies the second. Suppose $\tilde{\tau}([x],[y]) \geq 0$ and $\tilde{\tau}([y],[x]) \geq 0$. Then:

$$\tau(x,y) = \tilde{\tau}([x],[y]) \geq 0 \implies x \leq y$$

$$\tau(y,x) = \tilde{\tau}([x],[y]) \geq 0 \implies y \leq x$$

Hence $x \sim y \implies [x] = [y]$. Therefore for all $[x] \neq [y]$,

$$\tilde{\tau}([x],[y]) \geq 0 \implies \tilde{\tau}([y],[x]) = -\infty$$

□

Finally we give a simple result relating the topology on the precausal space with that of the causal space.

Remark 4.3.1. From the fact that $\tau(x,y) = \tilde{\tau}([x],[y])$, we have that the $\mathcal{H}_i$ topology on $(\tilde{X},\tilde{\tau})$ coincides with the quotient topology.

Chapter 5

Hausdorff time separation

In this short section, we introduce a causal variation of the Hausdorff distance, called a Hausdorff time separation. We will see that this induces a precausal structure on the power set $\mathcal{P}(X)$ of a causal space X, and a causal structure on $\mathcal{J}(X)$.

We begin by defining the following:

Definition 5.0.1. Given $A \subset X$ and $r \geq 0$, we write

$$I_r^{\pm}(A) := \bigcup_{a \in A} I_r^{\pm}(a) \qquad J_r^{\pm}(A) := \bigcup_{a \in A} J_r^{\pm}(a)$$

Definition 5.0.2. Let (X, τ) be a causal space. For $A, B \subset X$, we define the *Hausdorff time separation* to be:

$$\tau_H(A, B) := \sup\{r \geq 0 : A \subset J_r^-(B), B \subset J_r^+(A)\}$$

with the convention $\sup(\varnothing) = -\infty$

Proposition 4. *$(\mathcal{P}(X), \tau_H)$ is a precausal space.*

Proof. First it is clear that $\tau_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \{-\infty\} \sqcup [0, \infty]$, since τ_H is only negative if the supremum is empty, in which case it is $-\infty$. Next, for any $A \in \mathcal{P}(X)$, $A \in J^{\pm}(A)$, since $\tau(x, x) \geq 0$ for all $x \in X$. Hence $\tau_H(A, A) \geq 0$. This verifies the first axiom of causality.

Now let $\tau_H(A, B) = r, \tau_H(B, C) = s$, and suppose $r, s \geq 0$. Then for each $\varepsilon > 0$,

$$A \subset J_{r-\varepsilon}^-(B); \quad B \subset J_{r-\varepsilon}^+(A)$$
$$B \subset J_{s-\varepsilon}^-(C); \quad C \subset J_{s-\varepsilon}^+(B)$$

Let $a \in A$. Then there exists $b \in B$ such that $\tau(a,b) \geq r - \varepsilon$. Furthermore there is $c \in C$ such that $\tau(b,c) \geq s - \varepsilon$. Then

$$\tau(a,c) \geq \tau(a,b) + \tau(b,c) \geq r + s - 2\varepsilon \tag{5.1}$$

And therefore $A \subset J^-_{r+s-2\varepsilon}(C)$. Similarly, for each $c \in C$, there is $b \in B$ such that $\tau(b,c) \geq s - \varepsilon$ and there is $a \in A$ such that $\tau(a,b) \geq r - \varepsilon$. Then

$$\tau(a,c) \geq \tau(a,b) + \tau(b,c) \geq r + s - 2\varepsilon \tag{5.2}$$

And therefore $C \subset J^+_{r+s-2\varepsilon}(A)$. This holds for all sufficiently small $\varepsilon > 0$, and therefore $A \subset J^-_{r+s}(C)$ and $C \subset J^+_{r+s}(A)$. Hence:

$$\tau_H(A,C) \geq r + s = \tau_H(A,B) + \tau_H(B,C) \tag{5.3}$$

This verifies the third axiom of causality, so $(\mathcal{P}(X), \tau_H)$ is a precausal space. $\square$

We then know that $(\widetilde{\mathcal{P}(X)}, \tilde{\tau_H})$ is a causal space, but in this case we can say more.

Lemma 5.0.1. *Let A be causally convex. Then $B \sim A \implies B \subset A$.*

Proof. Let $B \sim A$. Then for every $b \in B$, there is $a, a' \in A$ such that $a \leq b \leq a'$. Then $b \in J(a,a')$, and hence $b \in A$, since A is causally convex. Then $B \subset A$. $\square$

Lemma 5.0.2. *For all $A \subset X$, $A \sim \mathrm{conv}_\tau(A)$.*

Proof. Since $A \subset \mathrm{conv}_\tau(A)$, $A \subset J_0^\pm(\mathrm{conv}_\tau(A))$. It is left to show that $\mathrm{conv}_\tau(A) \subset J_0^\pm(A)$. Let $a \in \mathrm{conv}_\tau(A)$. Then either $a \in A \cap S$, or $a \in J(b,b')$ for some $b, b' \in A$. If the former is true, then $a \in A$, so $a \in J_0^\pm(A)$. If the latter is true, then $b \leq a \leq b'$, so thus $a \in J_0^\pm(A)$. Hence $\mathrm{conv}_\tau(A) \subset J_0^\pm(A)$. Hence $A \sim \mathrm{conv}(A)$. $\square$

Recall that $\mathcal{J}(X)$ denotes the collection of all causally convex subsets of X. The above lemmas give the following result.

Theorem 5.0.1. *$(\mathcal{J}(X), \tau_H)$ is a causal space.*

Proof. Let $A \sim B$. By lemma 5.0.2, $\mathrm{conv}_\tau(A) \sim A \sim B \sim \mathrm{conv}_\tau(B)$. By lemma 5.0.1, $\mathrm{conv}_\tau(A) \subset \mathrm{conv}_\tau(B) \subset \mathrm{conv}_\tau(A)$. Therefore $A \sim B \implies \mathrm{conv}(A) = \mathrm{conv}(B)$. On the other hand, if $\mathrm{conv}_\tau(A) = \mathrm{conv}_\tau(B)$, then by lemma 5.0.2, $A \sim \mathrm{conv}_\tau(A) = \mathrm{conv}_\tau(B) \sim B$. Therefore $A \sim$

$B \iff \mathrm{conv}_\tau(A) = \mathrm{conv}_\tau(B)$. It follows that on $\mathcal{J}(X)$, every equivalence class contains a single set. $\square$

Bibliography

[Ale51] AD Alexandrov. "A theorem on triangles in a metric space and some of its applications". In: *Trudy Mat. Inst. Steklov.* 38 (1951), pp. 5–23.

[Top59] Viktor Andreevich Toponogov. "Riemann spaces with curvature bounded below". In: *Uspekhi Matematicheskikh Nauk* 14.1 (1959), pp. 87–130.

[KP67] E. H. Kronheimer and R. Penrose. "On the structure of causal spaces". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 63.2 (1967), pp. 481–501. DOI: 10.1017/S030500410004144X.

[Kel75] J.L. Kelley. *General Topology.* Graduate Texts in Mathematics. Springer New York, 1975. ISBN: 9780387901251. URL: https://books.google.ca/books?id=-goleb9Ov3oC.

[GLP81] M. Gromov, J. Lafontaine, and P. Pansu. *Structures métriques pour les variétés riemanniennes.* Textes mathématiques. Recherche. CEDIC/Fernand Nathan, 1981. ISBN: 9782712407148. URL: https://books.google.ca/books?id=TxNOQgAACAAJ.

[BGP92] Yu Burago, M Gromov, and G Perel'man. "A.D. Alexandrov spaces with curvature bounded below". In: *Russian Mathematical Surveys* 47.2 (Apr. 1992), pp. 1–58. DOI: 10.1070/rm1992v047n02abeh000877. URL: https://doi.org/10.1070/rm1992v047n02abeh000877.

[Per95] G. Perelman. "Spaces with Curvature Bounded Below". In: *Proceedings of the International Congress of Mathematicians.* Ed. by S. D. Chatterji. Basel: Birkhäuser Basel, 1995, pp. 517–525. ISBN: 978-3-0348-9078-6.

[BBI01] D. Burago, Y. Burago, and S.V. Ivanov. *A Course in Metric Geometry.* Crm Proceedings & Lecture Notes. American Mathematical Society, 2001. ISBN: 9780821821299. URL: https://books.google.ca/books?id=dRmIAwAAQBAJ.

[GS05] Alfonso Garcéa-Parrado and José Senovilla. "TOPICAL REVIEW: Causal structures and causal boundaries". In: *Classical and Quantum Gravity* 22 (May 2005), pp. 1–. DOI: `10.1088/0264-9381/22/9/R01`.

[Spe07] Timothy Speer. *A Short Study of Alexandroff Spaces.* 2007. DOI: `10.48550/ARXIV.0708.2136`. URL: `https://arxiv.org/abs/0708.2136`.

[PZ11] Victor Pambuccian and Tudor Zamfirescu. "Paolo Pizzetti: The forgotten originator of triangle comparison geometry". In: *Historia Mathematica* 38.3 (2011), pp. 415–422. ISSN: 0315-0860. DOI: `https://doi.org/10.1016/j.hm.2010.11.001`. URL: `https://www.sciencedirect.com/science/article/pii/S0315086010000911`.

[KS18] Michael Kunzinger and Clemens Sämann. "Lorentzian length spaces". In: *Annals of Global Analysis and Geometry* 54 (Oct. 2018). DOI: `10.1007/s10455-018-9633-1`.

[BS22] Tobias Beran and Clemens Sämann. *Hyperbolic angles in Lorentzian length spaces and timelike curvature bounds.* 2022. DOI: `10.48550/ARXIV.2204.09491`. URL: `https://arxiv.org/abs/2204.09491`.