WEAK LAGRANGIAN SOLUTIONS TO A ONE DIMENSIONAL MODEL OF THE MOIST SEMI-GEOSTROPHIC EQUATIONS

by

Dorian Goldman

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Graduate Department of Mathematics

University of Toronto

Supervisor: Robert J. McCann

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Abstract

The semi-geostrophic equations with additional terms incorporating the effects of moisture are introduced. A one dimensional model of these equations which is equivalent to the full three dimensional in the spatially homogenous case is studied and a weak formulation of this model is defined. A new stability condition that strengthens the Cullen-Purser stability condition is introduced and shown to be required for the the dynamics to be energy minimizing. A time stepping procedure is used to construct difference equations in discrete space and time which are shown to converge to weak solutions as the size of the mesh tends to zero under certain monotonicity assumptions on the initial data.

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Contents

List of Symbols

Variables

θ	potential temperature
q	moisture content
q^{sat}	saturation curve in atmosphere
$F(t,z)$	Lagrangian map
ω	velocity measure
(u,v,w)	full semi-geostrophic velocity
(u_g,v_g)	geostrophic velocity
ϕ	geo-potential
$\tilde{\theta}$	Lagrangian potential temperature
$\tilde{q}$	Lagrangian moisture content
θ_h	Discrete approximation of potential temperature
q_h	Discrete approximation of moisture content
$\tilde{\theta}_h$	Discrete Lagrangian approximation of potential temerature
$\tilde{q}_h$	Discrete Lagrangian approximation of moisture content
t	time
z	height

Constants

L	Latent heat of condensation
α	velocity of column of air
β	slope of saturation curve q_{sat}

1 Introduction

The semi-geostrophic equations are a valid approximation of the Navier Stokes equations on large (horizontal) scales which model rotationally dominated, stratified flows in the atmosphere [4]. Work by Brenier and Benamou, [2], Cullen and Gangbo, [5], Cullen and Maroofi, [6] and Loeper [11] established weak existence of solutions in a set of *dual variables* first introduced by Hoskins [10]. Smoothness and uniqueness of solutions for the semi-geostrophic equations remain unknown, except for Loeper's results [11] on the torus. A recent result of Ambrosio [1] concerning the existence of solutions to ODEs with BV vector fields, combined with the results of [2] is used by Cullen and Feldman to prove weak existence of the semi-geostrophic equations in a Lagrangian reformulation of the original physical variables [4]. Holt [9] introduced a geometric algorithm to study the effects of moisture convection in the semi-geostrophic equations which is similar to the difference scheme we use, however no rigorous analysis showing the consistency with the classical equations was established. This paper is in some sense a first step towards rigorously extending the result of [4] to include the effects of moist convection studied in [9].

We study the semi-geostrophic equations with the effect of moisture included. This corresponds to introducing an additional term q representing moisture in the semi-geostrophic system studied by Cullen and Feldman in [4], and replacing conservation of potential temperature θ, with equations (4)–(5) below. The eight equations written in Eulerian form are for the eight unknowns $\mathbf{u} = (u, v, w), \phi, \theta, q, u_g, v_g$, on a bounded domain $[0, T) \times \Omega$ where $\Omega \subset \mathbf{R}^3$ are

$$\frac{D(u_g, v_g)}{Dt} + (-v, u) = (\partial_1 \phi, \partial_2 \phi) \tag{1}$$

$$\text{where } \frac{D}{Dt} := \partial_t + (u, v, w) \cdot \nabla,$$

$$(u_g, v_g) := (-\partial_2 \phi, \partial_1 \phi)$$

$$\nabla \cdot (u, v, w) = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial z} + \theta = 0, \tag{3}$$

$$\frac{Dq}{Dt} = \begin{cases} L\left[\frac{Dq^{sat}}{Dt}\right]^{-} & \text{if } q = q^{sat} \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$\frac{D\theta}{Dt} + L\frac{Dq}{Dt} = 0, \tag{5}$$

where $[f]^{-} \leq 0$ denotes the negative part of the function f.

In what follows we choose units so that $L \equiv 1$, where L is the latent heat of condensation. Above, q^{sat} represents the saturation moisture level which is the maximum moisture content a particular air parcel may have. This is observed to be a monotone decreasing function of height [3]. The function q^{sat} and coefficient L are the *exogenous* variables that appear in equations (1)–(5).

The *endogenous* variables are $(u, v, w, \phi, \theta, q, u_g, v_g)$, where ϕ is a geopotential and θ is the *potential temperature*. This is the temperature an air parcel would have if it were brought adiabatically to a standard reference pressure, such as the pressure at the surface of the earth [3]. Since the potential temperature is measured with respect to a standard reference pressure, fluctuations in height of an air parcel do not affect it in the absence of condensation, which will occur when a parcel becomes saturated with moisture. This phenomenon is described by equations (4)–(5) which yield a conservation law for the potential temperature away from saturation. We are most interested in the vertical component of the above equations which corresponds to (4)–(5) since this is where the effects of moisture are incorporated into the equations. For this reason we study a one dimensional version of (1)–(5) where we assume $u \equiv v \equiv 0$ and study (4)–(5) exclusively restricted to a one dimensional column. This is equivalent to studying the three dimensional problem in the spatially homogenous case, meaning all variables depend only on (z, t) instead of (x, y, z, t). The incompressibility condition (2) then forces the vertical component of the velocity w to be constant and we denote this velocity as α below. This condition however, will be replaced with the requirement that there is a measure preserving flow compatible with the Lagrangian form of equations (4)–(5).

When air parcels become saturated ($q = q^{sat}$), precipitation occurs and this phase transition causes the liberation of heat. The heat liberated then causes the parcel to rise while further changes to the parcel's potential temperature occur according to equation (4)–(5) as the parcel rises along the saturated adiabat q^{sat}. Physically it is observed that parcels quickly arrange themselves so that the parcels with highest potential temperature are at the top of the column. This amounts to assuming $\theta(t, \cdot)$ is monotone increasing at all times which is a physical constraint we add to equations (1)–(5). This is a known as the *Cullen-Purser stability criterion* and was first proposed in [7]. Without the effects of moisture (when $q \equiv 0$), θ is conserved in time and hence monotonicity of θ is conserved. When moisture effects are included, a stronger constraint is required on $\theta(t, \cdot)$ which we introduce below in Definition 1.

2 One dimensional moist semi-geostrophic model

We assume horizontally uniform potential temperature and moisture throughout the column which is rising at a constant rate $\alpha > 0$. In the notation below t represents time and $z \in [0,1]$ represents the position in the column of air of total height 1. For simplicity we assume the following saturation curve,

$$q^{sat}(t,z) := q^{sat}(0,0) + \frac{dq_{sat}}{dz}(z + \alpha t), \tag{6}$$

where $\frac{dq_{sat}}{dz} := -\beta < 0$ is a negative constant and $\alpha > 0$ is the constant speed of ascent of the column. The time dependence of q^{sat} reflects the vertical displacement of the column.

We now introduce a Lagrangian reformulation of (4)–(5). Given functions $\theta, q : [0,T) \times [0,1] \to \mathbf{R}^+$ satisfying (4)–(5), we seek a volume preserving diffeomorphism $F : [0,T) \times [0,1] \to [0,1]$ such that, setting $\tilde{\theta}(t,z) = \theta(t, F(t,z))$, $\tilde{q}(t,z) = q(t, F(t,z))$ and $\tilde{q^{sat}}(t,z) = q^{sat}(t, F(t,z))$ the following equations are satisfied:

$$\partial_t \tilde{q}(t,z) = -\beta 1_{[1,\infty)}(\tilde{q}/\tilde{q^{sat}})[\partial_t F(t,z) + \alpha]^+ \tag{7}$$

$$\partial_t \tilde{\theta}(t,z) + \partial_t \tilde{q}(t,z) = 0. \tag{8}$$

In addition we require the stability condition that $z \mapsto \theta(t,z)$ be monotone non-decreasing for all times. It turns out this goal is too ambitious and we are forced to introduce a weak formulation of (6)–(8) in section 2.2. We now introduce an additional physical requirement on $z \mapsto \theta(t,z)$ that is a strengthening of the monotonicity condition and which we will require on all solutions.

2.1 A stronger stability condition

We will continue to require that $z \mapsto \theta(t,z)$ be monotone non-decreasing for all times since the effects of moisture do not change the physical observation that the parcels with largest potential temperature quickly rise to the top of the column. This physical observation can be interpreted as an energy minimization condition for the semi-geostrophic energy:

$$E_{SG} := -\int_{[0,1]} z\theta(z)dz, \tag{9}$$

since the above integral will be smaller when parcels with larger values of θ are distributed higher in the column.

Saturated air

When the effects of moisture are included, this stability condition must be modified since there is an ambiguity in the dynamics of equations (6)–(8) if only the monotonicity of $z \mapsto \theta(t,z)$ is imposed as a physical requirement on the solutions. We demonstrate this ambiguity below and show how it is resolved by choosing the state which minimizes the integral (9).

Consider a column with only two parcels z_1 and z_2 in $[0,1]$ and assume they have potential temperature profiles $\theta_1 = \theta_2$ and moisture profiles q_1, q_2 with $q_1 - q_2 = -\beta(z_2 - z_1)$. Then time stepping equation (6) forward, we see that z_1 and z_2 will both become saturated at the same instant in time. However now there is a choice in the dynamics. Both z_1 and z_2 could increase their potential temperature equally (since they become saturated simultaneously) and the monotonicity of the potential temperature profile would be preserved. However, there is an alternative evolution. If $z_1 \mapsto z_2$ and $z_2 \mapsto z_1$ then instead of z_1 and z_2 sharing the increase in potential temperature, all of the increase is absorbed into z_1 as it rises above z_2 along the saturated adiabat q^{sat}. By studying (9) it is clear that the second scenario does indeed result in a lower energy state since when z_1 and z_2 swap locations, more of the available potential temperature is distributed at the top of the column. Hence we wish to introduce a stability condition which favors distributing as much of the available heat released from the convection at the top of the column. This ambiguity is eliminated if we introduce the stronger stability condition that saturated parcels quickly arrange themselves so that $z \mapsto \theta(z)+q(z)$ is a monotone non-decreasing function. We now make this condition precise.

Definition 1 *(Strong stability) We say that the function $\theta : [0,T) \times [0,1] \to \mathbf{R}^+$ is strongly stable if for each $t \in [0,T)$*

$$
\begin{aligned}
&z \mapsto \theta(t,z) - \beta z \text{ is monotone increasing on } Z_t = \{z : q(t,z) = q^{sat}(t,z)\},\\
&z \mapsto \theta(t,z) \text{ is monotone increasing on } [0,1].
\end{aligned}
$$

This condition resolves the ambiguity in the above example and requires the evolution to favor the second scenario, where z_1 and z_2 swap locations. The above stability condition is well known amongst meteorologists and can be found in the work of Cullen [3]. Equations (6)–(8) along with the stability condition in Definition 1 are what we will refer to as the *moist one-dimensional semi-geostrophic model*.

2.2 Motivation for weak formulation

As the column of air rises in the atmosphere, it is possible that at some instant in time the stability condition in Definition 1 will fail to be satisfied. An instantaneous and discontinuous rearrangement of the fluid particles may then occur to correct the lack of stability. As a simple example, if the column of air has uniform potential temperature and the moisture in a parcel at the bottom of the column becomes saturated and heat is released, the parcel will jump discontinuously to the top of the column to a stable configuration. Hence we expect very little regularity from the Lagrangian map $F : [0, T] \times [0, 1]$ and at best we could hope for a measure valued velocity field. We therefore introduce a weak formulation as follows. As a weakening of a volume preserving diffeomorphism F of $[0, 1]$ to itself, we seek a map $F : [0, T) \times [0, 1] \to [0, 1]$ that is Lebesgue measure preserving in the sense that

$$F(t, .)_{\#}\mathcal{L}^1_{[0,1]} = \mathcal{L}^1_{[0,1]}. \tag{10}$$

where here $F(t, \cdot)_{\#}\mathcal{L}^1_{[0,1]}$ is the *push forward* of one dimensional Lebesgue measure by the map $z \mapsto F(t, z)$, defined by

$$F(t, \cdot)_{\#}\mathcal{L}^1_{[0,1]}(E) = \mathcal{L}^1(F^{-1}(t, E)),$$

on $\mathcal{L}^1$ measurable sets $E \subset [0, 1]$. In addition we require F to define a flow in the space of measure preserving mappings. These requirements motivate conditions (i)–(ii) below in Definition 2. We can obtain sufficient regularity of the maps F to ensure that $\partial_t F(t, z)$ defines a Borel measure valued velocity field and hence our weak formulation involves understanding (7)–(8) in the sense of measures. In addition we require our solutions to satisfy the strong stability assumption (iii) given in Definition 1.

It turns out to be difficult to relate the velocity measure derived from the Lagrangian map $z \mapsto F(t, z)$ to the velocity on the right hand side of equation (7). This difficulty is encountered due to rapid oscillations of particles, which occurs when saturated and unsaturated air mix. In such a scenario, it is physically observed that a parcel's moisture may not change according to equation (7) but may become unsaturated upon mixing with dry air parcels when it rises in the atmosphere. These physical observations are incorporated into the assumptions on the measure ω below. We will show in Theorem 4 that our weak notion of a solution is indeed consistent with a smooth solution of (6)–(8) in the absence of the mixing described above.

Definition 2 *(Weak Lagrangian Formulation) Given $T > 0$ we seek a triplet*

$$\tilde{\theta}, \tilde{q}, F \in BV([0,T) \times [0,1]),$$

along with a positive measure $d\omega(t,z)$ with spt $\omega \subset \{(t,z) : \tilde{q} = \tilde{q}_{sat}\}$. We then require the following equations to be satisfied:

$$-\int_0^1 \int_0^T \partial_t \phi(t,z) \tilde{q}(t,z) dt dz = -\beta \int_0^1 \int_0^T \phi(t,z) d\omega(t,z) + \int_0^1 \phi(0,z) q(0,z) dz$$

$$\int_0^1 \int_0^T \partial_t \phi(t,z) [\tilde{\theta}(t,z) + \tilde{q}(t,z)] dt dz = -\int_0^1 \phi(0,z)[\tilde{\theta}(0,z) + \tilde{q}(0,z)] dz$$

for all $\phi \in C_c^\infty([0,T) \times [0,1])$.

We require the following conditions on the solution for it to be physically admissible:

(i) $F(t,\cdot)$ pushes forward *$\mathcal{L}^1([0,1])$ to $\mathcal{L}^1([0,1])$ and yields the monotone rearrangement $\theta(t,\cdot)$ of $\tilde{\theta}(t,\cdot)$ for a.e $t \in [0,T)$.*
(ii) $F(0,z) = z$ for a.e $z \in [0,1]$
(iii) (stability) θ is stable in the sense of Definition 1.
(iv) $\tilde{q}(t,z) \leq \tilde{q}_{sat}(t,z)$ for a.e $(t,z) \in [0,T) \times [0,1]$.

Remark 3 *Note that $t \mapsto F(t,z)$ is only responsible for rearrangements of the column relative to itself. If the map $z \mapsto \theta(t,z)$ is strictly increasing and either $z \mapsto \theta(t,z) + q(t,z)$ is strictly increasing or $\tilde{q}(t,z) < \tilde{q}_{sat}(t,z)$ is satisfied for all times on $[0,1]$, then $F(t,z) = z$ for all times and no rearrangements occur. When $F(t,\cdot) \neq Id$ we say* mixing *has occurred.*

In the following theorem, we show that in the absence of the mixing described in the previous remark, a smooth weak solution in the sense of Definition 2 is indeed a smooth solution of (6)–(8).

Theorem 4 *(Consistency of weak formulation in the absence of mixing) A smooth weak solution* $(\tilde{\theta}, \tilde{q}, F, \omega)$ *to* (6)–(8) *in the sense of Definition 2 which satisfies* $\partial_z\theta(t,z) > 0$ *and either* $\partial_z\theta(t,z) + \partial_z q(t,z) > 0$ *or* $q(t,z) < q^{sat}(t,z)$ *on* $[0,1]$ *for each* $t \in [0,T]$ *satisfies equations* (6)–(8).

Proof: The case $q(t,z) < q^{sat}(t,z)$ is clear by noting that spt $\omega \subset \{(t,z) : q^{sat}(t,z) = q(t,z)\}$. We deal with the first case now.

To preserve strict monotonicity of $z \mapsto \theta(t,z)$ and $z \mapsto \theta(t,z) + q(t,z)$, it is clear that $F(t,z) = z$ for each $t \in [0,T)$. We write $d\omega = g(t,z)dtdz$ where g is a smooth positive function. Then the following equations are satisfied:

$$\partial_t\tilde{q}(t,z) = -\beta g(t,z) \tag{11}$$

$$\partial_t\tilde{\theta}(t,z) + \partial_t\tilde{q}(t,z) = 0. \tag{12}$$

It is clear that $g(t,z) \geq 1_{[1,\infty)}(\tilde{q}/\tilde{q^{sat}})[\partial_t F(t,z)+\alpha]^+$ since otherwise condition (v) would be violated in Definition 2. Assume there exists a $z_0 \in [0,1]$ and $t_0 \in [0,T)$ such that $g(t_0,z) > 1_{[1,\infty)}(\tilde{q}/\tilde{q^{sat}})\alpha$. Because $z \mapsto g(t,z)$ is smooth however, for any $\epsilon > 0$, equation (11) along with the fact that spt $\omega \subset \{(t,z) : q^{sat}(t,z) = q(t,z)\}$ guarantees that $q(t_0+\epsilon, z_0) = q^{sat}(t_0+\epsilon, z_0)$. Since $g(t,z) \neq 0$ if and only if $q(t,z) = q^{sat}(t,z)$, it follows from (6) and (8) that $g(t_0,z_0) = \alpha$, a contradiction. Consequently

$$\begin{aligned} g(t,z) &= 1_{[1,\infty)}(\tilde{q}/\tilde{q^{sat}})\alpha \\ &= 1_{[1,\infty)}(\tilde{q}/\tilde{q^{sat}})[\partial_t F(t,z)+\alpha]^+, \end{aligned}$$

which yields the desired result. □

3 Construction of the solutions

We construct the weak solutions by replacing the continuous equations (6)–(8) with difference equations with finite time step $\Delta t > 0$ and spatial partition size $h > 0$, while requiring that $\theta(t,\cdot)$ satisfies the stability condition in Definition 1 at all times. We then let the size of the time and space partition tend to zero and obtain weak solutions to the equations (6)–(8) in the sense of Definition 2.

3.1 Time stepping procedure

We discretize equations (6)–(8) in time yielding,

$$q_h^{sat}(t_{k+1}, z_i) = q_h^{sat}(0,0) - \beta(z_i + \alpha t_{k+1}), \tag{13}$$

$$\tilde{q}_h(t_{k+1}, z_i) - \tilde{q}_h(t_k, z_i) = -\beta 1_{[1,\infty)} \left(\frac{\tilde{q}_h(t_k, z_i)}{\tilde{q}_h^{sat}(t_k + \Delta t, z_i)} \right) [F_h(t_{k+1}, z_i) - F_h(t_k, z_i) + \alpha \Delta t]^+ \tag{14}$$

$$\tilde{\theta}_h(t_{k+1}, z_i) - \tilde{\theta}_h(t_k, z_i) + \tilde{q}_h(t_{k+1}, z_i) + \tilde{q}_h(t_k, z_i) = 0. \tag{15}$$

along with the requirement that $\theta_h(t_k, \cdot)$ satisfy Definition 1 and $F_h(t_{k+1}, \cdot)_{\#}\mathcal{L}^1_{[0,1]} = \mathcal{L}^1_{[0,1]}$ for each $h > 0$. Hence the algorithm described below will involve taking a time step $\Delta t > 0$ and determining $q^{sat}(t_k + \Delta t, \cdot)$. Subsequent iterations will then involve computing new values of $\tilde{\theta}_h, \tilde{q}_h$ from the last two equations and the performing rearrangements of $\theta_h + q_h$ on the saturated regions as to satisfy the stability condition in Definition 1.

4 Main result

We now prove our main existence result for very specific initial data. Specifically we prove existence when $z \mapsto \theta_0(z) + q_0(z)$ is monotone non-increasing. This assumption, although specific, still demonstrates important aspects of the dynamics of weak solutions to (6)–(8) as defined by Definition 2.

Theorem 5 *(Global in time existence) Fix θ_0, q_0 such that $z \mapsto \theta_0(z) + q_0(z)$ is monotone non-increasing, $z \mapsto q_0(z) + (\beta - \epsilon)z$ is strictly increasing for some $0 < \epsilon < \beta$, and θ_0 is stable in the sense of Definition 1. Given $T > 0$ there exists a weak solution to (6)–(8), $\tilde{\theta}$, F, $\tilde{q}$, ω in the sense of Definition 2 on $[0, T) \times [0, 1]$, which satisfies $\theta_0 = \tilde{\theta}(0, \cdot)$ and $q_0 = \tilde{q}(0, \cdot)$ a.e.*

Proof: The proof procedure will involve taking time steps that correct the lack of monotonicity of $\theta + q$ on the saturated regions during each time step, while allowing further exchanges between the parcel's potential temperature and moisture to occur according to equations (14)–(15). We will make an inductive argument on the time step and show that certain properties of θ and q are preserved. A main ingredient is the compact embedding $BV \subset\subset L^1$ (see [8]), combined with the Riesz representation theorem to prove the convergence of our difference scheme to a weak solution to (6)–(8) in the sense of Definition 2.

Partition $[0, T)$ into N intervals $[kT, (k + 1)T/N)$ and $[0, 1]$ into M intervals $[k/M, (k + 1)/M)$ where $\Delta t := T/N$ satisfies $\Delta t < \epsilon h/\alpha\beta$. This assumption ensures that only one parcel will become saturated at a time, in sequence from the top of the column. We approximate θ_0 and q_0 by functions $\theta_h(0, \cdot)$ and $q_h(0, \cdot)$ by defining $\theta_h(0, z) := \theta_0((k + 1)/M)$ and $q_h(0, z) := q_0((k + 1)/M)$ for $z \in [z_k, z_{k+1})$ where $z_k := k/M$. Our choice of approximation ensures that $z \mapsto \theta_h(0, z)$ satisfies Definition 1 and $z \mapsto q_h(0, z) + (\beta - \epsilon)z$ is monotone increasing. The latter property will be important in Step 2 of the proof. Below $\tilde{\theta}_h$ and $\tilde{q}_h$ will denote the solutions of the difference equations (14)–(15) which we construct below.

Step 1 (Rearrangements):

We now describe the time stepping procedure inductively. At an initial time $t_k := kT$, we assume that $\theta_h(t_k, z_i) = \theta(0, z_i)$, $F_h(t_k, z_i) = z_i$ and $q_h(t_k, z_i) = q_h(0, z_i)$ on $[0, b_k)$ for some $b_k \in [0, 1]$, that $z \mapsto \theta_h(t_k, z) + q_h(t_k, z)$ is monotone increasing on $[b_k, 1]$ and $q_h(t_k, z) \leq q_h^{sat}(t_k, z)$ on $[0, 1]$, with the the top particle remaining saturated ($q_h(t_k, z_M) = q_h^{sat}(t_k, z_M)$) at all times t_k after it becomes saturated initially. These properties will be preserved at the end of the time step to time t_{k+1}, with evolution of $\tilde{\theta}_h$, $\tilde{q}_h$ proceeding according to equations (14)–(15).

Compute $q_h^{sat}(t_k + \Delta t, z)$ from (13). We will call a parcel z_i *saturated at time t* if $q_h(t, z_i) \geq q_h^{sat}(t + \Delta t, z_i)$. Then since $\Delta t < \epsilon h/\alpha\beta$ and $z \mapsto q_h(t_k, z) + (\beta - \epsilon)z$ is increasing on $[0, b_k)$, at most one parcel z_i becomes saturated from $[0, b_k)$ and if so set $b_{k+1} = z_i$ and define the map $z \mapsto G_h(t_{k+1}, z)$ by

$$G_h(t_{k+1}, z_j) := \begin{cases} z_j & \text{if } z_j < b_{k+1} \\ z_{j-1} & \text{if } b_{k+1} < z_j \leq z_M \\ z_M & \text{if } j = i, \end{cases} \tag{16}$$

and extend $G_h(t_{k+1}, \cdot)$ to be piecewise constant on the intervals $z \in [z_j, z_{j+1})$.
Use $\tilde{q}_h^{sat}(t_{k+1}, z_j) = q_h^{sat}(t_{k+1}, G_h(t_{k+1}, z_j))$ to define the functions $\tilde{q}_h(t_{k+1}, z_j)$, $\tilde{\theta}_h(t_{k+1}, z_j)$ by

$$\tilde{q}_h(t_{k+1}, z_j) := \tilde{q}_h(t_k, z_j) - 1_{[1,\infty)}\left(\frac{\tilde{q}_h(t_k, z_j)}{\tilde{q}_n^{sat}(t_{k+1}, z_j)}\right)(\beta[G_h(t_{k+1}, z_j) - z_j] + \tilde{q}_h(t_k, z_j) - \tilde{q}_h^{sat}(t_{k+1}, z_j))),$$
$$\tilde{\theta}_h(t_{k+1}, z_j) := \tilde{\theta}_h(t_k, z_j) + \tilde{q}_h(t_k, z_j) - \tilde{q}_h(t_{k+1}, z_j)$$

for all $j = 0, \cdots, M - 1$, and the corresponding Eulerian functions are defined as:

$$\theta_h(t_{k+1}, G_h(t_{k+1}, z_j)) := \tilde{\theta}_h(t_{k+1}, z_j),$$
$$q_h(t_{k+1}, G_h(t_{k+1}, z_j)) := \tilde{q}_h(t_{k+1}, z_j).$$

Similarly to $G_h(t_{k+1}, \cdot)$, extend $\tilde{\theta}_h(t_{k+1}, \cdot)$, $\tilde{q}_h^{sat}(t_{k+1}, \cdot)$ and $\tilde{q}_h(t_{k+1}, \cdot)$ to be piecewise constant on the intervals $z \in [z_j, z_{j+1})$. The definition (16) maps the parcel z_i to the top position in the column so that $z \mapsto \theta_h(t_{k+1}, z) + q_h(t_{k+1}, z)$ is monotone non-decreasing on $[b_{k+1}, 1]$ where $b_{k+1} = z_i$. Hence the inductive hypotheses are satisfied for b_{k+1} at time t_{k+1} and satisfied for $t = 0$. The map $z \mapsto \theta_h(t_{k+1}, z)$ also remains monotone increasing on $[b_{k+1}, 1]$. To demonstrate this, note that since the particle initially at b_{k+1} ends up with less moisture content than the particle at z_M, $q_h(t_k, z_M) = q_h^{sat}(t_k, z_M) = \tilde{q}_h^{sat}(t_{k+1}, b_{k+1})$ and $z_M \mapsto z_{M-1}$ under (16), $\tilde{\theta}_h(t_k, b_{k+1}) + \tilde{q}_h(t_k, b_{k+1}) \geq \theta_h(t_k, z_M) + q_h(t_k, z_M)$ implies that $\theta_h(t_{k+1}, z_M) \geq \theta_h(t_{k+1}, z_{M-1})$ and $z \mapsto \theta_h(t_{k+1}, z)$ was already monotone on $[b_{k+1}, z_{M-1})$. Now define $F_h(t_{k+1}, z_j) := G_h(t_{k+1}, F_h(t_k, z_j))$ and proceed to Step 2.

Step 2 (Estimates):

(i) *Spatial estimates:*
We proceed by making an inductive argument about the behavior of $z \mapsto F_h(t_k, z)$ based on the time step done in Step 1 above, which controls the BV norm on $[0, 1]$.

Following the inductive time step in Step 1, it is clear that the map $z \mapsto F_h(t_k, z)$ remains the identity on $[0, b_k)$ and a decreasing function on $[b_k, 1]$ where b_k represents the point in the opening paragraph of Step 1. The map $z \mapsto F_h(t_k, z)$ remains decreasing on $[b_k, 1]$ since parcels swap positions in $[b_k, 1]$ fall as a group in any subsequent time steps. This is clear from definition (16).

These observations allow us to conclude the estimate:

$$||F_h(t_k, \cdot)||_{BV([0,1])} \leq 2, \text{ for all } k \in [1, N]. \tag{17}$$

(ii) *Time estimates*:
Definition (16) ensures that $F_h(t_k, z) = z$ on $[0, s_z)$ and $z \mapsto F_h(t_k, z)$ is non-increasing on $[s_z, T]$ for each $z \in [0, 1]$. This is true since for each $z \in [0, 1]$, there is a $k \in [1, M]$ such that $z < b_j$ for all $j \leq k$ so that $F_h(t, z) = z$ on $[0, b_k)$, and by looking at the second line in the definition (16), it is clear that $z \mapsto F_h(t_k, z)$ remains non-increasing for all future times. Extend $F_h(\cdot, z)$, $\tilde{q}_h(\cdot, z)$ and $\tilde{\theta}_h(\cdot, z)$ to be piecewise linear on $[t_k, t_{k+1})$ for each $z \in [0, 1]$. Hence we can conclude the estimate:

$$||F_h(\cdot, z_i)||_{BV([0,T)} \leq 2. \tag{18}$$

Combining estimates (17) and (18) and Lemma 7 we arrive at:

$$||F_h||_{BV([0,T)\times[0,1])} \leq C(\theta_0, q_0). \tag{19}$$

Notice the lack of time dependence of this BV estimate and that similar BV estimates hold for $\tilde{\theta}_h$ and $\tilde{q}_h$ since the composition of monotone functions is monotone.

We now recall the compact embedding $BV \subset\subset L^1$. This along with the estimates (17), (18), (19) ensure an $L^1([0, T) \times [0, 1])$ convergent subsequence of F_h $\tilde{\theta}_h$, $\tilde{q}_h$ as $h \to 0$. Due to the above estimates, it's also clear that

$$\left|1_{[1,\infty)}\left(\frac{\tilde{q}_h}{\tilde{q}_{sat_h}}\right)[\partial_t F_h+\alpha]^+dtdz\right|([0,T]\times[0,1])\le C(\theta_0,q_0).$$

Hence we can extract a weak-* limit of these measures, which we will call ω on $[0,T]\times[0,1]$.
Step 3: Convergence to the solution
We take the difference equations acquired in Step 1.2 and make them piecewise linear in time, connecting the endpoint at t_k to the endpoint at t_{k+1} of $\tilde{q}_h$, $\tilde{\theta}_h$ and F_h. This gives us two *differential* equations (7)–(8) satisfied almost everywhere in $[0,T)\times[0,1]$ except that the argument of the indicator function is evaluated at $\tilde{q}_h(t_k,z))/\tilde{q}_h^{sat}(t_{k+1},z)$ when $t\in[t_k,t_{k+1})$. Letting $\varphi\in C_c^\infty([0,T)\times[0,1])$ we multiply our equations by this test function and integrate by parts to arrive at the following equations:

$$-\int_0^1\int_0^T\partial_t\varphi(t,z)\tilde{q}_h(t,z)=\frac{dq_{sat}}{dz}\int_0^1\int_0^T\varphi(t,z)a_h(t,z)dtdz+\int_0^1\varphi(0,z)q_0(t,z)dtdz,$$

$$\int_0^1\int_0^T\partial_t\varphi(t,z)[\tilde{\theta}_h(t,z)+\tilde{q}_h(t,z)]dtdz=\int_0^1\varphi(0,z)[\theta_0(z)+q_0(z)],$$

where

$$a_h(t,z):=1_{[1,\infty)}\left(\frac{\tilde{q}_h(t,z)}{\tilde{q}_h^{sat}(t+\Delta t,z)}\right)[\partial_t F_h(t,z)+\alpha]^+.$$

From the estimates of section 2 we have that

$$\tilde{q}_h\to\tilde{q}\text{ in }L^1([0,T]\times[0,1])\tag{20}$$

$$\tilde{\theta}_h\to\tilde{\theta}\text{ in }L^1([0,T])\times[0,1])\tag{21}$$

$$F_h\to F\text{ in }L^1([0,T]\times[0,1])\tag{22}$$

$$\omega_h:=1_{[1,\infty)}\left(\frac{\tilde{q}_h(t,z)}{\tilde{q}_h^{sat}}(t+\Delta t,z)\right)[\partial_t F_h+\alpha]^+dtdz\overset{*}{\rightharpoonup}\omega\text{ on }[0,T]\times[0,1],\tag{23}$$

as h and $\Delta t<\epsilon h/\alpha\beta$ tend to zero. It is clear from the dominated convergence theorem and the Riesz representation theorem that the above convergence is sufficient so that ω, $\tilde{q}$, $\tilde{\theta}$ satisfy the equations in the weak formulation of equations (6)–(8) (Definition 2). Lemma 8 along with Lemma 9 ensure that $z\mapsto F(t,z)$ is indeed the measure preserving map which yields the monotone rearrangement of $\tilde{\theta}(t,z)$, $\theta(t,z)$ for $\mathcal{L}^1$ almost every time $t \in [0,T]$.

Step 4 (Properties of solution):
(i) spt ω concentrated on saturated parcels

The map $t \mapsto \tilde{q}(t,z_0)/\tilde{q}^{sat}(t,z_0)$ is continuous everywhere for each $z_0 \in [0,1]$. This can be seen by studying equation (16) as $h \to 0$ and noting that the only discontinuity in $t \mapsto G_h(t,z_0)$ which occurs when z_0 becomes saturated is not a discontinuity of $t \mapsto \tilde{q}(t,z_0)/\tilde{q}^{sat}(t,z_0)$ since this ratio approaches (and recedes from) saturation continuously. Therefore the set $\Omega_u(z_0) := \{t : 0 < \tilde{q}(t,z_0)/\tilde{q}^{sat}(t,z_0) < 1\}$ is open. Lemma 6 implies that there is a subsequence such that $\tilde{q}_{h_k}(t,z_0)/\tilde{q}^{sat}_{h_k}(t,z_0) \to \tilde{q}(t,z_0)/\tilde{q}^{sat}(t,z_0)$ in $L^1([0,T])$ as $h_k \to 0$ for a.e $z_0 \in [0,1]$. Note that $\partial_t \tilde{q}(t,z) = -\beta\omega$ in the weak sense of Definition 2. Since $\tilde{q}_h \to \tilde{q}$ on $L^1([0,T] \times [0,1])$, Lemma 6 ensures that $\tilde{q}_{h_k}(t,z_0) \to \tilde{q}(t,z_0)$ as $h_k \to 0$ for a.e $z_0 \in [0,1]$. Lemma 8 then tells us that $\partial_t \tilde{q}_{h_k}(t,z_0) \stackrel{*}{\rightharpoonup} \partial_t \tilde{q}(t,z_0)$ on $[0,T]$ for a.e. z_0 and that $\partial_t \tilde{q}_{h_k} \stackrel{*}{\rightharpoonup} \partial_t \tilde{q}$ on $[0,T) \times [0,1]$. Since $\tilde{q}_h(t,z_0)$ is constant on the open set $t \in \Omega_u(z_0)$, it follows that $\partial_t \tilde{q}_{h_k}(t,z_0) \stackrel{*}{\rightharpoonup} 0$ on $\Omega_u(z_0)$. Consequently $\partial_t \tilde{q}(t,z_0) = 0$ on $\Omega_u(z_0)$ which implies that the support of ω is concentrated on $\{(t,z) : \tilde{q}(t,z) = \tilde{q}^{sat}(t,z)\}$.

(ii) $t \mapsto F(t,z)$ measure preserving

Since $F_h \to F$ in $L^1([0,T) \times [0,1])$ it follows from Lemma 4that $F_h(t,\cdot) \to F(t,\cdot)$ for Lebesgue almost every $t \in [0,T)$. From the dominated convergence theorem, the L^1 limit of measure preserving maps is measure preserving.

(iii) Strong stability

This property is clear from the L^1 convergence of θ_h and q_h.

(iv) $q \leq q^{sat}$ on $[0,T] \times [0,1]$

That this property is satisfies is also clear from the L^1 convergence of $\tilde{q}_h$, F_h and $\tilde{q}^{sat}_h$.

Hence all of the requirements on $\tilde{\theta}, \tilde{q}, F$ and ω of Definition 2 are satisfies and hence constitutes a weak solution to (6)–(8). □

5 Appendix

The following lemmas are used to prove the results.

Lemma 6 *Let $f_h \to f$ in $L^1([0,1] \times [0,1]; [0,1])$. Then for Lebesgue a.e $x \in [0,1]$ $f_h(x, \cdot) \to f(x, \cdot)$ in $L^1([0,1])$.*

Proof: Since $f_h \to f$ in $L^1([0,1] \times [0,1]; [0,1])$,

$$0 = \liminf_{h \to 0} \int_0^1 \int_0^1 |f_h(x,y) - f(x,y)| dx dy$$

$$\geq \int_0^1 \liminf_{h \to 0} \int_0^1 |f_h(x,y) - f(x,y)| dx dy.$$

Hence $f_h(x, \cdot) \to f(x, \cdot)$ for a.e $x \in [0,1]$ at least up to a subsequence. □

Lemma 7 *Let $\Omega = [0,1] \times [0,1]$ be and $f : \Omega \to \mathbf{R}$. Then $f \in BV(\Omega; \mathbf{R})$ if and only if $||f(z, \cdot)||_{BV} \leq C$ and $||f(\cdot, t)||_{BV} \leq C$ for every $z, t \in [0,1]$ and some $C > 0$.*

Proof: Taking $\varphi \in C_c^1([0,1]^2\ \mathbf{R}^2)$, the estimates

$$\begin{aligned} ||Df||_{TV([0,1]^2)} &:= \sup_{||\varphi||_{L^\infty(\Omega;\mathbf{R}^2)} \leq 1} \int_0^1 \int_0^1 f(t,z) \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_2}{\partial z} \right) dz dt \\ &\leq \int_0^1 \left|\left| \frac{\partial f}{\partial z}(t, \cdot) \right|\right|_{TV([0,1])} dt + \int_0^1 \left|\left| \frac{\partial f}{\partial t}(z, \cdot) \right|\right|_{TV([0,1])} dz \end{aligned}$$

imply the lemma, since $||f||_{BV([0,1]^2)} := ||f||_{L^1([0,1]^2)} + ||Df||_{TV([0,1]^2}$. □

Lemma 8 *Let $f_k \in BV(\mathbf{R}; \mathbf{R})$ with $||f_k||_{BV} \leq C$ where $C > 0$ is independent of k. Then there exists a subsequence $\{f_{n_k}\}$ such that,*

$$f_{n_k} \to f \text{ in } L^1(X; \mathbf{R}),$$

$$f'_{n_k} \rightharpoonup f' \text{ weak-* as measures .}$$

Proof: Since f_{n_k} are BV functions they have measure valued derivatives, f'_{n_k}. So,

$$\int f_{n_k}\phi' dz = -\int f'_{n_k}\phi dz,$$

for some measure $f'_{n_k} \in M(\mathbf{R})$ when $\phi \in C_c^1$. Now use the weak-* compactness of $M(\mathbf{R})$ along with the uniform BV bound $|f'_{n_k}| \leq C$ to get a weak-* convergent subsequence $f'_{n_k} \to g$. The BV compactness theorem ensures that $f_{n_k} \to f \in L^1 \cap BV$ up to taking a further subsequence. Now, $f_{n_k}\phi'$ is uniformly bounded in L^1 by $C\phi'$. By ensuring we take a subsequence that converges pointwise Lebesgue a.e to f we can now use the dominated convergence theorem to conclude that,

$$\int f\phi' dz = -\int g\phi dz.$$

But $f \in BV$ and so we know that,

$$\int f\phi' dz = -\int f'\phi dz,$$

where f' is the weak measure valued derivative of f. Hence,

$$\int g\phi dz = \int f'\phi dz$$

for all $\phi \in C_c^1$. By choosing an interval and using Urysohns lemma it is easy to see that gdz and $f'dz$ must agree on all Borel sets and hence $f'_{n_k} \to f'$ weak $*$. □

Lemma 9 *Let (Ω, μ) be a topological space with finite borel measure μ and $f_h, f \in L^1(\Omega; d\mu)$. If $f_h \to f$, $f_h^* \to g$ pointwise μ-a.e where f_h^* is the increasing rearrangement of f_h, then $g = f^*$.*

Moreover, if $\mu = \mathcal{L}^1$ is Lebesgue measure, $f_h^* \to f^*$ in $L^1(\Omega; d\mu)$, s_h is a measure preserving map which yields the rearrangement between f_h^* and f_h, $s_h \to s$ in $L^1(\Omega; d\mu)$. Then s is a measure preserving map which satisfies $f^* \circ s = f$.

Proof: Recalling the definition of rearrangement by Brenier, we can say

$$\int_\Omega F(f_h(z))d\mu(z) = \int_\Omega F(f_h^*(z))d\mu(z), \tag{24}$$

for all $F \in C(\mathbb{R}^d)$ with $|F(\xi)| \leq K(1+|\xi|)$ for some $K > 0$. Hence $|F(f_h)| \leq K + |f_h|$ and $|F(f_h^*)| \leq K + |f_h^*|$. The continuity of F, the pointwise a.e convergence of $f_h \to f$ and the assumption that $f_h, f \in L^1(\Omega; d\mu)$ allows us to use the Lebesgue Dominated Convergence Theorem to let $h \to 0$ in the above equation, yielding

$$\int_\Omega F(f(z))d\mu(z) = \int_\Omega F(g(z))d\mu(z). \tag{25}$$

But since g is the pointwise limit of increasing functions, it is increasing. Since increasing rearrangements are unique it follows that $g(z) = f^*(z)$.

Now for the second part of the lemma, first note that a simple application of the dominated convergence theorem tells us that the pointwise limit of measure preserving mappings $\lim_{h\to 0} s_h := s$ is measure preserving. Given $\epsilon > 0$, choose $N > 0$ sufficiently large so that $|f_h^*(x) - f^*(x)| < \epsilon$ for all $x \in \Omega$ except on a set Ω^ϵ with $|\Omega^\epsilon| < \epsilon$. Since s_h is measure preserving, it follows that $|f_h^*(s_h(x)) - f^*(s_h(x))| < \epsilon$ uniformly except on a set $\Omega^\epsilon_{s_h} = s_h(\Omega^\epsilon)$ with $|\Omega^\epsilon_{s_h}| < \epsilon$ once again. Hence for all x except in a set with measure no greater than ϵ, we have that

$$|f_h^*(s_h(x)) - f^*(s(x))| \leq |f_h^*(s_h(x)) - f^*(s_h(x))| + |f^*(s_h(x)) - f^*(s(x))| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This of course means that $f^* \circ s$ and f agree everywhere except for an ϵ-set. But ϵ is arbitrary and so $f^* \circ s = f$. □

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