Applications of the signed distance function to surface geometry

by

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Abstract

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In studying the geometry of a submanifold, it is often convenient to represent the submanifold as the zero set of an appropriately chosen defining function. For a hypersurface, a natural function to consider is its signed distance function. In this thesis we study the differential geometry of surfaces embedded in $\mathbf{R}^3$ by expressing the curvatures and principal directions of a surface in terms of the derivatives of its signed distance function. This allows us to derive many established and new results using simple multivariable calculus.

After first defining the signed distance function to a surface and demonstrating its basic properties, we prove several integral formulas involving the surface's principal curvatures, including a generalization of the divergence theorem. We then derive a complex linear differential equation in the principal radii and directions of the surface that is based on the Mainardi-Codazzi equations, and establish a connection between the function in this equation and the surface's support function. These are used to obtain a characterization of the principal directions around an umbilical point on a surface, and to reduce the Loewner index conjecture for real analytic surfaces to a problem about the location of the roots of a specific class of polynomials. Finally, we extend our generalized divergence theorem to submanifolds of arbitrary co-dimension equipped with a Riemannian metric.

Contents

Chapter 1

Introduction

In this thesis we study the differential geometry of surfaces embedded in $\mathbf{R}^3$, except for the final chapter in which we study general submanifolds embedded in a Riemannian manifold. We prove from first principles elementary new results concerning the principal curvatures and directions of a surface. All of the background necessary to understand chapters 2-7 is contained in [20], and most introductory texts about Riemannian geometry such as [7] contain the background necessary for chapter 8.

The geometry of surfaces embedded in $\mathbf{R}^3$ is a classical subject, one of whose highlights is the definition and study of a surface's principal curvatures and their associated lines of curvature. Many types of surfaces that are of interest, such as minimal surfaces and capillary surfaces, can be characterized directly in terms of their principal curvatures. The basic theory of curvature is simple and well understood. Nevertheless, open questions concerning this subject persist. One of these is the Carathéodory conjecture, which states that any smooth, closed, convex surface must contain at least two umbilical points (whose definition is recalled below). Although established for real analytic surfaces (most recently in the work of Ivanov [18]), it remains a vexing problem for smooth non-analytic surfaces. This thesis develops new results and methods for studying the principal curvatures and directions of a surface with a view to attacking such questions.

One consequence is a differential equation for embedded surfaces that reduces to the Weierstrass-Enneper formula when the embedding is minimal. Another consequence is to reduce the Loewner index conjecture for analytic surfaces (a conjecture that implies the Carathéodory conjecture) to a statement about the locations of the roots of certain complex polynomials. In addition, we develop new integral formulas involving the curvature of embedded surfaces that include generalizations of the Gauss-Bonnet theorem. All of our results are derived by applying multivariable calculus in $\mathbf{R}^3$ to a surface's signed distance function, whose zero level set gives the surface in question.

The main object of study in chapters 2-7 is an orientable C^∞ surface M in $\mathbf{R}^3$, with or without boundary, whose outward unit normal we denote by N. We need only a few basic definitions to describe our results. If $X(s)$ is a curve on M parametrized by arc length $s \in (-\epsilon, \epsilon)$ passing through the point $X_0 = X(0)$, then the normal curvature of X at X_0 is defined as:

$$\kappa_n = X''(0) \cdot N.$$

Among all the curves passing through X_0 there will be those that either maximize or minimize κ_n at X_0. If a curve $X(s)$ does so then the tangent vector $X'(0)$ is said to be a principal direction at X_0, and the corresponding maximum or minimum normal curvature is said to be a principal curvature at X_0. The average of the two principal curvatures at a point on M is called the mean curvature and is denoted by H, while the product of the two principal curvatures is called the Gaussian curvature and is denoted by K. The reciprocals of the principal curvatures are called the principal radii, and the average of the principal radii (equal to H/K) is called the mean radius. When the minimum curvature at a point is strictly less than the maximum curvature, any two principal directions corresponding to the two distinct principal curvatures will be perpendicular. If it should happen that the two principal curvatures at X_0 are equal, then all curves passing through X_0 have the same normal curvature, all tangent vectors at X_0 are principal directions, and X_0 is said to be an umbilical point.

The key tool we use to study the differential geometry of M is its signed distance function, defined in chapter 2, which we denote by n. The reason for naming the function this way is that:

$$N = \nabla n$$

on M and all other level surfaces of n. Differentiating n a second time, we find that all of the information about the principal curvatures and directions of M can be obtained by diagonalizing $\mathrm{Hess}\,(n)$. Thus n can be used as a substitute for fundamental forms in studying the geometry of a surface. In fact, the fundamental forms of classical differential geometry hardly appear in this thesis at all.

In chapter 3, we use the information about curvature embedded in the derivatives of n to prove theorem 3.1.1, which consists of two new integral formulas. These lead to simple and unified proofs of known integral formulas involving the curvatures of a surface, including the global version of the the Gauss-Bonnet theorem. We then prove two new classes of integral formulas in sections 3.4 and 3.5. The first gives a method for generalizing the Gauss-Bonnet theorem to the case where the integrand is the Gaussian curvature times an arbitrary function of N. The second involves integrands that include a term equal to the difference of the principal curvatures.

In chapter 4, we prove theorem 4.3.1, which consists of a complex differential equation that relates the mean radius of a surface to a function W that is the difference of the principal radii multiplied by a function of the principal directions. This equation is a concise restatement of the classical Mainardi-Codazzi equations. In section 5.1 we show that the equation generalizes the Weierstrass-Enneper representation for minimal surfaces. In section 5.2, we show that it is possible to use 4.3.1 to reduce problems that specify a function of the curvatures of a surface as a function of the normal to a first-order complex differential equation.

However, the most interesting application of the results in chapter 4 is to the Carathéodory conjecture, which states that any smooth closed convex surface must contain at least two umbilical points. This conjecture is implied by the Loewner index conjecture, which states that the index of an umbilical point on a surface is at most 1. A great deal of effort has been spent trying to prove these conjectures, starting with the work of H. L. Hamburger publised in the 1940's in [14] and [15]. The Loewner index conjecture is known to be true for real analytic surfaces, although the most recent proof contained in [18] is still quite lengthy. While we do not provide a complete proof here, we do reformulate the problem in terms of the location of the roots of polynomials in a single variable. Specifically, in section 5.3 we characterize those power series that can represent the function W on an analytic surface in terms of the second derivatives of self-inversive polynomials. This is relevant because the index of an umbilical point is equal to half of the winding number of W about the point. In chapter 6 we digress to prove a result about self-inversive polynomials. In chapter 7 we are then able to reformulate the Loewner conjecture for real analytic surfaces in terms of the number of roots lying inside or outside the unit disk of second derivatives of self-inversive polynomials, and to prove the conjecture in several specific cases.

Finally, in chapter 8, we branch out from surfaces in $\mathbf{R}^3$ to Riemannian submanifolds, and prove an identity that generalizes the divergence theorem for manifolds in the same way that (3.4) generalizes the divergence theorem for surfaces.

Chapter 2

Elementary properties of the signed distance function

In this chapter we define the signed distance function and prove all of the basic properties that we will need to use in subsequent chapters. Although the presentation here is customized to suit our purposes, all of this material exists in the literature in some form. In particular, section 12.6 of [24] introduces the idea of straining a surface by displacing all of its points in the direction δN, and section 14.6 of [8] derives results for the unsigned distance function that are similar to those in sections 2.2 and 2.3.

2.1 Definition and properties

As stated in the introduction, the main object of study will be an orientable C^∞ surface M, with or without boundary, that is embedded in $\mathbf{R}^3$ and has outward unit normal N.

The distance from a point $X = (x_1, x_2, x_3)^T \in \mathbf{R}^3$ to a set $\Omega \subset \mathbf{R}^3$ is defined by:

$$\operatorname{dist}(X, \Omega) = \inf_{Y \in \Omega} \|X - Y\|.$$

For any set Ω, a signed distance function n of Ω can be defined globally on $\mathbf{R}^3$ by:

$$n(X) = \begin{cases} \operatorname{dist}(X, \Omega), & \text{if } X \in \Omega^c \\ -\operatorname{dist}(X, \Omega^c), & \text{if } X \in \Omega. \end{cases}$$

The case most often considered (for example in [23], chapter 2) is where Ω is an open set with non-zero volume and is enclosed by a piecewise-smooth boundary. Then, $\mathbf{R}^3$ is split into three regions $\Omega^- = \Omega$, $\partial\Omega$, and $\Omega^+ = (\Omega^c)^o$, and the signed distance function is given by:

$$n(X) = \begin{cases} \operatorname{dist}(X, \partial\Omega), & \text{if } X \in \Omega^+ \\ -\operatorname{dist}(X, \partial\Omega), & \text{if } X \in \Omega^- \\ 0, & \text{if } X \in \partial\Omega. \end{cases}$$

However, n defined in this way is not guaranteed to be differentiable everywhere, particularly at points $X \in \mathbf{R}^3$ for which there exist two or more points on $\partial\Omega$ having the same minimal distance to X.

For our purposes, it is only necessary to define n in a small $\mathbf{R}^3$-neighborhood G of M, but in such a way that n is differentiable everywhere in G. Specifically, let G be small enough so that every point of G lies on some normal ray passing through a point on M, and so that no two normal rays passing through different points on M intersect in G. We then define the signed distance function for M on G by:

$$n(X) = \begin{cases} \operatorname{dist}(X, M), & \text{if } X \text{ lies on an outward normal ray of } M \\ -\operatorname{dist}(X, M), & \text{if } X \text{ lies on an inward normal ray of } M \\ 0, & \text{if } X \in M. \end{cases} \tag{2.1}$$

Given a point $X \in G$, let $Y(X)$ be the unique point on M whose normal ray passes through X, and let $\nu(X)$ be the outward unit normal to M at $Y(X)$. Since the line of shortest distance from X to M goes through $Y(X)$ and is normal to M, we have:

$$X - Y(X) = n(X)\nu(X).$$

Differentiating with respect to x_1 gives:

$$\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) - \frac{\partial Y}{\partial x_1} = \frac{\partial n}{\partial x_1}\nu + n\frac{\partial \nu}{\partial x_1}.$$

Taking the dot product with ν gives:

$$\nu_1 - \frac{\partial Y}{\partial x_1} \cdot \nu = \frac{\partial n}{\partial x_1} + n\frac{\partial \nu}{\partial x_1} \cdot \nu.$$

Since Y is constrained to lie on M, any derivative of Y must be tangent to M, and we have:

$$\frac{\partial Y}{\partial x_1} \cdot \nu = 0.$$

Since $\|\nu\| = 1$, we must also have:

$$\frac{\partial \nu}{\partial x_1} \cdot \nu = 0$$

and then:

$$\frac{\partial n}{\partial x_1} = \nu_1.$$

Repeating for x_2 and x_3 gives:

$$\nabla n = \nu$$

and finally:

$$\|\nabla n\| = 1. \tag{2.2}$$

Equation (2.2) is a special case of the eikonal equation (the general equation has an arbitrary function of X on the right side), and is a key property of n leading to many interesting consequences. This equation arises in the context of wave propagation and optics, and can be solved using the method of characteristics (see [3], part II, section 6).

Since ∇n is perpendicular to all level surfaces of n, $n = 0$ on M, $\|\nabla n\| = 1$, and the direction of increase of n is outwards to M, the vector ∇n coincides with the outward unit normal vector N on M. We can therefore extend N to a unit vector field defined in all of G by setting:

$$N = \nabla n = (n_1, n_2, n_3)^T. \tag{2.3}$$

The use of subscripts for the components of N is justified by the fact that they are derivatives in $\mathbf{R}^3$. Defined this way, the components of the extended unit normal of M can be differentiated like any other functions defined on an open subset of $\mathbf{R}^3$ without having to use intrinsic differentiation. One convenient consequence is that differentiation of the components of N can be done in any order so that $n_{ij} = n_{ji}$ for any $i,\ j \in \{1, 2, 3\}$.

2.2 Curvature

Let J_N denote the Jacobian matrix of N, or equivalently the Hessian matrix of n. Since this matrix is symmetric, it is diagonalizable, and we can find the eigenvectors and eigenvalues of J_N for points on M. By differentiating the relation $\langle N, N\rangle = 1$ we get:

$$J_N N = 0, \tag{2.4}$$

which gives us the first eigenvector (N) and eigenvalue (0). In order to find the remaining two, consider a curve $X(s)$ lying on M. For the unit normal vector parametrized by s we have $N'(s) = J_N X'(s)$, so that J_N, when restricted to tangent vectors on M, is simply the shape operator. If $X'(s)$ is a principal direction on M at $X(s)$ with principal curvature κ, then by Rodrigues' formula we have $N' = J_N X' = -\kappa X'$. It follows that the other two eigenvectors of J_N are the principal directions on M, with eigenvalues equal to the negative of the corresponding principal curvatures. The sum of all the eigenvalues of J_N is thus $-2H$, where H represents mean curvature, and we have (as also shown in [23]):

$$\operatorname{Tr} J_N = \operatorname{div} N = \Delta n = -2H. \tag{2.5}$$

Relation (2.4) implies that all of the rows of J_N are tangent vectors to M. By orthonormally diagonalizing J_N we can describe them more explicitly. If $P = (p_1, p_2, p_3)^T$ and $Q = (q_1, q_2, q_3)^T$ are any perpendicular unit principal directions at a point on M with curvatures κ_1 and κ_2 then we have:

$$J_N = \begin{pmatrix} n_1 & p_1 & q_1 \\ n_2 & p_2 & q_2 \\ n_3 & p_3 & q_3 \end{pmatrix} \begin{pmatrix} 0 & & \\ & -\kappa_1 & \\ & & -\kappa_2 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \tag{2.6}$$

so that:

$$\begin{aligned} \nabla n_1 = (n_{11}, n_{12}, n_{13})^T &= -\kappa_1 p_1 P - \kappa_2 q_1 Q \\ \nabla n_2 = (n_{12}, n_{22}, n_{23})^T &= -\kappa_1 p_2 P - \kappa_2 q_2 Q \\ \nabla n_3 = (n_{13}, n_{23}, n_{33})^T &= -\kappa_1 p_3 P - \kappa_2 q_3 Q \\ n_{ij} &= -p_i p_j \kappa_1 - q_i q_j \kappa_2. \end{aligned} \tag{2.7}$$

The set of vectors $\{P, Q, N\}$ forms an orthonormal frame on M. For the sake of definiteness, we fix the orientation of P and Q by imposing the condition:

$$P \times Q = N. \tag{2.8}$$

Note however that even with this condition, the vectors P and Q are not uniquely determined: if (P, Q) is an ordered pair of principal directions satisfying (2.8), then so are $(-P, -Q)$, $(Q, -P)$ and $(-Q, P)$. Furthermore, where M has umbilical points, P and Q can be chosen to be any pair of orthonormal tangent vectors oriented to satisfy (2.8). All of the equations that follow are invariant under orientation-preserving negation/switching transformation of P and Q, and all of the integral formulas in the next chapter are well defined even if M has umbilical points. However, some equations in chapter 4 only hold true at non-umbilical points, and how these equations behave around umbilical point singularities is a key focus of chapter 7.

With K denoting the Gaussian curvature on M, we can take cross products of the above identities to get:

$$\begin{aligned}
\nabla n_1 \times \nabla n_2 &= \kappa_1 \kappa_2 (p_1 q_2 - p_2 q_1) P \times Q \\
&= n_3 K N \\
\nabla n_2 \times \nabla n_3 &= n_1 K N \\
\nabla n_3 \times \nabla n_1 &= n_2 K N.
\end{aligned} \tag{2.9}$$

Taking the third, first and second components respectively of each identity gives:

$$\begin{aligned}
n_{11} n_{22} - n_{12}^2 &= n_3^2 K \\
n_{22} n_{33} - n_{23}^2 &= n_1^2 K \\
n_{33} n_{11} - n_{13}^2 &= n_2^2 K
\end{aligned} \tag{2.10}$$

and:

$$K = n_{11} n_{22} + n_{22} n_{33} + n_{33} n_{11} - n_{12}^2 - n_{23}^2 - n_{13}^2. \tag{2.11}$$

Henceforth, we will use lower case subscripts to denote the directional derivative of a function or a vector field in $\mathbf{R}^3$, so that, for example:

$$f_p = \nabla_P f = \langle \nabla f, P \rangle, \quad f_q = \nabla_Q f = \langle \nabla f, Q \rangle, \quad f_n = \nabla_N f = \langle \nabla f, N \rangle$$

A differentiable vector field Y on $\mathbf{R}^3$ can be viewed as an ordinary function $Y \colon \mathbf{R}^3 \to \mathbf{R}^3$, to which we can associate the Jacobian J_Y. Using our subscript notation we then have:

$$Y_p = J_Y P = \begin{pmatrix} Y_p^1 \\ Y_p^2 \\ Y_p^3 \end{pmatrix}, \quad Y_q = J_Y Q = \begin{pmatrix} Y_q^1 \\ Y_q^2 \\ Y_q^3 \end{pmatrix}, \quad Y_n = J_Y N = \begin{pmatrix} Y_n^1 \\ Y_n^2 \\ Y_n^3 \end{pmatrix}$$

2.3 Level surfaces of n

The level surfaces of n have normals, principal directions and principal curvatures just like M does. The quantities N, P, Q, κ_1 and κ_2 can be extended differentiably from M to the surrounding level surfaces because they are implicit functions of n and its derivatives, which are defined everywhere on G. Furthermore, all of the equations derived up to now that relate the derivatives of n to these quantities remain valid on any level surface of n because the only fact used to obtain them is that n is constant on M – the same reasoning will work if n is equal to a constant different from 0 on a surface. We therefore seek to determine how the principal curvatures and directions of the level surfaces of n evolve from M by finding the directional derivatives in the normal direction of the various quantities defined so far.

We start with N itself, which satisfies:

$$N_n = J_N N = 0$$

by (2.4). We next calculate $(J_N)_n$. The starting relation is:

$$\langle \nabla n_i, N \rangle = 0.$$

Differentiating with respect to component j gives:

$$\langle \nabla n_{ij}, N \rangle + \langle \nabla n_i, \nabla n_j \rangle = 0.$$

Equation (2.7) then implies:

$$\begin{aligned} \langle \nabla n_{ij}, N \rangle &= -p_i p_j \kappa_1^2 - q_i q_j \kappa_2^2 \\ (J_N)_n &= -\kappa_1^2 P P^T - \kappa_2^2 Q Q^T. \end{aligned}$$

Now we calculate P_n and $(\kappa_1)_n$. Since:

$$\begin{aligned} \langle P_n, N \rangle &= -\langle P, N_n \rangle = 0 \\ \langle P_n, P \rangle &= \frac{1}{2} \langle P, P \rangle_n = 0, \end{aligned}$$

there exists a scalar a such that $P_n = aQ$. If we differentiate the relation:

$$J_N P = -\kappa_1 P$$

we get:

$$(J_N)_n P + J_N P_n = -\kappa_1^2 P + J_N P_n = -(\kappa_1)_n P - \kappa_1 P_n$$

$$-\kappa_1^2 P - a\kappa_2 Q = -(\kappa_1)_n P - a\kappa_1 Q.$$

Applying the same reasoning to Q shows that $Q_n = -aP$ and:

$$-\kappa_2^2 Q + a\kappa_1 P = -(\kappa_2)_n Q + a\kappa_2 P.$$

From the linear independence of P and Q we conclude:

$$\begin{aligned}(\kappa_1)_n &= \kappa_1^2 \\ (\kappa_2)_n &= \kappa_2^2.\end{aligned} \tag{2.12}$$

At non-umbilical points we can also conclude that $a = 0$ and:

$$P_n = Q_n = 0 \qquad \text{if } \kappa_1 \neq \kappa_2. \tag{2.13}$$

At umbilical points, P and Q themselves are not well defined, hence neither are their normal directional derivatives.

What equations (2.12) and (2.13) show is that the level surfaces of n are dilations of the base surface M. As n increases or decreases from 0, a point starting on M moves along a fixed ray that is normal to all of the level surfaces of n along the trajectory of the starting point. The shape of M, as measured by the principal directions or the presence of an umbilical point, remains unchanged (equations (2.12) imply that umbilical starting points on M remain umbilical). However the shape of M, as measured by the principal curvatures, becomes flatter or more curved as n increases or decreases. Equations (2.12) imply that the principal radii decrease or increase at a constant rate as n increases or decreases.

Chapter 3

Integral formulas

As a first application of the properties of the signed distance function n shown in chapter 2, we prove several integral formulas involving the principal curvatures of M. Some of these are well established, while others appear to be new.

3.1 Two identities

In this section we prove two integral formulas (3.4) and (3.5) containing the mean and Gaussian curvatures of M. These generalize identities involving vector fields on M to vector fields on $\mathbf{R}^3$, leading to the addition of curvature terms to the formulas. Identity (3.4) (which is proven by different means in section 7 of [26]) is an extension of the divergence theorem in dimension 2, while (3.5) extends an identity that is proven in [31] for vector fields restricted to M using variational techniques.

We first state three basic identities. If Y and Z are vector fields on $\mathbf{R}^3$ and $f, g \colon \mathbf{R}^3 \to \mathbf{R}$, then the following can be proven by direct computation:

$$\operatorname{curl}(fY) = f \operatorname{curl}(Y) + \nabla f \times Y \tag{3.1}$$

$$\operatorname{curl}(f \nabla g) = \nabla f \times \nabla g \tag{3.2}$$

$$\operatorname{curl}(Y \times Z) = J_Y Z - J_Z Y - (\operatorname{div} Y) Z + (\operatorname{div} Z) Y. \tag{3.3}$$

Theorem 3.1.1. If V is any vector field on $\mathbf{R}^3$ and M is a surface patch in $\mathbf{R}^3$ with position vector X and other intrinsic quantities as defined before, then:

$$\int_{\partial M} (V \times N) \cdot dX = \int_M (\langle V_n, N \rangle - \operatorname{div} V - 2H\langle V, N \rangle)\, dA \tag{3.4}$$

$$\int_{\partial M} (V \times N) \cdot dN = \int_M (\kappa_2 \langle V_p, P \rangle + \kappa_1 \langle V_q, Q \rangle + 2K\langle V, N \rangle)\, dA. \tag{3.5}$$

Proof. By Stokes' Theorem:

$$\begin{aligned}
&\int_{\partial M} (V \times N) \cdot dX \\
&= \int_M \langle \operatorname{curl}(V \times N), N \rangle\, dA \\
&= \int_M \langle J_V N - J_N V - (\operatorname{div} V) N + (\operatorname{div} N)\langle V, N \rangle\, dA \qquad \text{by (3.3)} \\
&= \int_M (\langle V_n, N \rangle - \langle V, J_N N \rangle - \operatorname{div} V + (\operatorname{div} N)\langle V, N \rangle)\, dA \\
&= \int_M (\langle V_n, N \rangle - \operatorname{div} V - 2H\langle V, N \rangle)\, dA \qquad \text{by (2.4) and (2.5)},
\end{aligned}$$

which proves (3.4). To prove (3.5), we start with:

$$\begin{aligned}
&\int_{\partial M} (V \times N) \cdot dN \\
&= \int_{\partial M} \left(V^2 n_3 - V^3 n_2, V^3 n_1 - V^1 n_3, V^1 n_2 - V^2 n_1\right) \cdot (dn_1, dn_2, dn_3) \\
&= \int_{\partial M} \left(V^2 n_3 - V^3 n_2\right) \nabla n_1 \cdot dX + \left(V^3 n_1 - V^1 n_3\right) \nabla n_2 \cdot dX + \left(V^1 n_2 - V^2 n_1\right) \nabla n_3 \cdot dX.
\end{aligned}$$

Using Stokes' Theorem and (3.2) gives:

$$\begin{aligned}
&= \int_M \left\langle \nabla \left(V^2 n_3 - V^3 n_2\right) \times \nabla n_1, N \right\rangle dA + \left\langle \nabla \left(V^3 n_1 - V^1 n_3\right) \times \nabla n_2, N \right\rangle dA \\
&\quad + \left\langle \nabla \left(V^1 n_2 - V^2 n_1\right) \times \nabla n_3, N \right\rangle dA \\
&= \int_M \left\langle \nabla V^1 \times (n_2 \nabla n_3 - n_3 \nabla n_2), N \right\rangle dA + \left\langle \nabla V^2 \times (n_3 \nabla n_1 - n_1 \nabla n_3), N \right\rangle dA \\
&\quad + \left\langle \nabla V^3 \times (n_1 \nabla n_2 - n_2 \nabla n_1), N \right\rangle dA \\
&\quad + \left\langle 2V^1 \nabla n_2 \times \nabla n_3 + 2V^2 \nabla n_3 \times \nabla n_1 + 2V^3 \nabla n_1 \times \nabla n_2, N \right\rangle dA
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \left\langle \nabla V^1, (-\kappa_1 q_1 P + \kappa_2 p_1 Q) \times N \right\rangle \, dA + \left\langle \nabla V^2, (-\kappa_1 q_2 P + \kappa_2 p_2 Q) \times N \right\rangle \, dA \\
&\quad + \left\langle \nabla V^3, (-\kappa_1 q_3 P + \kappa_2 p_3 Q) \times N \right\rangle \, dA + 2K \langle V, N \rangle \, dA \qquad \text{(by (2.7) and (2.9))} \\
&= \int_M \left\langle \nabla V^1, \kappa_1 q_1 Q + \kappa_2 p_1 P \right\rangle \, dA + \left\langle \nabla V^2, \kappa_1 q_2 Q + \kappa_2 p_2 P \right\rangle \, dA \\
&\quad + \left\langle \nabla V^3, \kappa_1 q_3 Q + \kappa_2 p_3 P \right\rangle \, dA + 2K \langle V, N \rangle \, dA \\
&= \int_M \left(\kappa_2 \langle V_p, P \rangle + \kappa_1 \langle V_q, Q \rangle + 2K \langle V, N \rangle \right) \, dA.
\end{aligned}
$$

□

It is interesting to compare (3.5) with Stokes' Theorem, which can be written in the form:

$$
\int_{\partial M} V \cdot dX = \int_M \left(\nabla V^1 \cdot (0, -n_3, n_2)^T + \nabla V^2 \cdot (n_3, 0, -n_1)^T + \nabla V^3 \cdot (-n_2, n_1, 0)^T \right) \, dA. \tag{3.6}
$$

The three vector fields appearing above are tangent to the curves formed by the intersection of M with planes perpendicular to the x, y and z axes respectively. Since P and Q form an orthonormal basis for the tangent space at any point on M, any tangent vector at the same point can be written as the sum of its projection onto P and its projection onto Q. We therefore have:

$$
\begin{aligned}
(0, -n_3, n_2)^T &= [(0, -n_3, n_2)^T \cdot P] P + [(0, -n_3, n_2)^T \cdot Q] Q \\
&= q_1 P - p_1 Q \\
(n_3, 0, -n_1)^T &= q_2 P - p_2 Q \\
(-n_2, n_1, 0)^T &= q_3 P - p_3 Q
\end{aligned}
$$

and Stokes' Theorem is equivalent to:

$$
\int_{\partial M} V \cdot dX = \int_M \left(\langle V_p, Q \rangle - \langle V_q, P \rangle \right) \, dA. \tag{3.7}
$$

The derivatives of V appearing in the surface integral in (3.4) can be written as:

$$\left\langle \nabla V^1, (-n_2^2 - n_3^2, n_1 n_2, n_1 n_3)^T \right\rangle \\ + \left\langle \nabla V^2, (n_1 n_2, -n_1^2 - n_3^2, n_2 n_3)^T \right\rangle + \left\langle \nabla V^3, (n_1 n_3, n_2 n_3, -n_1^2 - n_2^2)^T \right\rangle .$$

These vector fields are the cross products of the standard vector fields that appear in Stokes' Theorem (3.6) and N. Thus, the derivatives of V in (3.4) are taken in the tangent directions to M that are perpendicular to the curves formed by the intersection of M with the planes perpendicular to the x, y and z axes.

The orthonormality of P, Q and N implies that:

$$\operatorname{div} V = \langle V_p, P\rangle + \langle V_q, Q\rangle + \langle V_n, N\rangle.$$

This leads to an alternative way to write (3.4) that highlights its similarity to (3.5):

$$\int_{\partial M} (V \times N) \cdot dX = \int_M \left(-\langle V_p, P\rangle - \langle V_q, Q\rangle - 2H\langle V, N\rangle\right) dA. \tag{3.8}$$

3.2 Familiar results

Many well known integral formulas can be derived easily from (3.4) and (3.5). While none of the formulas presented in this section are new, the proofs we present for some are much simpler than the existing proofs (e.g. compare the proof of the Minkowski-Steiner formula below with the proofs in [1] or [6]).

Our technique is to set V in (3.4) and (3.5) so that the terms involving derivatives of V reduce to a constant. The simplest choices for V are the unit component vectors $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$, which yield:

$$\int_{\partial M} n_3\, dx_2 - n_2\, dx_3 = 2\int_M n_1 H\, dA$$
$$\int_{\partial M} n_1\, dx_3 - n_3\, dx_1 = 2\int_M n_2 H\, dA$$
$$\int_{\partial M} n_2\, dx_1 - n_1\, dx_2 = 2\int_M n_3 H\, dA$$

$$\int_{\partial M} n_2\, dn_3 - n_3\, dn_2 = 2\int_M n_1 K\, dA$$
$$\int_{\partial M} n_3\, dn_1 - n_1\, dn_3 = 2\int_M n_2 K\, dA$$
$$\int_{\partial M} n_1\, dn_2 - n_2\, dn_1 = 2\int_M n_3 K\, dA.$$

These identities can be combined by writing them in vector form as:

$$\int_{\partial M} N \times dX = -2\int_M HN\, dA \tag{3.9}$$
$$\int_{\partial M} N \times dN = 2\int_M KN\, dA. \tag{3.10}$$

Another way to eliminate the terms involving derivatives of V is by setting its components to linear terms of opposite sign. For $V = (0, -x_3, x_2)^T$, $(x_3, 0, -x_1)^T$ and $(-x_2, x_1, 0)^T$ we get:

$$\int_{\partial M} n_1x_2\, dx_2 + n_1x_3\, dx_3 - (n_2x_2 + n_3x_3)\, dx_1 = 2\int_M (n_2x_3 - n_3x_2)H\, dA$$
$$\int_{\partial M} n_2x_1\, dx_1 + n_2x_3\, dx_3 - (n_1x_1 + n_3x_3)\, dx_2 = 2\int_M (n_3x_1 - n_1x_3)H\, dA$$
$$\int_{\partial M} n_3x_1\, dx_1 + n_3x_2\, dx_2 - (n_1x_1 + n_2x_2)\, dx_3 = 2\int_M (n_1x_2 - n_2x_1)H\, dA$$
$$\int_{\partial M} n_1x_2\, dn_2 + n_1x_3\, dn_3 - (n_2x_2 + n_3x_3)\, dn_1 = 2\int_M (n_3x_2 - n_2x_3)K\, dA$$
$$\int_{\partial M} n_2x_1\, dn_1 + n_2x_3\, dn_3 - (n_1x_1 + n_3x_3)\, dn_2 = 2\int_M (n_1x_3 - n_3x_1)K\, dA$$
$$\int_{\partial M} n_3x_1\, dn_1 + n_3x_2\, dn_2 - (n_1x_1 + n_2x_2)\, dn_3 = 2\int_M (n_2x_1 - n_1x_2)K\, dA$$

or in vector form:

$$\int_{\partial M} (X \times N) \times dX = 2\int_M H\, X \times N\, dA \tag{3.11}$$
$$\int_{\partial M} (X \times N) \times dN = -2\int_M K\, X \times N\, dA. \tag{3.12}$$

Setting $V = X$ reduces the derivative terms to non-zero constants. When substituted in (3.4), we get Jellet's formula:

$$\int_{\partial M} X \times N \cdot dX = -2\int_M (1 + \langle X, N\rangle H)\; dA \tag{3.13}$$

while substituting $V = X$ in (3.5) gives the Minkowski-Steiner formula:

$$\int_{\partial M} X \times N \cdot dN = 2 \int_M (H + \langle X, N \rangle K)\, dA. \tag{3.14}$$

If V is tangent to M then $\langle V, N \rangle = 0$ on M and equation (3.8) reduces to:

$$\int_{\partial M} N \times V \cdot dX = \int_M (\langle V_p, P \rangle + \langle V_q, Q \rangle)\, dA.$$

Since P and Q are an orthonormal basis for the tangent plane to M, the right integrand above is equal to the divergence of V when V is treated as a vector field on M rather than on $\mathbf{R}^3$. Equation (3.8) thus implies that for any vector field V on M we have:

$$\int_{\partial M} N \times V \cdot dX = \int_M \operatorname{div} V\, dA,$$

which is the standard divergence theorem in dimension 2.

The most familiar result of all is the Gauss-Bonnet theorem. Equation (3.5) can be used to derive a variation of this result quickly. We choose n_3 as an arbitrary component of N. If M does not contain any points at which $n_3 = \pm 1$, then substituting:

$$V = \left(0, 0, -\frac{n_3}{1 - n_3^2}\right)^T$$

into (3.5) yields:

$$\begin{aligned} \int_{\partial M} \frac{n_3}{n_1^2 + n_2^2} (n_2\, dn_1 - n_1\, dn_2) &= \int_M \left(\frac{1 + n_3^2}{(1 - n_3^2)^2} (p_3^2 + q_3^2) - \frac{2n_3^2}{1 - n_3^2} \right) K\, dA \\ &= \int_M K\, dA. \end{aligned} \tag{3.15}$$

If M does contain isolated points at which $n_3 = \pm 1$, the above equation will contain an additional term for each such point. Since the boundary integral is of "$d\theta$" type in the variables n_1 and n_2, the extra term at each point will be equal to -2π times the index of the vector field $(n_2, -n_1, 0)^T$ (formed by intersecting the planes $x_3 =$ constant with M) about the point:

$$\int_{\partial M} \frac{n_3}{n_1^2 + n_2^2} (n_2\, dn_1 - n_1\, dn_2) = \int_M K\, dA - 2\pi \sum_{\substack{Y \in M \\ n_3(Y) = \pm 1}} \operatorname{Index}_{(n_2, -n_1, 0)}(Y). \tag{3.16}$$

With the equation in this form, we can use Morse theory or the Poincaré-Hopf index theorem to deduce the global version of the Gauss-Bonnet theorem. In order to deduce the classical local version of theorem, it is necessary to show:

$$\int_{\partial M} \frac{n_3}{n_1^2 + n_2^2} (n_2 \, dn_1 - n_1 \, dn_2) + \kappa_g \, ds = 2\pi\chi(M) - 2\pi \sum_{\substack{Y \in M \\ n_3(Y) = \pm 1}} \mathrm{Index}_{(n_2, -n_1, 0)}(Y). \quad (3.17)$$

I am unable to provide a proof. However, the above equation may be taken as a corollary of combining (3.16) with the classical Gauss-Bonnet theorem.

3.3 The generalized volume form

In this section we exhibit a large set of differential 2-forms that are closed in G. The fact that these forms are closed leads us to ask whether it is possible to obtain integral formulas for these forms over M. The next two sections give explicit formulas for two special cases, while the next chapter is motivated by the goal of finding an explicit formula for the general case of closed forms involving the second-order derivatives of n.

Let us define a differential 2-form on G by:

$$\omega = n_1 \, dx_2 \wedge dx_3 + n_2 \, dx_3 \wedge dx_1 + n_3 \, dx_1 \wedge dx_2.$$

When restricted to a level surface of n, this form coincides with the volume form of the surface. However, since ω is defined on an open subset of $\mathbf{R}^3$, it has an exterior derivative that is not zero. In fact we have:

$$\begin{aligned} d\omega &= (n_{11} + n_{22} + n_{33}) \, dx_1 \wedge dx_2 \wedge dx_3 \\ &= -2H \, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

We can multiply ω by functions f and characterize those functions for which $f\omega$ is closed. For any such f this gives rise to the possibility of obtaining an integral formula for f

over M. In general we have:

$$\begin{aligned} d(f\omega) &= d\left(fn_1\, dx_2 \wedge dx_3 + fn_2\, dx_3 \wedge dx_1 + fn_3\, dx_1 \wedge dx_2\right) \\ &= (f_n - 2Hf)\, dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

so that:

$$\begin{aligned} d(2H\omega) &= \left((\kappa_1+\kappa_2)_n - (\kappa_1+\kappa_2)^2\right)\, dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left(\kappa_1^2+\kappa_2^2 - (\kappa_1+\kappa_2)^2\right)\, dx_1 \wedge dx_2 \wedge dx_3 \qquad \text{(by (2.12))} \\ &= -2K\, dx_1 \wedge dx_2 \wedge dx_3 \\ d(K\omega) &= (K_n - 2HK)\, dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left((\kappa_1\kappa_2)_n - \kappa_1\kappa_2(\kappa_1+\kappa_2)\right)\, dx_1 \wedge dx_2 \wedge dx_3 \\ &= 0 \\ d(fK\omega) &= (f_nK + fK_n - 2fHK)\, dx_1 \wedge dx_2 \wedge dx_3 \\ &= f_nK\, dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Thus, $f\omega$ is closed for some functions f but not for others. In order for the form $f\omega$ to be closed, f must satisfy $f_n = 2Hf$, which is a somewhat opaque differential equation. On the other hand, for the form $fK\omega$ to be closed, f must satisfy $f_n = 0$. Using the equations in section 2.3 we can come up with many such functions. Let us define the set S by:

$$S = \{f\colon G \to \mathbf{R}^3 \mid f_n = 0\}.$$

Claim 3.3.1. If the following functions are defined then:

(i) All components of N, P, and Q belong to S.

(ii)

$$f_1, \ldots, f_k \in S,\ g \in C^1(\mathbf{R}^k) \Longrightarrow g(f_1, \ldots, f_k) \in S.$$

(iii)

$$\frac{1}{\kappa_1} - \frac{1}{\kappa_2} \in S.$$

(iv)

$$\langle X, P\rangle, \quad \langle X, Q\rangle \in S.$$

(v)

$$\langle X, N\rangle + \frac{1}{\kappa_1}, \quad \langle X, N\rangle + \frac{1}{\kappa_2} \in S.$$

(vi) All components of $X \times N$ belong to S.

(vii) All components of:

$$X \times P + \frac{1}{\kappa_1}Q, \quad X \times P + \frac{1}{\kappa_2}Q, \quad X \times P - \langle X, N\rangle Q$$
$$X \times Q - \frac{1}{\kappa_1}P, \quad X \times Q - \frac{1}{\kappa_2}P, \quad X \times Q + \langle X, N\rangle P$$

belong to S.

(viii)

$$\frac{(\kappa_1)_p}{\kappa_1^3}, \quad \frac{(\kappa_1)_q}{\kappa_1^2\kappa_2}, \quad \frac{(\kappa_2)_p}{\kappa_1\kappa_2^2}, \quad \frac{(\kappa_2)_q}{\kappa_2^3} \in S.$$

(ix)

$$f \in S \Longrightarrow \frac{1}{\kappa_1}f_p, \ \frac{1}{\kappa_2}f_q \in S.$$

Proof. All of (i) through (vii) follow from (2.4), (2.12), (2.13) and the identity $X_n = N$. To show (viii) and (ix), note that (2.13) and the definitions of P and Q imply:

$$f_{pn} = f_{np} + \kappa_1 f_p$$
$$f_{qn} = f_{nq} + \kappa_2 f_q.$$

□

3.4 Gauss-Bonnet formulas with functions of the normal

The set S contains arbitrary differentiable functions of the components of N. We claim that it is possible to obtain an explicit integral formula for fK when f is any integrable function of n_1, n_2 and n_3, thereby generalizing the Gauss-Bonnet theorem to a much larger set of integrands. If M is an ovaloid, the Gauss map is bijective and all functions on M can be expressed in terms of the components of the normal, meaning that in this case it is possible to obtain an integral formula for the product of K and an arbitrary function on M.

We start by noting that:

$$\begin{aligned}
&dn_1 \wedge dn_2 \\
&\quad = (n_{12}n_{23} - n_{13}n_{22})\, dx_2 \wedge dx_3 + (n_{12}n_{13} - n_{11}n_{23})\, dx_3 \wedge dx_1 + (n_{11}n_{22} - n_{12}^2)\, dx_1 \wedge dx_2 \\
&\quad = n_1 n_3 K\, dx_2 \wedge dx_3 + n_2 n_3 K\, dx_3 \wedge dx_1 + n_3^2 K\, dx_1 \wedge dx_2 \qquad \text{(by (2.9))} \\
&\quad = n_3 K\omega \\
&dn_2 \wedge dn_3 = n_1 K\omega \\
&dn_3 \wedge dn_1 = n_2 K\omega. \qquad (3.18)
\end{aligned}$$

These identities are similar to the result that the determinant of the differential of the Gauss map is equal to K. One immediate consequence is a Gaussian curvature analog to Stokes' Theorem:

Theorem 3.4.1. If $V\colon \mathbf{R}^3 \to \mathbf{R}^3$ then:

$$\int_{\partial M} V(n_1, n_2, n_3) \cdot dN = \int_M K\,(\operatorname{curl} V)(n_1, n_2, n_3) \cdot N\, dA. \qquad (3.19)$$

However, we would still like a direct method for integrating the product of an arbitrary function $f(n_1, n_2, n_3)$ with K. To obtain a primitive for $fK\omega$, we start by combining the identities in (3.18) to get:

$$K\omega = n_1\, dn_2 \wedge dn_3 + n_2\, dn_3 \wedge dn_1 + n_3\, dn_1 \wedge dn_2. \qquad (3.20)$$

Depending on the signs of n_1, n_2 and n_3, we can then multiply by f to get:

$$\begin{aligned} f(n_1,n_2,n_3)K\omega = &\pm\sqrt{1-n_2^2-n_3^2}\, f\left(\pm\sqrt{1-n_2^2-n_3^2}, n_2, n_3\right) dn_2\wedge dn_3 \\ &\pm\sqrt{1-n_1^2-n_3^2}\, f\left(n_1, \pm\sqrt{1-n_1^2-n_3^2}, n_3\right) dn_3\wedge dn_1 \\ &\pm\sqrt{1-n_1^2-n_2^2}\, f\left(n_1, n_2, \pm\sqrt{1-n_1^2-n_2^2}\right) dn_1\wedge dn_2. \end{aligned}$$

It is then possible to obtain primitives for each of the three component forms and thus a primitive for $fK\omega$. Note however that these primitives may have singularities on M, which will give rise to extra terms in the final integral formula if they occur.

If we know that one of the normal components – say n_3 – is non-zero on M, then $fK\omega$ can be reduced to a single form with a readily obtainable primitive:

$$f(n_1,n_2,n_3)K\omega = \frac{f\left(n_1,n_2,\pm\sqrt{1-n_1^2-n_2^2}\right)}{\pm\sqrt{1-n_1^2-n_2^2}}\, dn_1\wedge dn_2.$$

As an example, if n_3 is positive on M and $f=1$, applying this method gives:

$$\begin{aligned} \int_M K\,dA &= \int_M \frac{1}{\sqrt{1-n_1^2-n_2^2}}\, dn_1\wedge dn_2 \\ &= -\int_{\partial M} \tan^{-1}\left(\frac{n_2}{\sqrt{1-n_1^2-n_2^2}}\right) dn_1 \\ &= -\int_{\partial M} \tan^{-1}\left(\frac{n_2}{n_3}\right) dn_1. \end{aligned}$$

The inverse tangent in the above boundary integral is one of the spherical angles of the image of M under the Gauss map. Integrating by parts on the boundary gives:

$$\int_M K\,dA = \int_{\partial M} \frac{n_1}{n_2^2+n_3^2}(n_2\,dn_3 - n_3\,dn_2),$$

which has the same form as (3.15) with n_1 as the chosen component instead of n_3.

3.5 Identities involving $\kappa_2 - \kappa_1$

All of the integral formulas presented so far have been developed using the signed distance function. However, the following corollary of Stokes' Theorem provides an alternative way of proving (3.4), (3.5) and (3.19).

Claim 3.5.1. If V and W are any vector fields on $\mathbf{R}^3$ then:

$$\int_{\partial M} V \cdot dW = \int_M \left(\langle V_p, W_q\rangle - \langle V_q, W_p\rangle\right) dA. \tag{3.21}$$

Proof. We start with:

$$\begin{aligned}
\int_{\partial M} V \cdot dW &= \int_{\partial M} \left(V^1 \nabla W^1 + V^2 \nabla W^2 + V^3 \nabla W^3\right) \cdot dX \\
&= \int_M \left\langle \nabla V^1 \times \nabla W^1 + \nabla V^2 \times \nabla W^2 + \nabla V^3 \times \nabla W^3, N \right\rangle dA \qquad \text{by (3.1)} \\
&= -\int_M \left(\langle \nabla W^1, \nabla V^1 \times N\rangle + \langle \nabla W^2, \nabla V^2 \times N\rangle + \langle \nabla W^3, \nabla V^3 \times N\rangle\right) dA.
\end{aligned}$$

Using projection onto P and Q gives:

$$\begin{aligned}
\nabla V^i \times N &= \langle \nabla V^i \times N, P\rangle P + \langle \nabla V^i \times N, Q\rangle Q \\
&= \langle \nabla V^i, N \times P\rangle P + \langle \nabla V^i, N \times Q\rangle Q \\
&= V^i_q P - V^i_p Q
\end{aligned}$$

so that the original integral becomes:

$$\begin{aligned}
&-\int_M \left(V^1_q W^1_p - V^1_p W^1_q + V^2_q W^2_p - V^2_p W^2_q + V^3_q W^3_p - V^3_p W^3_q\right) dA \\
&= \int_M \left(\langle V_p, W_q\rangle - \langle V_q, W_p\rangle\right) dA.
\end{aligned}$$

□

Note that the above claim is still true if P and Q are any (not necessarily continuous) orthonormal vector fields on M, provided that they are oriented so that $P \times Q = N$ everywhere on M. Replacing V with $V \times N$ and W with X in the above equation gives (3.8) that is equivalent to (3.4), while replacing W with N instead gives (3.5). Replacing $V(X)$ with $V(N)$ and W with N gives (3.19) after some manipulation.

The above claim is useful for proving the following two identities involving the difference of the principal curvatures and an arbitrary function F, which to the best of my knowledge are new. The first identity corresponds to a particular subset of S while the second does not. (Not all integral formulas involving curvature correspond to a member of S – for example (3.9) and (3.13) do not). While it is possible to prove these identities using the signed distance function, the proof below based on (3.21) is shorter.

Theorem 3.5.1. For any differentiable function F:

$$\int_{\partial M} \nabla F(x_1, x_2, x_3) \cdot dN = -\int_M (\kappa_2 - \kappa_1)\, Q^T \operatorname{Hess} F(x_1, x_2, x_3)\, P\, dA \tag{3.22}$$

$$\int_{\partial M} \nabla F(n_1, n_2, n_3) \cdot dX = \int_M (\kappa_2 - \kappa_1)\, Q^T \operatorname{Hess} F(n_1, n_2, n_3)\, P\, dA \tag{3.23}$$

Proof. We have:

$$\begin{aligned}
(\nabla F(x_1, x_2, x_3))_p &= \operatorname{Hess} F(x_1, x_2, x_3)\, X_p & (\nabla F(x_1, x_2, x_3))_q &= \operatorname{Hess} F(x_1, x_2, x_3)\, X_q \\
&= \operatorname{Hess} F(x_1, x_2, x_3)\, P & &= \operatorname{Hess} F(x_1, x_2, x_3)\, Q \\
(\nabla F(n_1, n_2, n_3))_p &= \operatorname{Hess} F(n_1, n_2, n_3)\, N_p & (\nabla F(n_1, n_2, n_3))_q &= \operatorname{Hess} F(n_1, n_2, n_3)\, N_q \\
&= -\kappa_1 \operatorname{Hess} F(n_1, n_2, n_3)\, P & &= -\kappa_2 \operatorname{Hess} F(n_1, n_2, n_3)\, Q
\end{aligned}$$

so that the results follow from (3.21). □

As a corollary, for any differentiable function f we have:

$$\begin{aligned}
\int_{\partial M} f(n_1)\, dx_1 &= \int_M (\kappa_2 - \kappa_1) p_1 q_1 f'(n_1)\, dA \\
\int_{\partial M} f(n_2)\, dx_2 &= \int_M (\kappa_2 - \kappa_1) p_2 q_2 f'(n_2)\, dA \\
\int_{\partial M} f(n_3)\, dx_3 &= \int_M (\kappa_2 - \kappa_1) p_3 q_3 f'(n_3)\, dA.
\end{aligned}$$

Chapter 4

A differential equation for the principal directions and radii

4.1 Bonnet co-ordinates

In section 3.4 we showed how to obtain an integral formula for $fK\omega$, where f is an arbitrary function of N. However, according to section 3.3, it should be possible to obtain an integral formula when f is an arbitrary function of P, Q and $1/\kappa_1 - 1/\kappa_2$ (which are all based on the second derivatives of n) as well as N. In order to obtain an integral formula for $f(N)K\omega$, we subsumed $K\omega$ into the wedge product of the differentials of two components of N, making it possible to find a primitive for $f(N)K\omega$. If we now want to introduce extra terms into the arguments of f while keeping the same strategy, we are led to investigate how these terms behave when expressed as functions of two components of N. That is, instead of considering N as a dependent variable with respect to some co-ordinate system, we instead take N itself (or equivalently two functions of its components since $\|N\| = 1$) as the co-ordinate system and try to determine how the second-order terms in n behave with respect to these co-ordinates.

For our two functions of N we will assume that $n_3 \neq -1$ and use Bonnet co-ordinates defined by:

$$\xi_1 = \frac{n_1}{1+n_3} \qquad\qquad \xi_2 = \frac{n_2}{1+n_3}. \tag{4.1}$$

For a point $X \in M$, these co-ordinates result from stereographically projecting the point $N(X)$, which lies on the unit sphere, from the sphere's south pole $(0,0,-1)$ onto the plane $z=0$ ([27] contains a detailed construction and discussion of these co-ordinates). We can express the normal vector N in terms of Bonnet coordinates as:

$$N = \frac{1}{1+\xi_1^2+\xi_2^2}\begin{pmatrix} 2\xi_1 \\ 2\xi_2 \\ 1-\xi_1^2-\xi_2^2 \end{pmatrix}. \tag{4.2}$$

With the surface co-ordinates determined, consider the transformation on G defined by:

$$\begin{aligned} &C\colon G \to \mathbf{R}^3 \\ &C(x_1,x_2,x_3) = (\xi_1(x_1,x_2,x_3), \xi_2(x_1,x_2,x_3), n(x_1,x_2,x_3))^T. \end{aligned}$$

The first two components of C serve as co-ordinates on M and the other level surfaces of n, while the third component of C specifies a particular level surface of n within G. We have:

$$\begin{aligned} \nabla\xi_1 &= \frac{(1+n_3)\nabla n_1 - n_1\nabla n_3}{(1+n_3)^2} \\ &= \frac{-(1+n_3)p_1 + n_1 p_3}{(1+n_3)^2}\kappa_1 P + \frac{-(1+n_3)q_1 + n_1 q_3}{(1+n_3)^2}\kappa_2 Q \quad \text{(by (2.7))} \\ &= -\frac{p_1+q_2}{(1+n_3)^2}\kappa_1 P + \frac{p_2-q_1}{(1+n_3)^2}\kappa_2 Q \\ \nabla\xi_2 &= \frac{q_1-p_2}{(1+n_3)^2}\kappa_1 P - \frac{p_1+q_2}{(1+n_3)^2}\kappa_2 Q \end{aligned}$$

and:

$$\begin{aligned} J_C &= \left(-\frac{p_1+q_2}{(1+n_3)^2}\kappa_1 P + \frac{p_2-q_1}{(1+n_3)^2}\kappa_2 Q, \quad \frac{q_1-p_2}{(1+n_3)^2}\kappa_1 P - \frac{p_1+q_2}{(1+n_3)^2}\kappa_2 Q, \quad N\right)^T \\ \det(J_C) &= \frac{(p_1+q_2)^2 + (p_2-q_1)^2}{(1+n_3)^4}K \\ &= \frac{K}{(1+n_3)^2}. \end{aligned}$$

Therefore, for this co-ordinate system to be valid, we require that $K \neq 0$ everywhere in G, along with the original assumption that $n_3 \neq -1$. With our new co-ordinate system, we seek to determine what differentiation with respect to the surface co-ordinates ξ_1 and ξ_2 means in terms of the original co-ordinates. We have:

$$\begin{aligned}\frac{\partial}{\partial \xi_1} &= \frac{\partial x_1}{\partial \xi_1}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \xi_1}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial \xi_1}\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial \xi_2} &= \frac{\partial x_1}{\partial \xi_2}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \xi_2}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial \xi_2}\frac{\partial}{\partial x_3}.\end{aligned}$$

By taking minors of J_C we get:

$$\begin{aligned}\frac{\partial x_1}{\partial \xi_1} &= \frac{(p_2 - q_1)(n_2 p_3 - n_3 p_2)\kappa_1 + (p_1 + q_2)(n_2 q_3 - n_3 q_2)\kappa_2}{(1+n_3)^2} \cdot \frac{(1+n_3)^2}{K} \\ &= \left(q_1(p_2 - q_1)\kappa_1 - p_1(p_1 + q_2)\kappa_2\right)/K \\ \frac{\partial x_2}{\partial \xi_1} &= \left(q_2(p_2 - q_1)\kappa_1 - p_2(p_1 + q_2)\kappa_2\right)/K \\ \frac{\partial x_3}{\partial \xi_1} &= \left(q_3(p_2 - q_1)\kappa_1 - p_3(p_1 + q_2)\kappa_2\right)/K \\ \frac{\partial x_1}{\partial \xi_2} &= \left(-q_1(p_1 + q_2)\kappa_1 - p_1(p_2 - q_1)\kappa_2\right)/K \\ \frac{\partial x_2}{\partial \xi_2} &= \left(-q_2(p_1 + q_2)\kappa_1 - p_2(p_2 - q_1)\kappa_2\right)/K \\ \frac{\partial x_3}{\partial \xi_2} &= \left(-q_3(p_1 + q_2)\kappa_1 - p_3(p_2 - q_1)\kappa_2\right)/K\end{aligned}$$

These equations can be summarized in vector form as:

$$\begin{aligned}\frac{\partial X}{\partial \xi_1} &= -\frac{p_1 + q_2}{\kappa_1}P + \frac{p_2 - q_1}{\kappa_2}Q \\ \frac{\partial X}{\partial \xi_2} &= -\frac{p_2 - q_1}{\kappa_1}P - \frac{p_1 + q_2}{\kappa_2}Q.\end{aligned} \tag{4.3}$$

In terms of the directional derivative operators in directions P and Q we then have:

$$\begin{aligned}\frac{\partial}{\partial \xi_1} &= -\frac{p_1 + q_2}{\kappa_1}\nabla_P + \frac{p_2 - q_1}{\kappa_2}\nabla_Q \\ \frac{\partial}{\partial \xi_2} &= -\frac{p_2 - q_1}{\kappa_1}\nabla_P - \frac{p_1 + q_2}{\kappa_2}\nabla_Q.\end{aligned} \tag{4.4}$$

Note that all of the derivatives with respect to ξ_1 and ξ_2 calculated above are well defined at umbilical points. Reversing the relations in (4.4) yields:

$$\begin{aligned}\nabla_P &= -\frac{p_1+q_2}{(1+n_3)^2}\kappa_1\frac{\partial}{\partial\xi_1} - \frac{p_2-q_1}{(1+n_3)^2}\kappa_1\frac{\partial}{\partial\xi_2}\\ \nabla_Q &= \frac{p_2-q_1}{(1+n_3)^2}\kappa_2\frac{\partial}{\partial\xi_1} - \frac{p_1+q_2}{(1+n_3)^2}\kappa_2\frac{\partial}{\partial\xi_2}.\end{aligned} \tag{4.5}$$

4.2 The Mainardi-Codazzi equations

In this section we present a different perspective on the Mainardi-Codazzi equations, a classical pair of equations from differential geometry. Although our result is mentioned in passing in [31], we provide detailed proofs here. These equations are usually stated as two of the compatibility conditions necessary for a first and second fundamental form to be consistent with each other (i.e. correspond to the same surface); the third compatibility condition is the Gauss equation. However, the Mainardi-Codazzi equations actually determine the paths followed by the lines of curvature on M. In the way we present them here they are invariant under co-ordinate transformations so that they do not depend on the coefficients of the fundamental forms, or on any non-invariant co-ordinates used to parametrize M.

Claim 4.2.1. At non-umbilical points, the directional derivatives of the unit principal directions P and Q satisfy:

$$\begin{aligned}P_p &= -\frac{(\kappa_1)_q}{\kappa_2-\kappa_1}Q + \kappa_1 N\\ Q_p &= \frac{(\kappa_1)_q}{\kappa_2-\kappa_1}P\\ P_q &= -\frac{(\kappa_2)_p}{\kappa_2-\kappa_1}Q\\ Q_q &= \frac{(\kappa_2)_p}{\kappa_2-\kappa_1}P + \kappa_2 N.\end{aligned} \tag{4.6}$$

Proof. By differentiating the orthonormality relations between N, P and Q in the P and Q directions, we find that there exist scalar functions α and β such that at non-umbilical points:

$$P_p = \alpha Q + \kappa_1 N$$
$$Q_p = -\alpha P$$
$$P_q = -\beta Q$$
$$Q_q = \beta P + \kappa_2 N.$$

We present two ways to solve for α and β:

1) If u and v are co-ordinates that parametrize M by lines of curvature corresponding to principal directions P and Q, then (see [20], problem 10.6) the Mainardi-Codazzi equations in terms of the first fundamental form reduce to:

$$\frac{E_v}{E} = \frac{2(\kappa_1)_v}{\kappa_2 - \kappa_1} \qquad \frac{G_u}{G} = -\frac{2(\kappa_2)_u}{\kappa_2 - \kappa_1}.$$

The result then follows from:

$$\begin{aligned} -\alpha &= Q_p \cdot P & -\beta &= P_q \cdot Q \\ &= \frac{1}{\sqrt{E}}\left(\frac{X_v}{\sqrt{G}}\right)_u \cdot \frac{X_u}{\sqrt{E}} & &= \frac{1}{\sqrt{G}}\left(\frac{X_u}{\sqrt{E}}\right)_v \cdot \frac{X_v}{\sqrt{G}} \\ &= \frac{X_{uv} \cdot X_u}{E\sqrt{G}} & &= \frac{X_{uv} \cdot X_v}{G\sqrt{E}} \\ &= \frac{E_v}{2E\sqrt{G}} & &= \frac{G_u}{2G\sqrt{E}} \\ (\kappa_1)_v &= \sqrt{G}(\kappa_1)_q & (\kappa_2)_u &= \sqrt{E}(\kappa_2)_p. \end{aligned}$$

2) Here is a self-contained proof based on the function n that is slightly simpler than the proof of the classical equations. Since at least one component of N must be non-zero at every point on M, we assume without loss of generality that $n_3 \neq 0$. We determine α and β by equating the third derivatives of n calculated in two different ways. Notice first that:

$$\frac{\partial}{\partial x_1} = p_1 \nabla_P + q_1 \nabla_Q + n_1 \nabla_N$$
$$\frac{\partial}{\partial x_2} = p_2 \nabla_P + q_2 \nabla_Q + n_2 \nabla_N.$$

The relation:

$$(n_{11})_2 = (n_{12})_1$$

thus implies:

$$(p_2\nabla_P + q_2\nabla_Q + n_2\nabla_N)\left(p_1^2\kappa_1 + q_1^2\kappa_2\right) = (p_1\nabla_P + q_1\nabla_Q + n_1\nabla_N)\left(p_1p_2\kappa_1 + q_1q_2\kappa_2\right)$$

$$\begin{aligned}
&p_2\left[2p_1(q_1\alpha + n_1\kappa_1)\kappa_1 + p_1^2(\kappa_1)_p - 2p_1q_1\alpha\kappa_2 + q_1^2(\kappa_2)_p\right]\\
&+ q_2\left[-2p_1q_1\beta\kappa_1 + p_1^2(\kappa_1)_q + 2q_1(p_1\beta + n_1\kappa_2)\kappa_2 + q_1^2(\kappa_1)_q\right]\\
&+ p_1^2n_2\kappa_1^2 + q_1^2n_2\kappa_2^2\\
&= p_1\left[p_2(q_1\alpha + n_1\kappa_1)\kappa_1 + p_1(q_2\alpha + n_2\kappa_1)\kappa_1 + p_1p_2(\kappa_1)_p - p_1q_2\alpha\kappa_2 - p_2q_1\alpha\kappa_2 + q_1q_2(\kappa_2)_p\right]\\
&+ q_1\left[-p_2q_1\beta\kappa_1 - p_1q_2\beta\kappa_1 + p_1p_2(\kappa_1)_q + q_2(p_1\beta + n_1\kappa_2)\kappa_2 + q_1(p_2\beta + n_2\kappa_2)\kappa_2 + q_1q_2(\kappa_2)_q\right]\\
&+ p_1p_2n_1\kappa_1^2 + q_1q_2n_1\kappa_2^2.
\end{aligned}$$

Canceling terms gives:

$$\begin{aligned}
&- p_1p_2q_1(\kappa_2 - \kappa_1)\alpha + p_1q_1q_2(\kappa_2 - \kappa_1)\beta + p_2q_1^2(\kappa_2)_p + p_1^2q_2(\kappa_1)_q\\
&= -p_1^2q_2(\kappa_2 - \kappa_1)\alpha + p_2q_1^2(\kappa_2 - \kappa_1)\beta + p_1q_1q_2(\kappa_2)_p + p_1p_2q_1(\kappa_1)_q.
\end{aligned}$$

From the assumption that $n_3 \neq 0$ we then get:

$$p_1(\kappa_2 - \kappa_1)\alpha + q_1(\kappa_2 - \kappa_1)\beta = -p_1(\kappa_1)_q + q_1(\kappa_2)_p.$$

Repeating the process starting from the relation:

$$(n_{12})_2 = (n_{22})_1$$

leads to:

$$p_2(\kappa_2 - \kappa_1)\alpha + q_2(\kappa_2 - \kappa_1)\beta = -p_2(\kappa_1)_q + q_2(\kappa_2)_p.$$

Finally, another invocation of $n_3 \neq 0$ gives:

$$\alpha = -\frac{(\kappa_1)_q}{\kappa_2 - \kappa_1} \qquad \beta = \frac{(\kappa_2)_p}{\kappa_2 - \kappa_1}.$$

□

4.3 Principal radii and directions

This section proves the deepest new result in this thesis, namely that the Mainardi-Codazzi equations of the previous section reduce to a single complex first-order linear partial differential equation (4.9), in terms of the principal radii and principal directions, when written relative to Bonnet co-ordinates. While there do exist concise reductions of the Mainardi-Codazzi equations into complex form in the literature (see for example chapter 6 of [17]), our result is novel in that, like the equations in the previous section, it does not depend on any non-invariant co-ordinates used to parameterize M and does not refer to any coefficients of the surface's fundamental forms. All of the dependent and independent variables in (4.9) are explicit invariant functions of the surface.

Given the normal vector N, the two principal directions P and Q can be written in terms of a single additional parameter θ representing the angle that P and Q make relative to another reference pair of orthonormal vector fields on M. It turns out that the reference vector fields that are easiest to work with computationally are given by:

$$V^1 = \frac{1}{1+n_3}\begin{pmatrix} -1-n_3+n_1^2 \\ n_1 n_2 \\ n_1+n_1 n_3 \end{pmatrix} = \frac{1}{1+\xi_1^2+\xi_2^2}\begin{pmatrix} -1+\xi_1^2-\xi_2^2 \\ 2\xi_1\xi_2 \\ 2\xi_1 \end{pmatrix}$$
$$V^2 = \frac{1}{1+n_3}\begin{pmatrix} n_1 n_2 \\ -1-n_3+n_2^2 \\ n_2+n_2 n_3 \end{pmatrix} = \frac{1}{1+\xi_1^2+\xi_2^2}\begin{pmatrix} 2\xi_1\xi_2 \\ -1-\xi_1^2+\xi_2^2 \\ 2\xi_2 \end{pmatrix}. \tag{4.7}$$

The above vector fields can be obtained by removing the curvature terms from the vectors in (4.3) and normalizing. With these reference vectors in hand we define θ by the relations:

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \cos\theta\, V^1 + \sin\theta\, V^2, \qquad Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\sin\theta\, V^1 + \cos\theta\, V^2. \tag{4.8}$$

At non-umbilical points θ is defined only up to an integer multiple of 2π, but $\cos\theta$ and $\sin\theta$ are well defined, and so are the derivatives of θ. Let us further define:

$$W = \left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right) e^{2i\theta}$$
$$z = \xi_1 + i\xi_2.$$

While P, Q and θ may have discontinuities at umbilical points, the function W can be extended continuously to all of M by setting it to 0 at these points. We are now ready to state the main result:

Theorem 4.3.1. If z and $\bar{z}$ are taken as complex local co-ordinates on a surface M for which $K \neq 0$ and $n_3 \neq -1$ everywhere, then at all points on M there holds:

$$W_z - \frac{2\bar{z}}{1+z\bar{z}} W = \left(\frac{2H}{K}\right)_{\bar{z}}. \tag{4.9}$$

Proof. In this section we prove the theorem only at non-umbilical points on M. The remainder of the proof will be given at the end of section 4.5. We prove the result by writing equations (4.6) in terms of the parameter θ, and changing from directional derivatives in $\mathbf{R}^3$ to co-ordinate derivatives with respect to ξ_1 and ξ_2 (denoted by subscripts). We start by using (4.4) and (4.5) to get:

$$\begin{aligned}
P_{\xi_1} &= -\frac{p_1+q_2}{\kappa_1}P_p + \frac{p_2-q_1}{\kappa_2}P_q \\
&= \left[\frac{(p_1+q_2)(\kappa_1)_q}{\kappa_1(\kappa_2-\kappa_1)} - \frac{(p_2-q_1)(\kappa_2)_q}{\kappa_2(\kappa_2-\kappa_1)}\right] Q - (p_1+q_2)N \\
P_{\xi_1}\cdot Q &= \frac{(p_1+q_2)(p_2-q_1)\kappa_2(\kappa_1)_{\xi_1} - (p_1+q_2)^2\kappa_2(\kappa_1)_{\xi_2}}{(1+n_3)^2\kappa_1(\kappa_2-\kappa_1)} \\
&\quad + \frac{(p_1+q_2)(p_2-q_1)\kappa_1(\kappa_2)_{\xi_1} + (p_2-q_1)^2\kappa_1(\kappa_2)_{\xi_2}}{(1+n_3)^2\kappa_2(\kappa_2-\kappa_1)} \\
P_{\xi_2} &= -\frac{p_2-q_1}{\kappa_1}P_p - \frac{p_1+q_2}{\kappa_2}P_q \\
&= \left[\frac{(p_2-q_1)(\kappa_1)_q}{\kappa_1(\kappa_2-\kappa_1)} + \frac{(p_1+q_2)(\kappa_2)_p}{\kappa_2(\kappa_2-\kappa_1)}\right] Q - (p_2-q_1)N \\
P_{\xi_2}\cdot Q &= \frac{(p_2-q_1)^2\kappa_2(\kappa_1)_{\xi_1} - (p_1+q_2)(p_2-q_1)\kappa_2(\kappa_1)_{\xi_2}}{(1+n_3)^2\kappa_1(\kappa_2-\kappa_1)} \\
&\quad - \frac{(p_1+q_2)^2\kappa_1(\kappa_2)_{\xi_1} + (p_1+q_2)(p_2-q_1)\kappa_1(\kappa_2)_{\xi_2}}{(1+n_3)^2\kappa_2(\kappa_2-\kappa_1)}.
\end{aligned}$$

If we define:

$$R = \frac{1}{\kappa_1} - \frac{1}{\kappa_2} = \frac{\kappa_2 - \kappa_1}{\kappa_1 \kappa_2} \tag{4.10}$$

and notice that:

$$\begin{aligned} p_1 + q_2 &= -(1 + n_3) \cos\theta \\ p_2 - q_1 &= -(1 + n_3) \sin\theta \end{aligned} \tag{4.11}$$

then the above equations simplify to:

$$\begin{aligned}
2R\, P_{\xi_1} \cdot Q &= \sin 2\theta\, \frac{(\kappa_1)_{\xi_1}}{\kappa_1^2} - (1 + \cos 2\theta) \frac{(\kappa_1)_{\xi_2}}{\kappa_1^2} + \sin 2\theta\, \frac{(\kappa_2)_{\xi_1}}{\kappa_2^2} + (1 - \cos 2\theta) \frac{(\kappa_2)_{\xi_2}}{\kappa_2^2} \\
&= \cos 2\theta \left(\frac{2H}{K} \right)_{\xi_2} - \sin 2\theta \left(\frac{2H}{K} \right)_{\xi_1} + R_{\xi_2} \\
2R\, P_{\xi_2} \cdot Q &= (1 - \cos 2\theta) \frac{(\kappa_1)_{\xi_1}}{\kappa_1^2} - \sin 2\theta\, \frac{(\kappa_1)_{\xi_2}}{\kappa_1^2} - (1 + \cos 2\theta) \frac{(\kappa_2)_{\xi_1}}{\kappa_2^2} - \sin 2\theta\, \frac{(\kappa_2)_{\xi_2}}{\kappa_2^2} \\
&= \cos 2\theta \left(\frac{2H}{K} \right)_{\xi_1} + \sin 2\theta \left(\frac{2H}{K} \right)_{\xi_2} - R_{\xi_1} \\
-2iR\, P_z \cdot Q &= R(iP_{\xi_1} + P_{\xi_2}) \cdot Q \\
&= e^{-2i\theta} \left(\frac{2H}{K} \right)_{\bar{z}} - R_z.
\end{aligned}$$

We now recalculate the same dot product using (4.8). Direct differentiation of (4.7) gives:

$$\begin{aligned}
V^1_{\xi_1} &= \frac{2\xi_2}{1 + \xi_1^2 + \xi_2^2} V^2 + \frac{2}{1 + \xi_1^2 + \xi_2^2} N \\
V^1_{\xi_2} &= -\frac{2\xi_1}{1 + \xi_1^2 + \xi_2^2} V^2 \\
V^2_{\xi_1} &= -\frac{2\xi_2}{1 + \xi_1^2 + \xi_2^2} V^1 \\
V^2_{\xi_2} &= \frac{2\xi_1}{1 + \xi_1^2 + \xi_2^2} V^1 + \frac{2}{1 + \xi_1^2 + \xi_2^2} N
\end{aligned}$$

so that:

$$
\begin{aligned}
P_{\xi_1} &= \left(\theta_{\xi_1} + \frac{2\xi_2}{1+\xi_1^2+\xi_2^2}\right)(-\sin\theta\, V^1 + \cos\theta\, V^2) + \frac{2\cos\theta}{1+\xi_1^2+\xi_2^2}N \\
&= \left(\theta_{\xi_1} + \frac{2\xi_2}{1+\xi_1^2+\xi_2^2}\right) Q + \frac{2\cos\theta}{1+\xi_1^2+\xi_2^2}N \\
P_{\xi_2} &= \left(\theta_{\xi_2} - \frac{2\xi_1}{1+\xi_1^2+\xi_2^2}\right) Q + \frac{2\sin\theta}{1+\xi_1^2+\xi_2^2}N \\
-2iR\, P_z \cdot Q &= 2iR\theta_z - \frac{2\bar{z}}{1+z\bar{z}}R.
\end{aligned}
$$

Since $W = Re^{2i\theta}$, the result follows from equating the two dot products and multiplying by $e^{2i\theta}$. □

Note that it is possible to obtain a differential equation using the above theorem if θ is defined relative to any pair of orthonormal reference vector fields that are specified in terms of the components of N. The effect of redefining θ will be to multiply the function W by a function of z and $\bar{z}$ having modulus 1, so that the new function will satisfy a linear differential equation similar to (4.9).

4.4 Brief review of complex co-ordinates

Equation (4.9) is written in terms of complex co-ordinates z and $\bar{z}$ representing two independent real variables. Let us call the real variables x and y so that $z = x + iy$. Before proceeding further, we gather some simple statements about these co-ordinates that we will need to refer to throughout the remainder of this chapter and the next.

Lemma 4.4.1. For any function $f(z, \bar{z})$:

$$
\begin{aligned}
\overline{f_z} &= \bar{f}_{\bar{z}} \\
\overline{f_{\bar{z}}} &= \bar{f}_z.
\end{aligned}
$$

Corollary 4.4.2. For real functions f:

$$f_{\bar{z}} = \overline{f_z} \tag{4.12}$$

$$f_{\bar{z}\bar{z}} = \overline{f_{zz}}. \tag{4.13}$$

In particular:

$$\left(\frac{H}{K}\right)_{\bar{z}} = \overline{\left(\frac{H}{K}\right)_z}. \tag{4.14}$$

Corollary 4.4.3. If $f(z)$ is a holomorphic function, then $\overline{f(z)}$ is an anti-holomorphic function (i.e. a function of $\bar{z}$ alone).

Corollary 4.4.4. If $f(z) + g(\bar{z})$ is real then it is equal to $2\,\mathrm{Re}\, f(z)$ plus a constant.

Proof. Since $f + g = \bar{f} + \bar{g}$, we have:

$$g(\bar{z}) - \overline{f(z)} = \overline{g(\bar{z})} - f(z).$$

Since the left side is anti-holomorphic and the right side is holomorphic, both sides must be equal to the same real constant c. Then $g = \bar{f} + c$ and $f + g = f + \bar{f} + c$. □

Lemma 4.4.5. For any function $f(z, \bar{z})$:

$$f_{z\bar{z}} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y}\right) = \frac{1}{4}\Delta f \tag{4.15}$$

so that $f_{z\bar{z}}$ is real if f is real.

Lemma 4.4.6. If $f(z, \bar{z})$ is a given function on a star-shaped domain then the equation:

$$g_{\bar{z}} = f \tag{4.16}$$

has a real solution g iff f_z is real.

Proof. If f_z is real then:

$$\frac{\partial\, \mathrm{Im}\, f}{\partial x} = \frac{\partial\, \mathrm{Re}\, f}{\partial y}$$

and the differential form:

$$\eta = \mathrm{Re}\, f\, dx + \mathrm{Im}\, f\, dy$$

is closed. Thus by the Poincaré lemma there is a real function g on the domain such that $dg = \eta$ and $g_{\bar{z}} = f$. □

4.5 Relation of W to the support function

In this section we extend, using our own notation, a known result relating the principal directions of a surface to its support function $\langle X, N\rangle$. The existing result is proven in [12] and is stated at the beginning of [18]. As in these papers, the identity will serve as the basis for our study of the Loewner index conjecture. However, a more immediate application will be to complete the proof of theorem 4.3.1.

We start by rewriting (4.3) in our defined notation:

Claim 4.5.1.

$$\frac{\partial X}{\partial z} = \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\frac{H}{K} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}\frac{\overline{W}}{2} \tag{4.17}$$

$$\frac{\partial X}{\partial \bar{z}} = \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}\frac{H}{K} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\frac{W}{2}. \tag{4.18}$$

Proof.

$$\begin{aligned}\frac{\partial X}{\partial z} &= \frac{1}{2}\left(X_{\xi_1} - iX_{\xi_2}\right)\\ &= -\frac{1}{2}(p_1 + q_2 - i(p_2 - q_1))\left(\frac{P}{\kappa_1} - i\frac{Q}{\kappa_2}\right)\\ &= \frac{e^{-i\theta}}{1+z\bar{z}}\left(\frac{H}{K}(P-iQ) + \frac{R}{2}(P+iQ)\right) \qquad \text{(by (4.11) and (4.10)).}\end{aligned}$$

From (4.7) and (4.8) we have:

$$\begin{aligned}P - iQ &= e^{i\theta}(V^1 - iV^2)\\ &= \frac{e^{i\theta}}{1+z\bar{z}}\left[\begin{pmatrix} -1+\frac{1}{2}(\bar{z}^2+z^2) \\ \frac{i}{2}(\bar{z}^2-z^2) \\ \bar{z}+z \end{pmatrix} - i\begin{pmatrix} \frac{i}{2}(\bar{z}^2-z^2) \\ -1-\frac{1}{2}(\bar{z}^2+z^2) \\ i(\bar{z}-z) \end{pmatrix}\right]\end{aligned}$$

$$= \frac{e^{i\theta}}{1+z\bar{z}} \begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}$$

$$P + iQ = \frac{e^{-i\theta}}{1+z\bar{z}} \begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}$$

which leads to the result. □

We next relate W to the support function of a surface. The real-valued function $(1+z\bar{z})\langle X, N\rangle$ appearing in the equation below is referred to in the literature as the Bonnet function of a surface.

Claim 4.5.2.

$$[(1+z\bar{z})\langle X, N\rangle]_{\bar{z}\bar{z}} = -\frac{W}{1+z\bar{z}}. \tag{4.19}$$

Proof. Note that N can be written as:

$$N = \frac{1}{1+z\bar{z}} \begin{pmatrix} z+\bar{z} \\ -i(z-\bar{z}) \\ 1-z\bar{z} \end{pmatrix}. \tag{4.20}$$

We then have:

$$N_{\bar{z}} = \frac{1}{(1+z\bar{z})^2} \begin{pmatrix} 1-z^2 \\ i(1+z^2) \\ -2z \end{pmatrix}$$

$$N_{\bar{z}\bar{z}} = -\frac{2z}{1+z\bar{z}} N_{\bar{z}}.$$

Because $\langle X_{\bar{z}}, N\rangle = 0$ we have:

$$\begin{aligned} [(1+z\bar{z})\langle X, N\rangle]_{\bar{z}\bar{z}} &= (1+z\bar{z})\langle X_{\bar{z}}, N_{\bar{z}}\rangle + (1+z\bar{z})\langle X, N_{\bar{z}\bar{z}}\rangle + 2z\langle X, N_{\bar{z}}\rangle \\ &= (1+z\bar{z})\langle X_{\bar{z}}, N_{\bar{z}}\rangle \\ &= -\frac{1}{1+z\bar{z}} W \end{aligned}$$

by (4.18). □

The Bonnet function is a smooth function on smooth surfaces, and an analytic function on analytic surfaces. The above claim thus implies that the continuous function W is also smooth or analytic on smooth or analytic surfaces. This makes completing the proof of theorem 4.3.1 very simple:

Claim 4.5.3. Equation (4.9) holds at all umbilical points on M.

Proof. Since W is a smooth function, the left side of (4.9) is well defined everywhere on M. By continuity, (4.9) is true at all points in the closure of the set of non-umbilical points of M. If the complement of this set is non-empty, each of its components will be open and totally umbilical. Since $K \neq 0$, each such component must be part of a sphere, and (4.9) is satisfied at all points in the component because both sides are identically 0. □

4.6 Existence results

Equation (4.9) provides a necessary condition for the functions W and H/K to correspond to an actual surface. It is natural to ask whether this condition is also sufficient. The answer is affirmative, subject to one additional constraint stated in the claim below. Our method for constructing a surface from given functions W and H/K will be to integrate equation (4.18).

Claim 4.6.1. Let $\mathcal{W}(z, \bar{z})$ be a complex function and $\mathcal{H}/\mathcal{K}(z, \bar{z})$ a real function defined in a star-shaped region Ω of the complex plane. Suppose that:

$$\left(\frac{2\mathcal{H}}{\mathcal{K}}\right)^2 - |\mathcal{W}|^2 \neq 0 \tag{4.21}$$

and the functions satisfy (4.9) everywhere in Ω. Then there exists a surface parametrized over Ω with $K \neq 0$, $W = \mathcal{W}$ and $H/K = \mathcal{H}/\mathcal{K}$.

Proof. Define the vector function $\mathcal{V}(z,\bar{z})$ by:

$$\mathcal{V} = \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}\frac{\mathcal{H}}{\mathcal{K}} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\frac{\mathcal{W}}{2}.$$

We then have:

$$\begin{aligned}
\mathcal{V}_z = {} & \frac{2}{(1+z\bar{z})^3}\begin{pmatrix} z+\bar{z} \\ -i(z-\bar{z}) \\ 1-z\bar{z} \end{pmatrix}\frac{\mathcal{H}}{\mathcal{K}} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}\left(\frac{\mathcal{H}}{\mathcal{K}}\right)_z \\
& - \frac{\bar{z}}{(1+z\bar{z})^3}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\mathcal{W} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\frac{\mathcal{W}_z}{2} \\
= {} & \frac{2}{(1+z\bar{z})^3}\begin{pmatrix} z+\bar{z} \\ -i(z-\bar{z}) \\ 1-z\bar{z} \end{pmatrix}\frac{\mathcal{H}}{\mathcal{K}} \\
& + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z \end{pmatrix}\overline{\left(\frac{\mathcal{H}}{\mathcal{K}}\right)_{\bar{z}}} + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z} \end{pmatrix}\left(\frac{\mathcal{H}}{\mathcal{K}}\right)_{\bar{z}} \\
& \in \mathbf{R}^3.
\end{aligned}$$

Thus by lemma 4.4.6 there exists a real vector X defined on Ω satisfying:

$$X_{\bar{z}} = \mathcal{V}, \qquad\qquad X_z = \overline{\mathcal{V}}$$

for which we have:

$$\begin{aligned}
\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} &= \left(\frac{\partial X}{\partial z} + \frac{\partial X}{\partial \bar{z}}\right) \times \left(i\frac{\partial X}{\partial z} - i\frac{\partial X}{\partial \bar{z}}\right) \\
&= 2i\mathcal{V} \times \overline{\mathcal{V}}
\end{aligned}$$

$$= \frac{1}{(1+z\bar{z})^3}\left[\left(\frac{2\mathcal{H}}{\mathcal{K}}\right)^2 - |\mathcal{W}|^2\right]\begin{pmatrix} z+\bar{z} \\ -i(z-\bar{z}) \\ 1-z\bar{z}\end{pmatrix}.$$

Therefore X defines a regular surface over Ω and the normal to this surface satisfies (4.20). By back solving, we find that the real and imaginary parts of z are Bonnet co-ordinates for the surface. Using (4.18) we now have:

$$\frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+z^2 \\ -i(1+z^2) \\ 2z\end{pmatrix}\left(\frac{H}{K}-\frac{\mathcal{H}}{\mathcal{K}}\right) + \frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -1+\bar{z}^2 \\ i(1+\bar{z}^2) \\ 2\bar{z}\end{pmatrix}\left(\frac{W}{2}-\frac{\mathcal{W}}{2}\right) = 0$$

which implies that $W = \mathcal{W}$ and $H/K = \mathcal{H}/\mathcal{K}$ because the cross product of the two vectors in the above equation is non-zero. □

Remark 4.6.2. In fact we have:

$$\left(\frac{2H}{K}\right)^2 - |W|^2 = \frac{4}{K}$$

so that condition (4.21) is equivalent to stating that the Gaussian curvature of the surface implied by $\mathcal{W}$ and $\mathcal{H}/\mathcal{K}$ is finite.

Corollary 4.6.3. If $\mathcal{W}$ is a complex function defined in a compact star-shaped region of the complex plane, there exists a surface parametrized over the region with $K \neq 0$ and $W = \mathcal{W}$ if:

$$\left[\mathcal{W}_z - \frac{2\bar{z}}{1+z\bar{z}}\mathcal{W}\right]_z = \mathcal{W}_{zz} - \frac{2\bar{z}}{1+z\bar{z}}\mathcal{W}_z + \frac{2\bar{z}^2}{(1+z\bar{z})^2}\mathcal{W} \in \mathbf{R}. \tag{4.22}$$

Proof. By claim 4.6.1, $\mathcal{W}$ will correspond to an actual surface if we can find a real function $\mathcal{H}/\mathcal{K}$ so that both functions satisfy (4.9) and (4.21). By lemma 4.4.6, the condition (4.22) is sufficient for finding a real solution $\mathcal{H}/\mathcal{K}$ to (4.9). Given any such solution, it can be made to satisfy (4.21) by adding a sufficiently large constant to it. □

Chapter 5

Applications

Theorem 4.3.1 was originally developed with the hope that it might lead to integral formulas involving products of K with arbitrary functions of the members of S that are first- or second-order derivatives of n, namely N, R, P and Q. In our defined notation these correspond to functions of z, $\bar{z}$, W and $\overline{W}$ multiplied by the form $dz \wedge d\bar{z}$. This hope has not yet been realized. However, we describe in this chapter two other applications of the theorem that are of interest in their own right. We then set the groundwork for our study of umbilical points based on claim 4.5.2.

5.1 The Weierstrass-Enneper representation

The Weierstrass-Enneper representation (see for example section 2.3 of [22] and chapter 9 of [29]) states that, up to translation, any minimal surface can be represented locally in terms of a holomorphic function F by:

$$\begin{aligned} x_1 &= \operatorname{Re} \int (1-\tau^2) F(\tau)\, d\tau \\ x_2 &= \operatorname{Re} \int i(1+\tau^2) F(\tau)\, d\tau \\ x_3 &= \operatorname{Re} \int 2\tau F(\tau)\, d\tau. \end{aligned} \tag{5.1}$$

We can use theorem 4.3.1 to derive the above equations, which will demonstrate that (4.9) is consistent with and generalizes this established result.

If M is a minimal surface then $H/K = 0$ and the solution to (4.9) is given by:

$$W = (1 + z\bar{z})^2 f(\bar{z})$$
$$\overline{W} = (1 + z\bar{z})^2 g(z) \tag{5.2}$$

for some arbitrary anti-holomorphic function $f(\bar{z})$ or holomorphic function $g(z)$. Equation (4.17) then gives:

$$\frac{\partial X}{\partial z} = \begin{pmatrix} -1 + z^2 \\ -i(1 + z^2) \\ 2z \end{pmatrix} g(z).$$

If we replace the variable z by the variable $\tau = -z$ and let $F(\tau) = 2g(-\tau)$ then we obtain the Weierstrass-Enneper representation from corollary 4.4.4 and the fact that X is real.

The solution for W given by (5.2) remains the same if H/K is equal to any constant λ. It is therefore possible to generalize the Weierstrass-Enneper representation slightly to obtain a representation for any surface having constant mean radius. If we integrate the H/K terms in (4.17) and use the same reasoning as in the proof of corollary 4.4.4 we obtain the representation:

$$\begin{aligned} x_1 &= \mathrm{Re}\left(\frac{2\lambda\tau}{1 + \tau\bar{\tau}} + \int (1 - \tau^2) F(\tau)\, d\tau\right) \\ x_2 &= \mathrm{Re}\left(\frac{2i\lambda\tau}{1 + \tau\bar{\tau}} + \int i(1 + \tau^2) F(\tau)\, d\tau\right) \\ x_3 &= \mathrm{Re}\left(\frac{2\lambda}{1 + \tau\bar{\tau}} + \int 2\tau F(\tau)\, d\tau\right). \end{aligned}$$

5.2 Christoffel's problem and other curvature problems in $\mathbf{R}^3$

One application for which (4.9) is tailor made is Christoffel's problem. In its classical formulation in $\mathbf{R}^3$, the problem posits a function ϕ on the unit sphere, and asks us to determine whether a surface enclosing a convex region exists such that the mean radius on the surface is $\phi(N)$ when the outward unit normal to the surface is N. In our notation this would be written as:

$$\frac{H}{K} = \phi(n_1, n_2, n_3) = f(z, \bar{z}).$$

It was recognized early that this problem is equivalent to a Poisson problem, and William J. Firey proved the existence of a global solution to the problem in general Euclidean space in [4] and [5]. However, (4.9) reduces this problem to a first-order linear complex ODE in W for which a local solution can be constructed explicitly, with an arbitrary anti-holomorphic function in the solution allowing a degree of freedom. By claim 4.6.1, a surface that solves the problem locally can then be obtained by integrating (4.18).

A more general application of (4.9) may be to curvature problems in $\mathbf{R}^3$ of the type described in [9]. Here we are asked to find a surface such that a given symmetric function of its principal curvatures is equal to a given function of the surface's normal vector, that is:

$$f(\kappa_1, \kappa_2) = g(n_1, n_2, n_3).$$

Since we have:

$$\kappa_1 = \frac{1}{\frac{H}{K} + \frac{|W|}{2}}, \quad \kappa_2 = \frac{1}{\frac{H}{K} - \frac{|W|}{2}}$$

the problem is equivalent to:

$$f\left(\frac{H}{K}, |W|\right) = g(z, \bar{z})$$

in our notation. Particular cases include the Minkowski problem, where K is specified as a function of the normal:

$$\left(\frac{H}{K}\right)^2 - \frac{|W|^2}{4} = g(z, \bar{z})$$

and the constant mean curvature equation:

$$\left(\frac{H}{K}\right)^2 - \frac{|W|^2}{4} = \lambda \frac{H}{K}.$$

If we solve for H/K in a given problem and substitute into (4.9), we obtain a non-linear complex first-order differential equation in W. Furthermore, if we use (4.14), then the resulting differential equation will have derivatives taken with respect to z alone.

5.3 Series expansions of W

When (4.9) is applied to curvature problems other than the Christoffel problem, the $e^{2i\theta}$ term in W is extraneous and must be removed by taking the modulus. However, if the object of study is the principal directions themselves then the $e^{2i\theta}$ term is welcome. In this section we characterize the possible power series expansions for W on a real analytic surface, with the eventual goal being to study the behavior of the principal directions around an isolated umbilical point.

We start by considering the Bonnet function of section 4.5 for a real analytic surface M with $K \neq 0$. Suppose that this function has the power series expansion:

$$(1 + z\bar{z})\langle X, N\rangle = \sum_{k \geq k_0} U_k(z, \bar{z}) \tag{5.3}$$

about the point $z = \bar{z} = 0$, where each U_k is a homogeneous polynomial of degree k in z and $\bar{z}$. Suppose further that, for a particular value of k:

$$U_k(z, \bar{z}) = a_k\bar{z}^k + a_{k-1}\bar{z}^{k-1}z + \cdots + a_0 z^k. \tag{5.4}$$

Given this expansion of U_k, let S_k be the complex polynomial in the complex variable w given by:

$$S_k(w) = a_k w^k + a_{k-1}w^{k-1} + \cdots + a_0 \tag{5.5}$$

so that:

$$\begin{aligned} U_k(z, \bar{z}) &= z^k\, S_k\left(\frac{\bar{z}}{z}\right) \\ (1 + z\bar{z})\langle X, N\rangle &= \sum_{k \geq k_0} z^k\, S_k\left(\frac{\bar{z}}{z}\right). \end{aligned} \tag{5.6}$$

Since the Bonnet function appearing on the left side of (5.3) is real, all of the U terms on the right side must also be real, which implies:

$$\begin{aligned} a_k\bar{z}^k + a_{k-1}\bar{z}^{k-1}z + \cdots + a_1\bar{z}z^{k-1} + a_0z^k &= \overline{a_k\bar{z}^k + a_{k-1}\bar{z}^{k-1}z + \cdots + a_1\bar{z}z^{k-1} + a_0z^k} \\ &= \bar{a_0}\bar{z}^k + \bar{a_1}\bar{z}^{k-1}z + \cdots + \bar{a_{k-1}}\bar{z}z^{k-1} + \bar{a_k}z^k. \end{aligned}$$

Thus the coefficients of S_k satisfy the condition :

$$a_j = \overline{a_{k-j}} \tag{5.7}$$

for $0 \le j \le k$.

Polynomials that satisfy (5.7) have been defined and studied in the literature (see for example section 6.8 of [16]). If:

$$p(z) = c_m z^m + c_{m-1}z^{m-1} + \cdots + c_0$$

is a polynomial in the complex variable z, then the reciprocal polynomial of p is defined by:

$$\begin{aligned} p^*(z) &= \overline{c_0}z^m + \overline{c_1}z^{m-1} + \cdots + \overline{c_m} \\ &= z^m\,\overline{p(1/\bar{z})}. \end{aligned} \tag{5.8}$$

If p is equal to its own reciprocal polynomial p^*, then p is said to be self-reciprocal or self-inversive. The key property of such polynomials p is that their roots are symmetric about the unit circle: if z is a root of p then so is $1/\bar{z}$. Chapter 7 of [25] provides an overview of results about self-inversive polynomials.

The condition (5.7) is equivalent to stating that each S_k term in the series expansion (5.6) is a self-inversive polynomial. If we differentiate twice with respect to $\bar{z}$, invoke equation (4.19) and shift the index, we get:

Claim 5.3.1. About a point on an analytic surface at which $z = \bar{z} = 0$ there exists a series expansion:

$$\frac{W(z,\bar{z})}{1+z\bar{z}} = \sum_{k \ge k_0} z^k S_k''\left(\frac{\bar{z}}{z}\right) \tag{5.9}$$

where each of the functions S_k is a self-inversive polynomial of degree $k+2$.

The necessary condition given in the above claim for a series expansion to correspond to a function W of a surface is also sufficient: Any series of the form in (5.9) is the second derivative with respect to $\bar{z}$ of a real function. Thus, two applications of lemma 4.4.5 show that the second derivative of such a series with respect to z is also a real function. However if a function $\mathcal{W}$ satisfies:

$$\left(\frac{\mathcal{W}}{1+z\bar{z}}\right)_{zz} \in \mathbf{R}$$

then by corollary 4.6.3 there exists a surface with $W = \mathcal{W}$.

In the next chapter we prove a new result about self-inversive polynomials that will be necessary for our intended study of umbilical points.

Chapter 6

A result about self-inversive polynomials

Motivated by the series expansion (5.9), in this chapter we prove a new theorem about the location of the roots of the second derivative of a self-inversive polynomial. This theorem extends the final theorem in [21], which deals with the location of the roots of the first derivative of a self-inversive polynomial. This chapter focuses solely on complex polynomials, not on surfaces, and hence for convenience we have used some notation that conflicts with what appears in other chapters. All variables defined are for this chapter only.

6.1 Statement of the result and outline of the proof

We use the letter S to denote a self-inversive polynomial of degree $n \geq 3$ (note that the definition of a self-inversive polynomial depends on the degree specified). We allow for some of the leading terms of S to be 0. If m of the leading coefficients of S are zero, we still treat S as an n-th degree polynomial, and we adopt the convention that S has m roots at ∞ along with $n - m$ regular roots. This convention is used often in the study of polynomials, and particularly for self-inversive polynomials (see for example sections 6.5

and 6.8 of [16]).

The polynomial S' cannot have any zeros on the unit circle that are not also zeros of S. However, it is possible for S'' to have zeros on the unit circle that are not zeros of S. We define such zeros to be *spurious*. With these definitions in hand we can now state the theorem:

Theorem 6.1.1. If S is a self-inversive polynomial that has s zeros in the closed unit disk, then S'' has at least $s-2$ non-spurious zeros in the closed unit disk.

To prove this theorem we will define:

$$F(z) = \frac{z^2 S''(z)}{2S(z)}$$

and we will choose Γ to be a closed contour encircling a region Ω that contains the origin. According to the argument principle, if $F(z)$ is defined and non-zero on Γ, then the change in argument of $F(z)$ as z winds once counterclockwise over Γ is equal to 2π times the number of zeros of S'' in Ω, minus the number of zeros of S in Ω, plus 2. Therefore our goal is to show that the change in $\arg F(z)$ as z winds counterclockwise once around the unit circle C is zero or greater. However, $F(z)$ might have isolated zeros and poles on C, so to avoid these, we initially choose Γ to be C, but then modify its path so that Γ goes around the zeros and poles of F. Specifically, we will have Γ take detours around the singularities on C by traveling along small circular arcs that are centered at these singularities. We choose the radii of these arcs to be small enough so that the arcs do not surround any other singularities of F on C, and do not enclose any additional singularities that do not lie on C.

There are three types of singularities on C that must be circumvented:

(i) Simple zeros of S on C,

(ii) Zeros of S of order 2 or higher on C, and

(iii) Spurious zeros of S''.

For singularities of types (i) and (ii) we will have Γ travel around them on arcs lying outside of C, while for singularities of type (iii) we will have Γ travel around them on arcs lying inside of C. Doing so will exclude zeros of type (iii) from our root count of S'' since Γ will not surround these.

The theorem will follow from this claim:

Claim 6.1.2. If all of the singularity arcs are chosen sufficiently small, it is impossible for $F(z)$ to travel through the negative real axis $\mathbf{R}^-$ in a clockwise direction as z travels counterclockwise along Γ.

The above claim implies the theorem because the total change in $\arg F(z)$ along Γ must be an integral multiple of 2π. However, if $F(z)$ cannot cross $\mathbf{R}^-$ while moving clockwise, it will be impossible for it to complete a full clockwise loop. Therefore negative total values for the change in $\arg F$ over Γ are precluded. We will prove the above claim in the basic case when Γ travels along C, and then when Γ travels around each of the three listed types of singularities.

6.2 Paths along the unit circle

In this section we prove claim 6.1.2 when Γ coincides with C. Let $z_1, \ldots, z_n$ denote the roots of S (possibly including roots at ∞). We define the rational functions:

$$p_j(z) = \frac{z}{z - z_j}.$$

The following lemma recalls some basic facts about complex mappings relative to the unit circle:

Lemma 6.2.1. (i) If $|z_j| = 1$, then the map p_j takes the unit circle with z_j deleted onto the line $\mathrm{Re}\,(z) = 1/2$.

(ii) If $|z_j| \neq 1$ and $z_k = 1/\overline{z_j}$, then the two maps p_j and p_k take a point on C to points that are horizontally symmetric about the line $\mathrm{Re}\,(z) = 1/2$.

(iii) If $|z_j| = 1$ and γ is a circular arc centered at z_j lying outside C, then the map p_j takes γ to a circular arc centered at 1 that lies to the right of the line $\mathrm{Re}\,(z) = 1/2$.

For real angles θ, let:

$$p_j(e^{i\theta}) = \frac{1}{2} + r_j(\theta) + it_j(\theta)$$

where r_j and t_j are real. Since S is self-inversive, any root z_j satisfies $|z_j| = 1$, or else there exists a root z_k with $z_k = 1/\overline{z_j}$. If $|z_j| = 1$ then statement (i) in the above lemma implies that $r_j(\theta) = 0$, while if z_j and z_k are roots satisfying $z_k = 1/\overline{z_j}$ then statement (ii) implies that $r_j(\theta) + r_k(\theta) = 0$. In terms of the functions p_j we have:

$$\begin{aligned} F(e^{i\theta}) &= \sum_{1\le j<k\le n} p_j(e^{i\theta})p_k(e^{i\theta}) \\ &= \sum_{1\le j<k\le n} \left(\frac{1}{2} + r_j(\theta) + it_j(\theta)\right)\left(\frac{1}{2} + r_k(\theta) + it_k(\theta)\right). \end{aligned}$$

For pairs of roots z_j, z_k not lying on C that satisfy $z_k = 1/\overline{z_j}$, the terms r_j and $r_k = -r_j$ will cancel out when p_j and p_k are multiplied by the term p_l $(l \neq j, k)$ and summed. Thus, the only non-zero, non-canceling r terms that appear in the above expression for F arise from the terms $p_j p_k$, and these are equal to $r_j r_k = -(r_j^2 + r_k^2)/2$. Consequently:

$$F(e^{i\theta}) = \frac{n(n-1)}{8} - \sum_{1\le j<k\le n} t_j(\theta)t_k(\theta) - \frac{1}{2}\sum_{j=1}^{n} r_j(\theta)^2 + i\frac{n-1}{2}\sum_{j=1}^{n} t_j(\theta)$$

We next calculate the derivative of $t_j(\theta)$. For $z = e^{i\theta}$ we have:

$$\begin{aligned} \frac{dp_j(z)}{d\theta} &= iz\frac{dp_j(z)}{dz} \\ &= -\frac{izz_j}{(z-z_j)^2} \\ &= -ip_j(z)(p_j(z)-1) \\ &= 2r_j(\theta)t_j(\theta) + i\left(\frac{1}{4} + t_j(\theta)^2 - r_j(\theta)^2\right) \end{aligned}$$

and therefore:

$$\frac{dt_j(\theta)}{d\theta} = \frac{1}{4} + t_j(\theta)^2 - r_j(\theta)^2. \tag{6.1}$$

Note that for 0-∞ pairs of roots of S, we have $r(\theta) = \pm 1/2$ and $t(\theta) = 0$ so that the above equation remains valid.

We are now ready to prove our claim. Suppose that $F(e^{i\theta_0})$ lies on $\mathbf{R}^-$ for some θ_0. We then have:

$$\operatorname{Im} F(e^{i\theta_0}) = \frac{n-1}{2} \sum_{j=1}^{n} t_j(\theta_0) = 0 \tag{6.2}$$

$$\operatorname{Re} F(e^{i\theta_0}) = \frac{n(n-1)}{8} - \sum_{1 \le j < k \le n} t_j(\theta_0) t_k(\theta_0) - \frac{1}{2} \sum_{j=1}^{n} r_j(\theta_0)^2 < 0 \tag{6.3}$$

Equation (6.2) implies:

$$\sum_{j=1}^{n} t_j(\theta_0)^2 + 2 \sum_{1 \le j < k \le n} t_j(\theta_0) t_k(\theta_0) = 0. \tag{6.4}$$

In order to determine whether $F(e^{i\theta_0})$ turns clockwise or counterclockwise at $\mathbf{R}^-$, we need to examine the derivative:

$$\frac{d \operatorname{Im} F(e^{i\theta})}{d\theta} = \frac{n-1}{2} \sum_{j=1}^{n} \frac{dt_j(\theta)}{d\theta},$$

which will be positive if F turns clockwise and negative if F turns counterclockwise. But we have:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{dt_j}{d\theta}(\theta_0) &= \frac{n}{4} + \sum_{j=1}^{n} t_j(\theta_0)^2 - \sum_{j=1}^{n} r_j(\theta_0)^2 \qquad \text{(by 6.1)} \\
&< \frac{n}{4} + \sum_{j=1}^{n} t_j(\theta_0)^2 - \frac{n(n-1)}{4} + 2 \sum_{1 \le j < k \le n} t_j(\theta_0) t_k(\theta_0) \qquad \text{(by 6.3)} \\
&= \frac{n}{4} - \frac{n(n-1)}{4} \qquad \text{(by 6.4)} \\
&< 0,
\end{aligned}$$

which proves that F must turn counterclockwise.

6.3 Paths around singularities

In this section we prove claim 6.1.2 when Γ follows a small circular arc γ around a singularity of F on C.

(i) Simple zeros of S

Let z_j be a simple zero of S on C. We choose γ so that it travels in a small arc outside of C around z_j. Along such an arc the function F will obviously turn clockwise, but we show that it will not cross $\mathbf{R}^-$. Isolating the zero z_j we may write:

$$\begin{aligned} F(z) &= \frac{z}{z-z_j}\left[\sum_{k\neq j}\frac{z}{z-z_k}\right] + \sum_{k,l\neq j}\frac{z^2}{(z-z_k)(z-z_l)} \\ &= \frac{z}{z-z_j}g(z) + h(z) \end{aligned}$$

By lemma 6.2.1(iii) we have:

$$-\frac{\pi}{2} < \arg\frac{z}{z-z_j} < \frac{\pi}{2}$$

for all z along γ, regardless of how small the radius of γ is. Since z_j is a simple root of S, both $g(z)$ and $h(z)$ remain bounded in a neighborhood of z_j. Since the roots of S excluding z_j lie on or are symmetric about C, lemma 6.2.1(i) and (ii) imply that $\operatorname{Re} g(z_j) = (n-1)/2$, and there exists a positive δ such that:

$$\begin{gathered} -\frac{\pi}{2} + \delta < \arg g(z) < \frac{\pi}{2} - \delta \\ -\pi + \delta < \arg\frac{zg(z)}{z-z_j} < \pi - \delta \end{gathered}$$

for all z along γ.

The magnitude of $zg(z)/(z-z_j)$ along γ can be made arbitrarily large by choosing the radius of γ small enough. Since the distance from a point z in the sector:

$$\{-\pi + \delta < \arg(z) < \pi - \delta\}$$

to $\mathbf{R}^-$ becomes arbitrarily large as $|z| \to \infty$, there exists a positive radius for the curve γ below which we will have:

$$-\pi < \arg\left(\frac{zg(z)}{z-z_j} + h(z)\right) = \arg F(z) < \pi$$

for all z along γ. Along such curves $F(z)$ will not pass through $\mathbf{R}^-$.

(ii) Multiple zeros of S

Let z_j be a zero of S on C of multiplicity $m > 1$. As in (i), we choose γ so that it travels in a small arc outside of C around z_j. We may write:

$$\begin{aligned} F(z) &= \frac{m(m-1)}{2} \frac{z^2}{(z-z_j)^2} + \frac{z}{z-z_j}\left[\sum_{z_k \neq z_j} \frac{z}{z-z_k}\right] + \sum_{z_k,\, z_l \neq z_j} \frac{z^2}{(z-z_k)(z-z_l)} \\ &= \frac{m(m-1)}{2} \frac{z^2}{(z-z_j)^2} + \frac{z}{z-z_j} g(z) + h(z) \end{aligned}$$

where, as before, $g(z)$ and $h(z)$ are bounded in a neighborhood of z_j. Let:

$$c = \frac{2g(z_j)}{m(m-1)} = a + ib$$

where a and b are real. Lemma 6.2.1 implies that $a > 0$, and that the image of $z/(z-z_j)$ along γ lies in the half-plane $\operatorname{Re}(z) \geq 1/2$. The conformal map $z \mapsto z + c/2$ takes this half-plane onto the half-plane $\operatorname{Re}(z) \geq (a+1)/2$ that lies to the right of the y-axis, so that the conformal map $z \mapsto z^2 + cz = (z + c/2)^2 - c^2/4$ takes the half-plane $\operatorname{Re}(z) \geq 1/2$ into a region G bounded on the left by a parabola:

$$G = \left\{ x + iy \;\middle|\; x \geq \frac{b^2 + 2a + 1}{4} - \left(\frac{2y + ab}{2(a+1)}\right)^2 \right\}.$$

The magnitude of:

$$\frac{m(m-1)}{2} \frac{z^2}{(z-z_j)^2} + \frac{z}{z-z_j} g(z)$$

along γ can be made arbitrarily large by choosing the radius of γ small enough. Since the distance from a point $z \in G$ to $\mathbf{R}^-$ becomes arbitrarily large as $|z| \to \infty$, there exists a positive radius for γ below which we will have:

$$-\pi < \arg\left(\frac{m(m-1)}{2} \frac{z^2}{(z-z_j)^2} + \frac{z}{z-z_j} g(z) + h(z)\right) = \arg F(z) < \pi$$

for all z along γ, so that $F(z)$ will not pass through $\mathbf{R}^-$.

(iii) Spurious zeros of S''

Let z_0 be a spurious zero of S'' on C. Within a small neighborhood of z_0 we have:

$$F(z) = \frac{z_0^2 S'''(z_0)(z - z_0)}{2S(z_0)} + O\left((z - z_0)^2\right).$$

Moving clockwise along a small circular arc γ of radius ϵ centered at z_0 lying inside C, the term $z - z_0$ takes values in the set:

$$\left\{ \epsilon z_0 e^{i\psi} \,\middle|\, \frac{\pi}{2} < \psi < \frac{3\pi}{2} \right\}$$

so that the major term of F takes values in the set:

$$B = \left\{ \frac{\epsilon z_0^3 S'''(z_0)}{2S(z_0)} e^{i\psi} \,\middle|\, \frac{\pi}{2} < \psi < \frac{3\pi}{2} \right\}$$

while the remainder term is $O(\epsilon^2)$. The set B is formed by rotating the open unit semicircle having negative real part by:

$$\arg \left[\frac{z_0^3 S'''(z_0)}{S(z_0)} \right] \tag{6.5}$$

and then applying a dilation. We will show that real part of the fraction appearing above is strictly negative, so that there exists a positive δ with:

$$\frac{\pi}{2} + \delta < \arg \left[\frac{z_0^3 S'''(z_0)}{S(z_0)} \right] < \frac{3\pi}{2} - \delta.$$

It will then follow that the argument of any member of B lies between $\pi + \delta$ and $3\pi - \delta$, the distance of any point in B to $\mathbf{R}^-$ is $o(\epsilon)$, and F does not cross $\mathbf{R}^-$ for ϵ chosen sufficiently small.

To analyze the term in (6.5), we start with the relation:

$$S(z) = z^n \overline{S\left(\frac{1}{\bar{z}}\right)}.$$

If $z = e^{i\theta}$ for some real angle θ, then $z = 1/\bar{z}$ and we have:

$$S(e^{i\theta}) = e^{in\theta} \overline{S(e^{i\theta})}$$

$$e^{-in\theta/2} S(e^{i\theta}) = f(\theta) \in \mathbf{R}. \tag{6.6}$$

Differentiating (6.6) with respect to θ gives:

$$-\frac{in}{2}e^{-in\theta/2}S(e^{i\theta}) + ie^{-i(n-2)\theta/2}S'(e^{i\theta}) = f'(\theta) \in \mathbf{R}. \tag{6.7}$$

Dividing this by (6.6) gives:

$$\begin{aligned}
-\frac{in}{2} + \frac{ie^{i\theta}S'(e^{i\theta})}{S(e^{i\theta})} &= \frac{f'(\theta)}{f(\theta)} \in \mathbf{R} \\
-\frac{n}{2} + \frac{zS'(z)}{S(z)} &= -i\frac{f'(\theta)}{f(\theta)} \\
\mathrm{Re}\left(\frac{zS'(z)}{S(z)}\right) &= \frac{n}{2}. \qquad (6.8)
\end{aligned}$$

Differentiating (6.6) twice more with respect to θ gives:

$$-\frac{n^2}{4}e^{-in\theta/2}S(e^{i\theta}) + (n-1)e^{-i(n-2)\theta/2}S'(e^{i\theta}) - e^{-i(n-4)\theta/2}S''(e^{i\theta}) = f''(\theta) \in \mathbf{R} \tag{6.9}$$

$$\begin{aligned}
&\frac{in^3}{8}e^{-in\theta/2}S(e^{i\theta}) - \frac{i(3n^2-6n+4)}{4}e^{-i(n-2)\theta/2}S'(e^{i\theta}) \\
&\qquad + \frac{i(3n-6)}{2}e^{-i(n-4)\theta/2}S''(e^{i\theta}) - ie^{-i(n-6)\theta/2}S'''(e^{i\theta}) = f'''(\theta) \in \mathbf{R}. \qquad (6.10)
\end{aligned}$$

Dividing (6.10) by (6.6) gives:

$$\frac{n^3}{8} - \frac{3n^2-6n+4}{4}\mathrm{Re}\left(\frac{zS'(z)}{S(z)}\right) + \frac{3n-6}{2}\mathrm{Re}\left(\frac{z^2S''(z)}{S(z)}\right) - \mathrm{Re}\left(\frac{z^3S'''(z)}{S(z)}\right) = 0.$$

From (6.8) and the fact that $S''(z_0) = 0$ we finally get:

$$\mathrm{Re}\left(\frac{z_0^3S'''(z_0)}{S(z_0)}\right) = \frac{-n^3+3n^2-2n}{4} < 0.$$

Note that the above inequality implies that a spurious root of S'' can have order at most 1.

Chapter 7

The Loewner index conjecture

In this chapter we apply the results of section 5.3 and chapter 6 to the Loewner index conjecture for analytic surfaces. While we do not provide a new proof of the conjecture, we are able to reformulate the conjecture in terms of the location of the roots of a class of polynomials that is related to the self-inversive polynomials.

7.1 Background

The Loewner index conjecture is a statement about the index of the distribution of principal directions about an umbilical point. Defined informally, the index of a vector field about a point at which it has a singularity is the number of times that the vector field rotates around the point, signed according to whether the total rotation is clockwise or counterclockwise. Correspondingly, the index of an isolated umbilical point B is defined as the index of one of the principal direction vectors (for example P) about the point. However, the principal direction vector P may not form a proper vector field at B, since it could happen that in winding around B, the vector P ends up as the negative of what it started. Such behavior is fully consistent with the definition of P, since if P is a principal direction then so is $-P$. If this happens, it will not be possible to associate P with a continuous vector field around B, but it is still possible to define an index for P

at B. This index will have fractional part 1/2 at critical points around which it is not possible to construct a continuous vector field. The extended definition of the index still satisfies the Poincaré-Hopf theorem in that the sum of the indices over all critical points on a closed surface is equal to the surface's Euler characteristic. A detailed definition and proof is contained in [28], chapter 4, addendum 2.

Using this definition of the index, the Loewner index conjecture states that the index of the principal directions around any isolated umbilical point on a surface is at most 1. This conjecture implies the Carathéodory conjecture, which states that any smooth, convex, and closed surface in $\mathbf{R}^3$ contains at least two umbilical points. The Loewner conjecture implies the Carathédory conjecture via the following reasoning: A smooth, closed, and convex surface cannot have zero umbilical points, because if it did, the fields of principal directions would form continuous vector fields on the surface. This is impossible according to the hairy ball theorem, and so the surface must have at least one umbilical point. However, if the surface has only one umbilical point, then the index of this umbilical point must equal the surface's Euler characteristic, which is 2. So if the index of any umbilical point is at most 1, the surface necessarily has at least 2 umbilical points.

The Loewner index conjecture is known to be true for real analytic surfaces. H.L. Hamburger attempted the first proof of the conjecture for this special case beginning in the 1920's, with the papers [14] and [15] being published in 1940 and 1941. G. Bol published another proof [2] shortly afterwards, and T. Klotz published a shorter proof [19] in 1959. The next proof appeared in 1973 in [30], where C. Titus proved a generalized conjecture using operator-theoretic methods. Most recently, V. Ivanov published a new self-contained proof [18] in 2002 based partially on the previous work of Bol and Klotz. The intent of the new paper is to address perceptions that the problem "remains unsettled, at least psychologically" despite the numerous papers that have come before. All of the existing proofs of the conjecture for analytic surfaces are lengthy, and all of the proofs except that of C. Titus involve resolution of singularities. Thus, as stated in [13], it would still be worthwhile to find a simpler proof of the conjecture.

Proving the conjecture for smooth non-analytic surfaces is even more difficult. In [12], the authors show that the truth of the Loewner conjecture for analytic surfaces implies the truth of the conjecture for umbilical points on smooth surfaces if the points satisfy a particular inequality called the Lojasiewicz condition. In preprints [11] and [10], the authors claim to prove that the Carathéodory conjecture is true for $C^{3+\alpha}$ surfaces, and that the index of an umbilical point on such a surface is at most 3/2.

7.2 Index of an umbilical point

Our goal in this section is to study the index of an isolated umbilical point B on an analytic surface M at which $K \neq 0$. To do so, we study the index of the complex function W in terms of its series expansion at B. Without loss of generality we can assume that $z = \bar{z} = 0$ at B, because M can be transformed using rigid motions so that the umbilical point satisfies $n_3 = 1$. Since B is umbilical we have $W = 0$ at B, and we may assume that W has a series expansion about B given by:

$$\frac{W(z, \bar{z})}{1 + z\bar{z}} = \sum_{k \geq k_0} z^k T_k \left(\frac{\bar{z}}{z}\right) \tag{7.1}$$

where $k_0 \geq 1$ and each T_k is a polynomial of degree k (T_k corresponds to S''_k in (5.9), but we will not use this fact until the next section). Because $W = Re^{2i\theta}$ and the indexes of the reference vector fields V^1 and V^2 at B are zero, the index of the umbilical point B is equal to one half of the index of W at B (the index of W must be halved due to the presence of the 2 in the exponent of W).

Let γ be any small closed curve on M that turns once around B in a positive direction. Using the formula for the winding number from complex analysis, the index of W at B is simply:

$$\text{Index}_W(B) = \frac{1}{2\pi i} \int_\gamma \frac{dW}{W}. \tag{7.2}$$

Along γ the real vector (ξ_1, ξ_2) turns once about $(0,0)$ and hence z turns once about 0. Thus the Loewner index conjecture is equivalent to the statement that the index of W about 0 is at most 2.

The index of W at B is determined by the series expansion of W at B. Since $1/(1+z\bar{z}) = 1$ at B, multiplying W by this term does not affect its index. We can therefore substitute the series expansion (7.1) into (7.2) to get:

$$\mathrm{Index}_W(0) = \frac{1}{2\pi i} \int_\gamma \frac{d\left[\sum_{k\geq k_0} z^k T_k\left(\frac{\bar{z}}{z}\right)\right]}{\sum_{k\geq k_0} z^k T_k\left(\frac{\bar{z}}{z}\right)}.$$

Let us now fix γ on M so that z goes once around a small circle of radius ϵ along γ. For the above index to be well defined, there must exist a positive ϵ_0 such that the denominator has no zeros on γ for any positive $\epsilon < \epsilon_0$. However, such an ϵ_0 must exist because the umbilical point B is assumed to be isolated: if there does not exist such an ϵ_0 then W and hence R have a zero in all neighborhoods of B, which is a contradiction.

If we let $w = \epsilon/z$, then along γ the variable w turns clockwise once around the unit circle C, and we have $z = \epsilon/w$ and $\bar{z} = \epsilon w$. Thus:

$$\mathrm{Index}_W(0) = -\frac{1}{2\pi i} \int_C \frac{d\left[\sum_{k\geq k_0} \epsilon^k w^{-k} T_k(w^2)\right]}{\sum_{k\geq k_0} \epsilon^k w^{-k} T_k(w^2)}. \tag{7.3}$$

For all $\epsilon < \epsilon_0$ the denominator above is non-zero on C. There must therefore exist an integer k_1 such that the function:

$$\sum_{k=k_0}^{k_1} \epsilon^k w^{-k} T_k(w^2) \tag{7.4}$$

has no zeros on C for all $\epsilon < \epsilon_0$. If we replace the series expansions in (7.3) with this function, the value of the index will remain unchanged for ϵ sufficiently small, and we get:

$$\mathrm{Index}_W(0) = -\frac{1}{2\pi i} \int_C \frac{d\left[\sum_{k=k_0}^{k_1} \epsilon^k w^{-k} T_k(w^2)\right]}{\sum_{k=k_0}^{k_1} \epsilon^k w^{-k} T_k(w^2)}.$$

According to the argument principle, the index is then equal to the number of poles of the function in (7.4) inside the unit disk minus the number of zeros of this function inside the unit disk. If we define the polynomial F_{k_1} by:

$$F_{k_1}(w) = \sum_{k=k_0}^{k_1} \epsilon^{k-k_0} w^{k_1-k} T_k(w^2) \tag{7.5}$$

then we have:

$$\sum_{k=k_0}^{k_1} \epsilon^k w^{-k} T_k(w^2) = \epsilon^{k_0} w^{-k_1} F_{k_1}(w)$$

and the index is equal to k_1 minus the number of roots of $F_{k_1}(w)$ inside the open unit disk (by assumption F_{k_1} does not have any roots on C). The Loewner index conjecture is therefore equivalent to the statement that $F_{k_1}(w)$ has at least $k_1 - 2$ roots inside the unit disk.

The number of zeros of F_{k_1} inside the unit disk depends only on the number of zeros of $T_{k_0}(w^2)$ inside the open unit disk, and the number of zeros of $T_{k_0}(w^2)$ on C that get perturbed into the open unit disk through addition of the terms involving $T_{k_0+1}, \ldots, T_{k_1}$. This can be seen inductively: the number of zeros of F_{k_0} in the closed unit disk is equal to the number of zeros of $T_{k_0}(w^2)$ in the closed unit disk. If $k \geq k_0$, then we have:

$$F_{k+1}(w) = wF_k(w) + \epsilon^{k-k_0+1} T_{k+1}(w^2).$$

Noticing that $F_k(w)$ is a polynomial of degree $2k$, $T_{k+1}(w^2)$ is a polynomial of degree $2k+2$, and that the leading and constant coefficients of F_{k+1} are multiples of ϵ^{k-k_0+1}, we see that the roots F_{k+1} consist of perturbations (of order one power of ϵ higher) of the $2k$ roots of F_k, plus two new additional roots. One of these additional roots, arising from the near-zero constant term of F_{k+1}, is close to zero and hence lies inside the unit disk, while the other root, arising from the near-zero leading term of F_{k+1}, is close to ∞. If this process is carried on all the way up to $k = k_1$, there will be $k_1 - k_0$ roots close to zero added to the initial roots of $T_{k_0}(w^2)$ in the closed unit disk. Subtracting $k_1 - k_0$ from the minimum number of roots $k_1 - 2$ of F_{k_1} required to be in the unit disk, we have shown:

Claim 7.2.1. The Loewner index conjecture is true at B iff at least $k_0 - 2$ roots of the polynomial $T_{k_0}(w^2)$ lie inside the open unit disk, or are perturbed into the open unit disk through the addition of the perturbation terms contained in the polynomial F_{k_1}.

7.3 Partial results

In this section we prove the Loewner index conjecture for some particular cases, and outline two approaches to future work for attempting to obtain a simpler proof in the general case. The analysis presented here has several elements in common with the proof in [30], including the study of polynomials in z and $\bar{z}$ and their perturbations. However, the connection between the conjecture and the theory of self-inversive polynomials appears to be new.

The simplest case occurs when T_{k_0} has no roots on C. If this condition holds, we can take $k_1 = k_0$, and to prove the index conjecture all we have to show is that $T_{k_0}(w^2)$ has at least $k_0 - 2$ roots inside the unit disk, or equivalently that $T_{k_0}(w)$ has at least $\lceil k_0/2 \rceil - 1$ inside the disk. However, we know that $T_{k_0}(w) = S''_{k_0}(w)$ for some self-inversive polynomial S_{k_0} of degree $k_0 + 2$ that has at least $\lceil k_0/2 \rceil + 1$ roots in the closed unit disk. It then follows from theorem 6.1.1 that $T_{k_0}(w)$ has the required minimum number of roots in the open unit disk.

In cases where T_{k_0} does have roots on C, it will be necessary to show that a minimum number of these roots are perturbed into the open unit disk. We make the following conjecture:

Conjecture 7.3.1. If $T_{k_0}(w^2)$ has a root of order j at a point w_0 on C, then at least $\lfloor (j-1)/2 \rfloor = \lceil j/2 \rceil - 1$ of the zeros at w_0 are perturbed into the open unit disk after the addition of the subsequent perturbation terms in F_{k_1}.

Claim 7.3.2. Conjecture 7.3.1 implies the Loewner index conjecture.

Proof. Suppressing subscripts, write $T_{k_0}(w) = T(w)$, and suppose as before that $T(w) = S''(w)$ for some self-inversive polynomial S of degree $k_0 + 2$. By symmetry, the number of roots of order j of $T(w^2)$ on C is even, and each symmetric pair of roots of order j of $T(w^2)$ on C corresponds to a single root of order j of $T(w)$ on C. Recalling the definition of spurious roots in section 6.1, suppose that for each $j \geq 1$, $T(w^2)$ has $2a_j$ roots of order j on C that correspond to a_j non-spurious roots of order j of $T(w)$ on C. Equations (6.7) and (6.9) imply that any simultaneous root of S'' and S on C must also be a root of S', so $S(w)$ has at least $\sum(j+2)a_j$ roots on C (it necessarily follows that $\sum(j+2)a_j \leq k_0 + 2$). Of the remaining $k_0 + 2 - \sum(j+2)a_j$ roots of S, at least half of these must lie in the closed unit disk. The minimum number of roots of S in the closed unit disk is therefore:

$$\sum(j+2)a_j + \left\lceil \frac{k_0 + 2 - \sum(j+2)a_j}{2} \right\rceil.$$

By theorem 6.1.1, the minimum number of non-spurious roots of $T(w)$ in the closed unit disk is:

$$\sum(j+2)a_j + \left\lceil \frac{k_0 + 2 - \sum(j+2)a_j}{2} \right\rceil - 2.$$

Excluding the $\sum ja_j$ non-spurious roots of $T(w)$ on C, this implies that there must be at least:

$$2\sum a_j + \left\lceil \frac{k_0 + 2 - \sum(j+2)a_j}{2} \right\rceil - 2$$

roots of $T(w)$ in the open unit disk, and at least:

$$4\sum a_j + 2\left\lceil \frac{k_0 + 2 - \sum(j+2)a_j}{2} \right\rceil - 4$$

roots of $T(w^2)$ in the open unit disk. If we add in the number of roots of $T(w^2)$ on C that are assumed to be perturbed into the unit disk according to conjecture 7.3.1, this will give us a total of at least:

$$4\sum a_j + 2\left\lceil \frac{k_0 + 2 - \sum(j+2)a_j}{2} \right\rceil - 4 + 2\sum\left(\left\lceil \frac{j}{2} \right\rceil - 1\right)a_j$$

$$
\begin{aligned}
&= 2\sum\left(\left\lceil\frac{j}{2}\right\rceil + 1\right)a_j + 2\left\lceil\frac{k_0 - \sum(j+2)a_j}{2}\right\rceil - 2 \\
&\geq \sum\left(2\left\lceil\frac{j}{2}\right\rceil + 2\right)a_j - \sum(j+2)a_j + k_0 - 2 \\
&\geq k_0 - 2
\end{aligned}
$$

roots in the interior of the disk. Claim 7.2.1 now implies the Loewner conjecture. □

Given that conjecture 7.3.1 is trivial for $j \leq 2$, we have in fact proven the Loewner index conjecture in the case where T_{k_0} has roots on C that are all of order at most 2. Another case for which we have proven the Loewner conjecture is where the first non-zero perturbation polynomial appearing after T_{k_0} does not have zeros at any of the points on C where T_{k_0} does: conjecture 7.3.1 is true in this case because all of the roots of T_{k_0} at a particular point on C will be perturbed in angular directions that are equally spaced.

In order to prove 7.3.1 in the general case, it will be necessary to study the behavior of derivatives of self-inversive polynomials on the unit disk, such as is described by equations (6.6)-(6.10) and their higher derivatives. Since an analytic surface with an umbilical point can be constructed from any sequence of self-inversive polynomials of increasing degree, there can be no relation inferred between any of the T terms of different degrees that compose F_{k_1}.

There is however a polynomial related to F_{k_1} for which we have information about the perturbations of its roots on C. If we keep the definition $T_k = S_k''$ for all k with S_k self-inversive of degree $k+2$, we can define the polynomial:

$$
G_{k_1}(w) = \sum_{k=k_0}^{k_1} \epsilon^{k-k_0} w^{k_1-k} S_k(w^2).
$$

While the polynomials S_k are self-inversive of differing degrees, all of the polynomials that make up G_{k_1} are self-inversive of degree $2k_1+4$. Thus G_{k_1} is itself self-inversive of degree $2k_1+4$, and all of its roots are symmetric about C. However, unlike for F_{k_1}, there is nothing that prevents G_{k_1} from having roots on C. While F_{k_1} cannot be written solely in terms of G_{k_1}, for $m = k_1 - k$ the corresponding components of the two polynomials satisfy the relation:

$$\begin{aligned} w^m T_k(w^2) &= w^m S_k''(w^2) \\ &= \frac{1}{w^2}\left[w^m S_k(w^2)\right]'' - \frac{1}{w^3}(2m+1)\left[w^m S_k(w^2)\right]' + \frac{1}{w^4} m(m+2)\left[w^m S_k(w^2)\right]. \end{aligned}$$

Letting $m_0 = k_1 - k_0$ we thus have the approximation:

$$F_{k_1}(w) = \frac{w^2 G_{k_1}''(w) - (2m_0+1)wG_{k_1}'(w) + m_0(m_0+2)G_{k_1}(w)}{w^4} + O(\epsilon).$$

If all of the roots of G_{k_1} lying on C are fixed (i.e. they do not move along C as ϵ varies) then it is possible to prove a result similar to theorem 6.1.1 for the above function using the same techniques, and thereby prove the Loewner conjecture. The case where G_{k_1} has roots on C that vary with ϵ presents a greater challenge.

Chapter 8

A generalized divergence theorem for Riemannian submanifolds

8.1 The divergence theorem

In this chapter we prove an extension of the divergence theorem for Riemannian submanifolds of any co-dimension that generalizes identity (3.4). If M is a Riemannian manifold with volume form ω, then the divergence theorem as regularly understood (see [29], chapter 7, addendum 1 for example) states that if V is a vector field on M then:

$$\int_{\partial M} \iota_V(\omega) = \int_M L_V(\omega) = \int_M \operatorname{div} V \, \omega.$$

We extend this theorem to the case where M is a Riemannian submanifold and V is any vector field on the surrounding space in which M is embedded. A theorem similar to the one below is proven in [26] for the case where M is a Euclidean submanifold.

Theorem 8.1.1. Let M be an m-dimensional submanifold of an n-dimensional Riemannian manifold N with $m < n$, and let ω be the volume form on M. If ω is extended to an m-form on an N-neighborhood of M and V is a vector field on N then:

$$\iota_V(d\omega) = -m\langle V, H\rangle\omega \tag{8.1}$$

where H is the mean curvature vector on M. Consequently:

$$\int_{\partial M} \iota_V(\omega) = \int_M L_V(\omega) + m\langle V, H\rangle\omega. \tag{8.2}$$

In the divergence theorem for non-embedded manifolds, $d\omega$ is 0 because it is an m-form on the m-dimensional manifold M. If V is restricted to M in (8.1) then its interior derivative is 0 because H belongs to the perpendicular space of M. Therefore theorem 8.1.1 is consistent with the divergence theorem for non-embedded manifolds. Independent of any integral formulas, (8.1) provides a tidy characterization of H without the use of covariant derivatives or one-parameter variations.

We will prove (8.1) by calculating the two sides of the equation using reduction in local co-ordinates.

8.2 Definitions

In this section we define the notation that we will use to prove (8.1). Let $x^1, \ldots, x^n$ be local co-ordinates on N, with G the corresponding metric matrix. Let ∂_i denote the vector $\partial/\partial x^i$, and suppose that:

$$y^i = \sum_{j=1}^{n} a_{ij}\partial_j$$

for $1 \leq i \leq m$ form an orthonormal basis for the tangent space of M. We also assume that the a_{ij}'s are defined differentiably and orthonormally in an N-neighborhood of M. Define the matrix A by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and let:

$$B = AG$$

$$C = A^T A = G^{-1}B^T A$$

so that:

$$AGA^T = AB^T = BA^T = I.$$

We represent tangent vectors on N as $n \times 1$ column matrices, specifically:

$$\sum_{i=1}^{n} f_i \partial_i = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

and subject them to standard matrix operations. If F is a matrix, F_j denotes the j-th column of F, while $F^{\tilde{i}}$ denotes F with the i-th row deleted. If F is a matrix with k rows, define:

$$F_{j_1 \cdots j_k} = \det \begin{pmatrix} f_{1j_1} & f_{1j_2} & \cdots & f_{1j_k} \\ & \vdots & & \\ f_{kj_1} & f_{kj_2} & \cdots & f_{kj_k} \end{pmatrix}.$$

Since:

$$dy^i = \sum_{j=1}^{n} b_{ij} dx^i,$$

we have:

$$\begin{aligned} \omega &= dy^1 \wedge \cdots \wedge dy^m \\ &= \sum_{1 \le j_1 < \cdots < j_m \le n} B_{j_1 \cdots j_m} \, dx^{j_1} \wedge \cdots \wedge dx^{j_m}. \end{aligned} \tag{8.3}$$

Conversely, when restricted to M, we have:

$$dx^{j_1} \wedge \cdots \wedge dx^{j_m} = A_{j_1 \cdots j_m} \omega.$$

A crucial property of these determinants is that:

$$\sum_{1 \le j_1 < \cdots < j_{m-1} \le n} A^{\tilde{p}}_{j_1 \cdots j_{m-1}} B^{\tilde{r}}_{j_1 \cdots j_{m-1}} = \delta_{pr}$$

because the left-hand side is the determinant of $A^{\tilde{p}}(B^{\tilde{q}})^T$. Similarly:

$$\sum_{1 \le j_1 < \cdots < j_{m-2} \le n} A^{\widetilde{pq}}_{j_1 \cdots j_{m-2}} B^{\widetilde{rs}}_{j_1 \cdots j_{m-2}} = \delta_{\{p,q\}\{r,s\}}.$$

8.3 Proof

8.3.1 Right Side: $\langle \mathbf{V}, \mathbf{H} \rangle$

The projection matrix onto the perpendicular space of M is $I - A^T B$, so:

$$
\begin{aligned}
mH &= (I - A^T B) \sum_{i=1}^{m} \nabla_{a_{i1}\partial_1 + \cdots + a_{in}\partial_n} a_{i1}\partial_1 + \cdots + a_{in}\partial_n \\
&= (I - A^T B) \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(a_{ij}\partial_j a_{ik}\partial_k + a_{ij}a_{ik}\nabla_{\partial_j}\partial_k \right) \\
&= (I - A^T B) \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\partial_j(a_{ij}a_{ik})\partial_k - a_{ik}\partial_j a_{ij}\partial_k + \frac{1}{2} a_{ij}a_{ik} G^{-1}(\partial_j G_k + \partial_k G_j - \operatorname{grad} g_{jk}) \right) \\
&= (I - A^T B) \sum_{j=1}^{n} \left(-A^T \partial_j A_j + \sum_{k=1}^{n} \left(\partial_j c_{kj}\partial_k + \frac{1}{2} c_{kj} G^{-1}(\partial_j G_k + \partial_k G_j - \operatorname{grad} g_{jk}) \right) \right) \\
&= (I - A^T B) \sum_{j=1}^{n} \left(\partial_j C_j + \sum_{k=1}^{n} \frac{1}{2} c_{kj} G^{-1}(\partial_j G_k + \partial_k G_j - \operatorname{grad} g_{jk}) \right) \\
&= (I - A^T B) \sum_{j=1}^{n} \\
&\quad \left(-G^{-1}\partial_j G C_j + G^{-1}\partial_j B^T A_j + G^{-1} B^T \partial_j A_j + \sum_{k=1}^{n} \frac{1}{2} c_{kj} G^{-1}(\partial_j G_k + \partial_k G_j - \operatorname{grad} g_{jk}) \right) \\
&= (I - A^T B) G^{-1} \sum_{j=1}^{n} \left(\partial_j B^T A_j + B^T \partial_j A_j + \sum_{k=1}^{n} \frac{1}{2} c_{kj} (-\partial_j G_k + \partial_k G_j - \operatorname{grad} g_{jk}) \right) \\
&= G^{-1}(I - B^T A) \sum_{j=1}^{n} \left(\partial_j B^T A_j + B^T \partial_j A_j - \sum_{k=1}^{n} \frac{1}{2} c_{jk} \operatorname{grad} g_{jk} \right) \\
&= G^{-1}(I - B^T A) \sum_{j=1}^{n} \left(\partial_j B^T A_j - \sum_{k=1}^{n} \frac{1}{2} c_{jk} \operatorname{grad} g_{jk} \right).
\end{aligned}
$$

To deal with the last term, notice that:

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} g_{jk} = \operatorname{Tr}(AGA^T) = m
$$

so that:

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} \partial_r g_{jk} = - \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_r c_{jk} g_{jk}
$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{m}\partial_r(a_{ij}a_{ik})g_{jk} \\
&= -\sum_{i=1}^{m}\sum_{j=1}^{n}\partial_r a_{ij}\sum_{k=1}^{n}a_{ik}g_{kj} - \sum_{i=1}^{m}\sum_{k=1}^{n}\partial_r a_{ik}\sum_{j=1}^{n}a_{ij}g_{jk} \\
&= -\sum_{i=1}^{m}\sum_{j=1}^{n}\partial_r a_{ij}b_{ij} - \sum_{i=1}^{m}\sum_{k=1}^{n}\partial_r a_{ik}b_{ik} \\
&= -2\sum_{i=1}^{m}\sum_{j=1}^{n}\partial_r a_{ij}b_{ij} \\
&= 2\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\partial_r b_{ij}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
mH &= G^{-1}(I - B^T A)\sum_{j=1}^{n}\left(\partial_j B^T A_j - \sum_{i=1}^{m}a_{ij}\,\mathrm{grad}\,b_{ij}\right) \\
\langle V, mH\rangle &= V^T(I - B^T A)\sum_{j=1}^{n}\left(\partial_j B^T A_j - \sum_{i=1}^{m}a_{ij}\,\mathrm{grad}\,b_{ij}\right) \\
&= \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}f_{pqrs}\partial_r b_{pq}V^s
\end{aligned}
$$

where:

$$
f_{pqrs} = \sum_{i=1}^{m} b_{is}(a_{iq}a_{pr} - a_{ir}a_{pq}) + a_{pr}\delta_{qs} - a_{pq}\delta_{rs}. \tag{8.4}
$$

8.3.2 Left Side: $\iota_{\mathbf{V}}(\mathbf{d}\omega)$

We assume for this section that $m \geq 3$. The cases where $m = 1$ or $m = 2$ can be handled individually along lines similar to (but simpler than) the higher-dimensional cases. From (8.3) we have:

$$
d\omega = \sum_{r=1}^{n}\sum_{1\leq j_1<\cdots<j_m\leq n}\partial_r B_{j_1\cdots j_m}\,dx^r\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_m}.
$$

Using the Jacobi rule for the derivative of a determinant:

$$
\begin{aligned}
d\omega = &\sum_{r=1}^{n}\sum_{1\leq j_1<\cdots<j_m\leq n}\sum_{p=1}^{m}\sum_{k=1}^{m} \\
&(-1)^{p+1}\partial_r b_{pj_k}B^{\tilde{p}}_{j_1\cdots\widehat{j_k}\cdots j_m}\,dx^r\wedge dx^{j_k}\wedge dx^{j_1}\wedge\cdots\wedge\widehat{dx^{j_k}}\wedge\cdots\wedge dx^{j_m}
\end{aligned}
$$

$$= \sum_{r=1}^{n}\sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{1\le j_1<\cdots<j_{m-1}\le n}(-1)^{p+1}\partial_r b_{pq}B^{\tilde{p}}_{j_1\cdots j_{m-1}}dx^r\wedge dx^q\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_{m-1}}$$

$$\iota_V(d\omega) = \sum_{r=1}^{n}\sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{1\le j_1<\cdots<j_{m-1}\le n}(-1)^{p+1}\partial_r b_{pq}B^{\tilde{p}}_{j_1\cdots j_{m-1}}\omega$$

$$\left(V^r A_{qj_1\cdots j_{m-1}} - V^q A_{rj_1\cdots j_{m-1}} + \sum_{k=1}^{m-1}(-1)^{k+1}V^{j_k}A_{rqj_1\cdots\widehat{j_k}\cdots j_{m-1}}\right)$$

$$= \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\partial_r b_{pq}V^s\omega$$

$$\Bigg(\sum_{1\le j_1<\cdots<j_{m-1}\le n}\left[\delta_{rs}(-1)^{p+1}A_{qj_1\cdots j_{m-1}}B^{\tilde{p}}_{j_1\cdots j_{m-1}} - \delta_{qs}(-1)^{p+1}A_{rj_1\cdots j_{m-1}}B^{\tilde{p}}_{j_1\cdots j_{m-1}}\right]$$

$$+ \sum_{1\le j_1<\cdots<j_{m-2}\le n}(-1)^{p+1}A_{rqj_1\cdots j_{m-2}}B^{\tilde{p}}_{sj_1\cdots j_{m-2}}\Bigg). \tag{8.5}$$

Now using minor expansions along the first column of the A's gives:

$$\sum_{1\le j_1<\cdots<j_{m-1}\le n}A_{qj_1\cdots j_{m-1}}B^{\tilde{p}}_{j_1\cdots j_{m-1}}$$

$$= \sum_{1\le j_1<\cdots<j_{m-1}\le n}\sum_{i=1}^{m}(-1)^{i+1}a_{iq}A^{\tilde{i}}_{j_1\cdots j_{m-1}}B^{\tilde{p}}_{j_1\cdots j_{m-1}}$$

$$= (-1)^{p+1}a_{pq}. \tag{8.6}$$

Similarly:

$$\sum_{1\le j_1<\cdots<j_{m-1}\le n}A_{rj_1\cdots j_{m-1}}B^{\tilde{p}}_{j_1\cdots j_{m-1}} = (-1)^{p+1}a_{pr}. \tag{8.7}$$

Finally, using minor expansions along the first two columns of the A's and the first column of the B's gives:

$$\sum_{1\le j_1<\cdots<j_{n-2}}A_{rqj_1\cdots j_{m-2}}B^{\tilde{p}}_{sj_1\cdots j_{m-2}}$$

$$
\begin{aligned}
&= \sum_{1 \le j_1 < \cdots < j_{n-2}} \\
&\quad \Bigg(\sum_{i=1}^{p-1} (-1)^{i+p} (a_{ir}a_{pq} - a_{pr}a_{iq}) A^{\widetilde{pi}}_{j_1 \cdots j_{m-2}} + \sum_{i=p+1}^{m} (-1)^{i+p+1} (a_{ir}a_{pq} - a_{pr}a_{iq}) A^{\widetilde{pi}}_{j_1 \cdots j_{m-2}} \\
&\qquad + \sum_{i_1, i_2 \neq p} \pm a_{i_1 r} a_{i_2 q} A^{\widetilde{i_1 i_2}}_{j_1 \cdots j_{n-2}} \Bigg) \\
&\quad \left(\sum_{i=1}^{p-1} (-1)^{i+1} b_{is} B^{\widetilde{pi}}_{j_1 \cdots j_{n-2}} + \sum_{i=p+1}^{m} (-1)^{i} b_{is} B^{\widetilde{pi}}_{j_1 \cdots j_{n-2}} \right) \\
&= (-1)^{p+1} \sum_{\substack{i=1 \\ i \neq p}}^{m} b_{is} (a_{ir}a_{pq} - a_{pr}a_{iq}) \\
&= (-1)^{p+1} \sum_{i=1}^{m} b_{is} (a_{ir}a_{pq} - a_{pr}a_{iq}). \qquad (8.8)
\end{aligned}
$$

Substituting (8.6), (8.7) and (8.8) into (8.5) shows that all of the terms match those in (8.4), with the $(-1)^{p+1}$'s being eliminated through squaring.

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