Light Beams Refractor

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We prove the existence of a two dimensional lens that refracts all prescribed light beams into the origin and has a refractive index more than the index of refraction of air.

Introduction:

In this note, our goal is to find a refractor lens that refracts all prescribed light rays, in different directions, to the origin. Similar problems have been studied recently; see [2], [3] and [4]. The main difference between this problem and the near field refractor problem, which was studied in [4], is that now we are given the initial positions and directions of light beams and we are looking for a surface which refracts all of them to the origin. On the contrary, in the near field refractor problem we had a point which was emitting light with different illumination densities in different directions and our goal was to design a lens to illuminate a surface with a prescribed illumination density.

A similar problem to the near field refractor problem is the far field refractor problem in which the goal is to refract light beams emanating from the origin, with known illumination densities in different directions, to the given directions and illumination densities on the unit sphere. First Wang showed in a similar problem, see [6], [7], (far field reflector problem) that this problem is in essence an optimal transportation problem and he proved the existence and uniqueness of such reflector up to dilation. Then C. E. Guti´errez and Qingbo Huang in [2] obtained similar results in the far field refractor problem. Since the far field problems are optimal transportation problems, a linear programming approach could be applied to find a solution surface, see [6]. The regularity of far field solutions was studied in [1], [2] and [3].

In the near field refractor problem the main equation is a conservation of energy equation which could be written as the equality between the integral of incoming and outgoing light illumination densities, thus we are dealing with a Monge-Ampére partial differential equation. The existence of a weak solution for this problem has been proved recently by [4]. They introduced a geometrical object (oval) which makes the image of a point at the origin in another point in the space. Then by approximating the solution surface by these ovals, they proved the existence of a weak solution. Similar works had been done by [5] on near field reflector problem and by approximating the surface by ellipsoids. They also (in [5]) found precise conditions for regularity of the solution.

The Two-Dimensional Problem:

Assume that we have different light beams emanating from part of a convex body's surface, a curve in two-dimensions, and in the direction perpendicular (pointing outward) to the surface of the body. We show the convex body by **V** and part of its surface where light beams are coming out from by $\sum$. We choose a coordinate system and the origin in a way that for each G = $(g1,g2) \in \sum$, we have $g1>0$ and $g2>0$, see Figure 1. We also assume that our convex body is C^2 and each normal vector of $\sum$, pointing outward, has a negative first component and a positive second component.

The similar problem in two-dimensions is studied by [3], in which light beams were emanating from a flat curve and we wanted to refract them to different directions in the far field. With using Legendre transform, the author found the surface by an explicit formula.

We are assuming that V is convex because we do not want to have any two rays crossing each other.

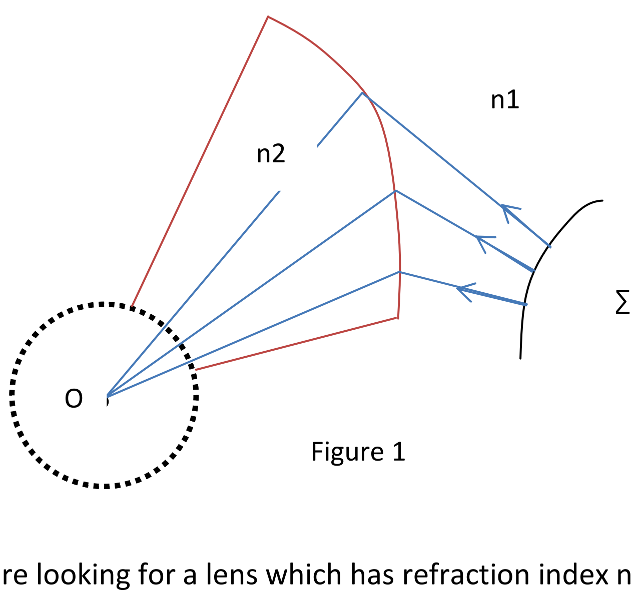

Figure 1

Now, we are looking for a lens which has refraction index n2, which is more than the refraction index of air, n1, where the rays are coming from, and refracts the light beams into the origin. It is sufficient to find the outer curve of the lens and then take an appropriate circle as the inner part of it, and connect the inner curve to the outer curve by radial lines.

First we parameterize the curve ∑ by the arc length, s. Therefore the direction of the light beam which is emanating from G(s) = ($g1(s)$, $g2(s)$), is:

N (s) = (- $g2'(s)$, $g1'(s)$) (1)

In which s ∈ $[0,1]$.

Geometrical Assumptions:

We will make these geometrical assumptions:

(**A1**) Our convex body is in a way that there exists ρ<min(|G|) such that for each line L(s), which is passing through G(s) and is in the direction of N(s) , crosses the circle, which is centered at the origin and has radius ρ , at least once. Therefore the intersection of the line and the circle is 1 or 2 points. We will call the point which has the lower second component, $M_{\rho,s}$. "| |" is the standard norm in $\boldsymbol{R}^2$ and the minimum is taken over all G ∈ ∑.

This could be written as:

There exists ρ$<_{s\in[0.1]} \min(|G(s)|) = G(0)$ such that:

$\{g(1) + t.N(1) | t \in \boldsymbol{R}\} \cap \{x \in \boldsymbol{R}^2$ | |x|=ρ} ≠Φ

In this situation, we would say that ρ has the property *.

(**A2**) min(<$-M_{\rho,s}/|M_{\rho,s}|$, N(s)>) **>** κ

In which the minimum is taken over all ρ and s. And "< , >" is the inner product in $\boldsymbol{R}^2$.

(**A3**) By Assumption A1 for the line L(1), there exists minimum ρ which has property *, We will call it ρ_1. There exists an ε in which the line, which is passing through (1-ε).G(0) and is in the direction of (-$\sqrt{1-\kappa^2}$,κ), crosses the line L(1) at a point which its distance from the origin is more than ρ_1. I will call (1-ε).|G (0)|, ρ_0.

Some Remarks on the Assumptions:

Assumption A1 Guarantees that for given ρ in a suitable range, and for any s in [0, 1], there exists a line which is passing through the origin and crosses the line L(s) at a point which has a distance ρ from the origin. Assumption A3 Guarantees the existence of that range for ρ. We will state and prove it precisely in lemma 2 in the next section.

Lemma 1:

As stated and proved in a lemma in [4], the assumption A2 is sufficient and necessary for having a surface which refracts a light beam in the direction N(s) to the direction, $-M_{\rho,s}/|M_{\rho,s}|$.

Derivation of the ODE in the Two-Dimensional Case:

We show the solution curve which we are looking for by X, we want to find it as a function of s.

We will write: X = ρ.x, in which ρ is a scalar and x is a unit vector.

From the Geometry of the problem, for some h, we have:

G(s) + h.N(s) = ρ.x (2)

And by Snell's law, for some λ, we have:

x + κ.N(s) = λ.v (3)

In which v is the unit normal vector to the solution curve, and κ=n1/n2. We will analyze the case 0<κ<1.

We can write x = (cos(θ), sin(θ)).

From (1) and (2):

N2(s)/N1(s) = - $g1'(s)/\ g2'(s)$= (ρ.sin(θ)- $g2(s)$)/(ρ.cos(θ)- $g1(s)$)

Therefore:

(1+$g1'(s)^2/g2'(s)^2$).ρ^2.cos $(\theta)^2$-2. $g1'(s)/\ g2'(s)$.($g1'(s)/\ g2'(s)$+ $g2(s)$) .ρ.cos(θ)+ +($g2(s)^2 + 2g1(s).g2(s).g1'(s)/g2'(s)$ -ρ^2) = 0 (4)

From (3) we have:

(sin(θ)+κ. $g1'(s)$) / (cos(θ)-κ. $g2'(s)$) = - (ρ.cos(θ))'/(ρ.sin(θ))' (5)

Derivatives are taken with respect to s.

Lemma 2:

For each solution curve which is passing through $\rho(0)$.G(0)/|G(0)| where $\rho_0 \leq \rho(0) < |G(0)|$, for any s, $\rho(s)$ = |X(s)|, has property * .i.e. for any ρ and s there will be a θ corresponds to it.

Proof:

If X(s) is the solution curve which is passing through $\rho(0)$.G(0)/|G(0)|, then by A2 we will see that, the slope of X(s) is at most $-1/\sqrt{1/\kappa^2-1}$. Thus if the line, which is passing through $\rho(0)$.G(0)/|G(0)| and is in the direction of $(-\sqrt{1-\kappa^2}, \kappa)$, crosses the line L(1) at a point, which its distance from the origin is more than ρ_1. Therefore $\rho(1)$ in the worst case has the property *, if $\rho(1)$ has the property *, then it is obvious that $\rho(s)$ for s in [0,1] would have property *.

From (4) and lemma 2, for a solution which is passing through ρ.G(0)/|G(0)| where $\rho_0 \leq \rho < |G(0)|$, θ could be written as a function of s and ρ. Thus:

cos(θ) = f(s,ρ) (6)

By (6) and (5) we can derive:

$$\rho' = \psi.\frac{\partial f}{\partial s}/(\phi - \psi.\frac{\partial f}{\partial \rho}) = H(s,\rho) \qquad (7)$$

In which:

$$\Psi(s,\rho) = \kappa.(g1(s)^2 + g2(s)^2)'/\sqrt{1-f(s,\rho)^2} \qquad (8)$$

$$\Phi(s,\rho) = 1+\kappa/\rho\,.(g2(s).\,g1(s)' + g1(s).\,g1(s)'^2/g2(s)') - \kappa.g2(s)'.f(s,\rho).(1+g1(s)'^2/g2(s)'^2) \quad (9)$$

We derived ρ’ explicitly as a function of s and ρ.

Remarks: From (4), (8) and (9), we can see that f, ψ and Φ are C^1 functions.

Existence and Uniqueness of the ODE in the Two-Dimensional Case:

Lemma 3:

In equation (7), $\left(\phi - \psi.\frac{\partial f}{\partial \rho}\right)$ is never zero.

Proof:

According to the geometry of the problem and equation (3), if the normal vector of the solution curve, v(s), is perpendicular to X(s), then we can see that <-N(s), x(s)> would be –κ. If we are working in the range, that assumption A3 provides for some initial conditions, then by assumption A2 we would get a contradiction. Thus, ρ' never could be larger than a specific value, therefore, $\left(\phi - \psi.\frac{\partial f}{\partial \rho}\right)$ is not going to be zero in our problem and we can state the following theorem about existence and uniqueness of such refractor.

Theorem:

Using the assumptions made in previous sections, there exists a refractor ($\mathbf{C^1}$ surface) lens which has refraction index n2 and refracts all prescribed light beams to the origin. If we want to have our outer surface of refractor to pass through the point ρ. G(0)/|G(0)|, $\rho_0 \leq \rho < |G(0)|$, and if more, our convex body was C^3, then our X(s) would be determined uniquely.

Proof:

According to the previous section and lemma 3, if we are looking for a solution which is passing through ρ.G(0)/|G(0)|, $\rho_0 \leq \ \rho < |G(0)|$, then we have equation (7), with H(s,ρ) as a C function. According to Peano Theorem in ODE, existence of the refractor is proved. If the convex body was C^3, then H(s,ρ) would be C^1 , therefore by Picard theorem in ODE, existence and uniqueness of X(s) is verified.

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