Multidimensional matching: theory and empirics*

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Abstract

We present a general analysis of multidimensional matching problems with transferable utility, paying particular attention to the case in which agents on one side of the market are multi-dimensional and agents on the other side are uni-dimensional. We describe a general approach to solve such problems. We discuss testability and identification of these models, and provide a new stochastic structure that we characterize in both uni- and multi-market contexts. In particular, we characterize situations when the structural matching model can be identified from OLS regressions of the characteristics of married individuals on the spouses'.

*The authors are grateful to Toronto's Fields' Institute for the Mathematical Sciences for its kind hospitality during part of this work. They acknowledge partial support of RJM's research by Natural Sciences and Engineering Research Council of Canada Grant 217006-08, -15 and -20. Pass is pleased to acknowledge support from Natural Sciences and Engineering Research Council of Canada Grants 412779-2012 and 04658-2018 and a University of Alberta start-up grant.

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1 Introduction

Matching problems under transferable utility have attracted considerable attention in recent years within economic theory. The general goal is to understand the stable equilibria of matches between distributions of agents on two sides of a 'market' (for example, husbands and wives in the marriage market, CEOs and firms in the labor market, producers and consumers in a market for differentiated commodities, etc.), as well as the resulting division of surplus between partners. Until recently, most work has focused on the setting in which a single characteristic is used to distinguish between the agents on each side; for example, in the marriage market, several models assume that individuals differ only by their income (or their human capital). These models have the advantage of being analytically tractable, and often allow explicit closed form solutions. Under the classical Spence-Mirrlees condition, the only stable matching is the positive assortative one, the nature of which ('who marries whom') is directly determined by the underlying distributions of the male and female characteristic. However, these one dimensional models are unsatisfactory in many situations, as both casual empiricism and factual evidence indicate that agents often match on several traits. In the marriage market, for instance, the suitability of a potential marriage between a woman and man typically depends on several characteristics of both, including income and education, but also age, tastes, ethnic background, physical attractiveness, etc.

It is therefore important to study and understand *multidimensional* matching problems, in which agents on both sides of the market are differentiated using several characteristics. These models have garnered increasing visibility in recent years, due to their wider applicability and flexibility, but their introduction brings forth serious theoretical challenges. The nature of the equilibrium matching is more interesting but also more complex; in contrast to the one dimensional case, it is no longer determined by the sole knowledge of the distributions of individual characteristics, even under (a generalization of) the Spence-Mirrlees condition. From a more technical perspective, it is generally not possible to derive closed form solutions; and discretising matching problems leads to a linear program, which often become numerically unwieldy when type spaces are multidimensional.

The purpose of this paper is to provide a general characterization of multidimensional matching models, in terms of existence, uniqueness and qualitative properties of stable matches, and to provide a new empirical approach for analyzing these models. Since the work of [Shapley & Shubik (1971)] in the discrete setting, and [Gretsky, Ostroy & Zame (1992)] in the continuum, it has been understood that transferable utility matching is equivalent to a variational problem; this problem is known in the mathematics literature as the *Monge-Kantorovich optimal transport* problem. We put a particular emphasis on the case in which the dimensions of heterogeneity on the two sides of the market are unequal (say, $m > n$). These sorts of problems have received relatively little attention from the mathematics community, but are quite natural economically; the dimension essentially reflects the number of attributes used to distinguish between agents and there is no compelling reason in general to expect this number to coincide for agents on the two different sides of the market (say, for consumers and producers, or for employees and firms). A typical pattern emerges in these situations, since for one side of the market (the one with a lower dimension), identical agents are typically matched with a continuum of different partners. We explore the properties of the 'indifference sets' thus defined, and argue that since such indifference sets can often be empirically recovered, these properties can provide testable consequences of multidimensional matching theory.

Of specific interest are the so-called 'multi-to-one dimensional matching problems', in which agents on one side of the market are assumed to be multidimensional, while those on the other side are unidimensional. They include an economically important class of examples (for instance, in [Chiappori, Oreffice & Quintana-Domeque (2012)] and [Low (2014)]), for which we show one can often obtain explicitly the stable matchings; see [Chiappori, McCann & Pass (2017)]. In this context, we describe our general approach aimed at characterizing the equilibrium matching. We describe a robust methodology that allows, under suitable conditions, to explicitly characterize its solutions. We discuss some interesting features that the indifference sets exhibit, and which are typically absent in purely unidimensional problems. For instance, the optimal mapping may be discontinuous, and so women of similar types may marry men of very different types.

Lastly, we discuss the empirical properties of multidimensional matching models. We first provide a theoretical discussion of the two main issues, namely testability of matching models and identifiability of the underlying structure. Next, we introduce a novel, stochastic structure aimed at capturing the presence of unobservable traits in the matching game. Our structure generalizes existing models (including the seminal contribution by [Choo & Siow (2006)]) by allowing for a continuous distribution of observable traits; in particular, following a formulation that has become standard in empirical IO, we allow for individual-specific valuations of a partner's characteristics. In addition, our approach clarifies the connections between two commonly used approaches in the empirical matching literature,[1] one based on a direct analysis of matching patterns as in [Choo & Siow (2006)] and the other relying on cross-sectional regressions of individual traits on the spouse's characteristics (as pioneered by [Chiappori, Oreffice & Quintana-Domeque (2012)], from now on COQ).

A basic distinction, in all contexts, is between 'single-market' and 'multi-market' frameworks. In the former case, we only observe the matching patterns corresponding to one particular matching game. In the latter, matching patterns are observed in several 'markets'; while the distributions of observable characteristics may freely differ across markets, some aspects of the underlying structural models (namely, the 'supermodular core' of the surplus function and the distribution of unobservable characteristics) are assumed to be identical across markets. We show that, in the single-market context, the simplest version of the model (in which the distribution of unobservable characteristics is known a priori) is exactly identified. In a multi-market case, however, testable restrictions are generated. In a specific, normal-quadratic version of

[1] See for instance the surveys by [Graham (2011)] and [Chiappori & Salanié (2016)].

the empirical model, in the line of [Tinbergen (1956)], one can derive closed-form solutions for the surplus function (up to a normalization); moreover, the model can be identified either directly or by OLS regressions of individual characteristics.

The next section summarizes the main theoretical results, with a particular emphasis on the unequal dimensions case and the (new) notion of 'nestedness'. General results on testability and identification are provided in Section 3, whereas Section 4 is devoted to the general stochastic structure and the specific, normal quadratic application.

2 Multidimensional matching under transferable utility: theory

2.1 Basic properties

2.1.1 The framework

Matching: definition We consider sets $X \subseteq R^m$ and $Y \subseteq R^n$, parametrizing populations of agents on two sides of a market. In what follows, we shall stick to the language of the marriage market interpretation (so that X and Y will denote the set of potential wives and husbands respectively), although alternative interpretations are obviously possible. They are distributed according to probability measures μ on X and ν on Y, respectively. In the transferable utility framework, a potential matching of agents $x \in X$ and $y \in Y$ generates a combined surplus $s(x, y)$, where $s : X \times Y \to \mathbf{R}$. This surplus can be divided in any way between the agents x and y. For simplicity, we assume that s and its derivatives are smooth and bounded unless otherwise remarked; many of the results we describe can also be extended to surpluses with less smoothness, as in [Chiappori, McCann & Nesheim (2010)] and [Noldeke & Samuelson (2015)] for example.

A *matching* is characterized by a probability measure γ on the product $X \times Y$, whose marginals are μ and ν, that is

$$\gamma(A \times Y) = \mu(A) \quad \text{and} \quad \gamma(X \times B) = \nu(B) \tag{1}$$

for all Borel $A \subset X, B \subset Y$. Intuitively, a matching is an assignment of the agents in the sets X and Y into pairs, and $\gamma(x, y)$ is related to the probability that x will be matched to y; in particular, $(x, y) \notin \operatorname{spt} \gamma$ implies[2] that agents x and y are not matched together. The marginal condition is often called the *market clearing* criterion. We denote the set of all matchings by $\Gamma(\mu, \nu)$.[3]

Payoff functions and stability Integrable functions $u : X \to \mathbf{R}$ and $v : Y \to \mathbf{R}$ are called payoff functions corresponding to γ if they satisfy the *budget constraint*:

$$u(x) + v(y) \leq s(x, y) \tag{2}$$

γ almost everywhere — i.e., for any pair of agents who match with positive probability. For such a pair $(x, y) \in \operatorname{spt} \gamma$, the functions $u(x)$ and $v(y)$ are interpreted respectively as the indirect utilities derived from the match by agents x and y; the constraint (2) ensures that the total indirect utility $u(x) + v(y)$ collected by the two agents does not exceed the total surplus $s(x, y)$ available to them.

A matching γ is called *stable* if there exist payoff functions $u(x)$ and $v(y)$ satisfying both (2) and the reverse inequality

$$u(x) + v(y) - s(x, y) \geq 0 \tag{3}$$

for all $(x, y) \in X \times Y$. Condition (3) expresses the stability of the matching in the following sense; if we had $u(x) + v(y) < s(x, y)$ for any (currently unmatched) pair of agents, it would be desirable for each of them to leave their current partners and match together, dividing the excess surplus $s(x, y) - u(x) - v(y) > 0$ in such a way as to increase the payoffs to both x and y. Note that (2) and (3) together ensure

[2] Here $\operatorname{spt} \gamma$ refers to the support of γ, i.e. the smallest closed set containing the full mass of γ.

[3] For simplicity, we shall assume that μ and ν have equal mass and that all agents are matched. When this assumptions are violated it is well-known that they can be restored by augmenting both sides of the market with a fictitious type representing the outside option of remaining unmatched (see for instance [Chiappori, McCann & Nesheim (2010)]).

$u(x) + v(y) = s(x, y)$, γ almost everywhere: if two agents match with positive probability, then they split the surplus generated between them.

Although u and v will not generally be everywhere differentiable, some mild regularity condition guarantees differentiability almost everywhere. Specifically, if the surplus function s is Lipschitz, so are the payoffs u and v — and with the same Lipschitz constant; if $s \in C^2(X \times Y)$, then u and v have second-order Taylor expansions Lebesgue almost-everywhere (see e.g. [Villani (2009)] or [Santambrogio (2015)]). When the probability measures μ and ν come from Lebesgue densities, this almost-everywhere differentiability proves sufficient for many analytic purposes. We use $\mathrm{Dom}\, Du$ (respectively $\mathrm{Dom}\, D^2u$) to denote those $x \in X$ at which u has a first- (respectively second-)order Taylor expansion, and $\mathrm{Dom}_0\, D^i u := \left(\overline{X}\right)^0 \cap \mathrm{Dom}\, D^i u$ where $\overline{X}$ and X^0 denote the closure and interior of X, respectively.

First order conditions for stability Given a stable match γ and associated matching functions u, v, the set

$$S = \{(x, y) \in X \times Y \mid u(x) + v(y) = s(x, y)\}$$

is of particular interest; as $\mathrm{spt}\, \gamma \subset S$, it tells us which agents can match together. If S is concentrated on a graph $\{(x, F(x)) \mid x \in S\}$ of some function $F : X \longrightarrow Y$, the stable matching is called *pure*, the interpretation being that almost all agents of type x must match with agents of the same type $y = F(x)$; in particular, purity excludes the presence of *randomization*, whereby an agent x may be randomly assigned to different partners. In this case, the distribution ν agrees with the image $F_{\#}\mu$ of μ under F, which assigns mass

$$(F_{\#}\mu)(V) := \mu[F^{-1}(V)] \tag{4}$$

to each $V \subset Y$.[4]

Note that this notion of purity in not symmetrical with respect to X and Y. This asymmetry is unavoidable in the unequal dimensional

[4]Also called the *push-forward* $F_{\#}\mu$ of μ through F; see e.g. [Ahmad, Kim & McCann (2011)].

cases $m > n$ of particularly interest; in this case each male type y will typically match with an $m - n$ dimensional continuum of female types x, so randomization is necessary in the male to female direction.

The fact that S is the zero-set of the non-negative function (3) enters crucially. It implies in particular the first-order and second order conditions

$$(Du(x), Dv(y)) = (D_x s(x,y), D_y s(x,y)) \tag{5}$$

and

$$\begin{pmatrix} D^2u(x) & 0 \\ 0 & D^2v(y) \end{pmatrix} \geq \begin{pmatrix} D^2_{xx}s(x,y) & D^2_{xy}s(x,y) \\ D^2_{yx}s(x,y) & D^2_{yy}s(x,y) \end{pmatrix} \tag{6}$$

are satisfied at each $(x,y) \in S \cap (X \times Y)^0$ for which the derivatives in question exist; here X^0 denotes the interior of X, and inequality (6) should be understood to mean the difference of these $(m+n) \times (m+n)$ symmetric matrices is non-negative definite.

The equality

$$Du(x) = D_x s(x,y) \tag{7}$$

has an interesting, economic interpretation for the case where characteristics are not exogenously given but result from some investment made by individuals before the beginning of the game (human capital being an obvious example). Indeed, if both x and y are endogenously chosen by the agent before the matching game starts, then (7) implies that the marginal gross return, for the individual, of an investment in characteristics is exactly equal to its gross social return (defined as the contribution of the investment to aggregate surplus, which is the natural definition in a TU context). In other words, one expects that for some equilibria such investments will be efficient, despite being made non-cooperatively before the matching game; their impact on global welfare is internalized by matching mechanisms, a point made by [Cole, Mailath & Postlewaite (2001)], [Makowski (2004)] and [Iyigun & Walsh (2007)] and generalized by [Noldeke & Samuelson (2015)].

2.1.2 Variational interpretation: optimal transport and duality

The Monge-Kantorovich problem The problem of identifying stable matches turns out to have a variational formulation, known as the optimal transport, or Monge-Kantorovich, problem in the mathematics literature (see for instance [Villani (2009)], [Santambrogio (2015)] and [Galichon (2016b)]). This is the problem of matching the measures μ and ν so as the maximize the total surplus; that is, to find γ among the set $\Gamma(\mu, \nu)$ which maximizes

$$s[\gamma] := \int_{X \times Y} s(x, y) d\gamma(x, y). \tag{MK}$$

The following theorem can be traced back to [Shapley & Shubik (1971)] for finite type spaces X and Y, and to [Gretsky, Ostroy & Zame (1992)] more generally. It asserts an equivalence between (MK) and stable matchings.

Theorem 1 (Stable matching via linear optimization) *A matching measure $\gamma \in \Gamma(\mu, \nu)$ is stable if and only if it maximizes* (MK)*.*

As the maximization of a linear functional over a convex set, problem (MK) has a dual problem, which is useful both in studying it maximizers, and in clarifying its relation with stable matching. The dual problem to (MK) is to minimize

$$\mu[u] + \nu[v] := \int_X u(x) d\mu(x) + \int_Y v(y) d\nu(y). \tag{MK$_*$}$$

among functions $u \in L^1(\mu)$ and $v \in L^1(\nu)$ satisfying the stability condition (3). It is well known that under mild conditions, Kantorovich-Koopmans duality holds (see, for instance, [Villani (2009)]), that is:

$$\max_{\gamma \text{ satisfying } (1)} s[\gamma] = \min_{(u,v) \text{ satisfying } (3)} \big(\mu[u] + \nu[v]\big). \tag{8}$$

Note that for any u and v satisfying the stability constraint (3) and any matching $\gamma \in \Gamma(\mu, \nu)$, the marginal condition implies

$$\mu[u] + \nu[v] = \int_{X\times Y} (u(x) + v(y))\, d\gamma(x, y) \geq \int_{X\times Y} s(x, y) d\gamma(x, y)$$

and we can have equality if and only if $u(x) + v(y) = s(x, y)$ holds γ-almost everywhere. It then follows from the duality theorem that γ is a maximizer in (MK) (and hence a stable match) and u, v are minimizers in the dual problem (MK_*), precisely when $u(x) + v(y) = s(x, y)$ holds γ almost everywhere; in other words, the solutions to (MK_*) coincide with the payoff functions. As a consequence, for any solution to (MK), one can define a stable matching by considering the optimal measure γ and the minimizers of the dual problem as payoff functions.

Existence of the stable matching easily follows from the previous findings. Indeed, one only need to prove existence of a solution to the associated, Monge-Kantorovich problem, which is a linear maximization problem. For instance, if $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ are bounded and s is continuous, then there exists an optimizer γ to (MK), and therefore a stable match.[5]

Moreover, an immediate corollary that has important empirical applications is the following:

Corollary 2 (Additive ambiguities in surplus identification) *Let s and $\bar{s}$ be two surplus functions. Assume there exists two functions f and g, mapping $\mathbf{R}^m$ to $\mathbf{R}$ and $\mathbf{R}^n$ to $\mathbf{R}$ respectively, such that*

$$s\,(x, y) = \bar{s}\,(x, y) + \psi\,(x) + \phi\,(y)$$

For any measures μ and ν, any stable matching for s is a stable matching for $\bar{s}$ and conversely.

Proof. *Any stable measure γ for s solves the surplus maximization problem:*

$$\max_{\gamma \text{ satisfying } (1)} \int_{X\times Y} s(x, y) d\gamma(x, y). \tag{9}$$

[5]The literature on the topic is large in both mathematics and economics. The interested reader is referred to (see [Villani (2009)], [Santambrogio (2015)]) or [Galichon (2016b)] for recent surveys.

which is equivalent to:

$$\max_{\gamma \text{ satisfying } (1)} \int_{X \times Y} \bar{s}(x,y) d\gamma(x,y) + \int_X \phi(x) d\mu(x) + \int_Y \psi(y) d\nu(y)$$

The last two integrals are given (by the marginal conditions on γ), and so any γ that maximizes (9) also solves (10):

$$\max_{\gamma \text{ satisfying } (1)} \int_{X \times Y} \bar{s}(x,y) d\gamma(x,y). \tag{10}$$

implying that the stable measure is associated with a stable matching for $\bar{s}$. ■

An important consequence of this result is that *the observation of matching patterns can only (at best) identify the surplus up to two additive functions of x and y respectively.* We shall see later on that, in general, s cannot be identified even up to two such additive functions. A second implication is that if the surplus function is additively separable in x and y:

$$s(x,y) = \psi(x) + \phi(y)$$

for some mappings ϕ and ψ, then any matching is stable; indeed, any matching is stable for the degenerate surplus $\bar{s}(x,y) = 0$, and by the Corollary the set of stable matchings is the same for s and $\bar{s}$.

Uniqueness, purity, and the twist condition The issues of uniqueness and purity are slightly more complex. Aside from its theoretical interest, uniqueness of the optimal matching γ plays an important computational role, as in its absence more sophisticated techniques must be employed. In practice, solutions are often assumed to be pure in empirical studies. Since this conclusion is not generically satisfied [McCann & Rifford (2016)], it is desirable to know conditions on s, μ and ν which guarantee it. Similarly, uniqueness is not guaranteed in general; for instance, as noted above, if the surplus function is additively separable in x and y then any matching γ is optimal and hence stable. It is therefore clear that certain structural conditions on s are indeed needed to ensure purity and uniqueness.

A sufficient condition for purity of the optimal matching, that in turn implies uniqueness, is a nonlocal generalization of the Spence-Mirrlees condition, known as the *twist* condition:

Definition 3 (Twist) *The function $s \in C^1$ satisfies the* twist condition *provided*

$$D_x s(x,y) \neq D_x s(x,y_0) \tag{11}$$

for all x and distinct $y \neq y_0$.

The twist condition is therefore equivalent to the *injectivity* of $y \mapsto D_x s(x,y)$, for each fixed x. For instance, in a one-dimensional context ($m = 1 = n$), the classical Spence-Mirrlees condition imposes that either $\frac{\partial^2 s}{\partial x \partial y} > 0$ or $\frac{\partial^2 s}{\partial x \partial y} < 0$ over $X \times Y$, which implies that $y \mapsto \frac{\partial s}{\partial x}(x,y)$ is strictly monotone (and hence injective) for each fixed x. It is in this sense that the twist condition can be viewed as a non-local generalization of the Spence-Mirrlees condition.[6]

It should be noted that our definition of twist breaks the symmetry between X and Y. One could call it x-twist, and define y-twist in a similar way. When both hold we say s is bi-twisted. However, if Y has non-empty interior, then x-twist can hold only if $m \geq n$, because it asserts the existence of a continuous injection (11) from an open subset Y^0 of R^n into R^m. Similarly, invariance of domain shows y-twist requires $n \geq m$ unless X^0 is empty, so bi-twist cannot hold unless $m = n$.

A well-known result[7] is that the twist condition is sufficient to guarantee purity; specifically, if μ is absolutely continuous with respect to Lebesgue measure and the surplus s satisfies the twist condition, then any solution γ to (MK) is pure; that is, almost all women with a given type x are matched with the same man y.[8]

[6] The twist condition is quite different from the non-local generalization of the same condition by [McAfee & McMillan (1988)], as discussed in Remark 9 below.

[7] See for instance [Gangbo (1995)] or [Levin (1999)].

[8] The absolute continuity of μ is a technical condition required to ensure the utilities can be differentiated on a set of full μ measure; the payoff functions are guaranteed to be Lipschitz if the surplus is, and are hence differentiable Lebesgue almost everywhere by Rademacher's theorem (but not everywhere in general). The condition on the measure can be weakened somewhat, but some regularity is needed: as a simple counterexample, if $\mu = \delta_{x_0}$ is a Dirac mass but ν is not, then the optimal

Two further remarks can be made at this point. First, in many relevant situations, the twist condition does indeed fail; for example, if we replace X and Y with compact smooth manifolds, it fails for *any* smooth surplus function s. Second, the twist condition is *not* necessary for purity. For instance, [Kitagawa & Warren (2012)] provide a setting in which purity holds in the absence of twist.

A standard result is that purity implies uniqueness:

Corollary 4 (Purity yields uniqueness) *Given μ, ν and $s \in C(\overline{X} \times \overline{Y})$, if all stable matches in $\Gamma(\mu,\nu)$ are pure, then the stable match is unique.*

Proof. *Theorem 1 asserts γ is stable if and only if it maximizes* (MK) *on $\Gamma(\mu,\nu)$. Suppose two maximizers γ_0 and γ_1 exist. Convexity of the problem makes it clear that $\gamma_2 = (\gamma_0 + \gamma_1)/2$ is again* ***maximal. Purity*** *asserts that γ_2 concentrates on the graph of a map $F : X \longrightarrow Y$, and vanishes outside this graph. Non-negativity ensures the same must be true for γ_0 and γ_1. But then $\gamma_0 = (F \times id)_{\#}\mu = \gamma_1$ by Lemma 3.1 of [Ahmad, Kim & McCann (2011)]* ■

The converse to this Corollary is not true; i.e., one can easily find situations where the optimal matching is unique but not pure.[9] Lastly, even when s satisfies the twist condition, and the optimal matching is therefore pure and unique, the mapping $F : X \to Y$ generating the matching is not continuous in general, as a wide range of examples throughout the literature on optimal transport show (see for instance [Ma, Trudinger & Wang (2005)]). In practice, thus, even when the stable matching is pure, one cannot generally expect that two women x and x' whose types are 'close' will marry men with similar characteristics.

2.1.3 Recovering individual utilities

The stability condition allows information on individual utilities at the stable match to be recovered. To see why, note first that stability implies

matching (indeed, the only measure in $\Gamma(\mu,\nu)$) is product measure $\delta_{x_0} \otimes \nu$, which pairs every point y with x_0 and is certainly not pure.

[9] Additional conditions which ensure uniqueness, but not purity, of the optimal matching can be found in [Chiappori, McCann & Nesheim (2010)] and [McCann & Rifford (2016)].

that, for μ almost every x,

$$u(x) = \max_y \left(s(x,y) - v(y) \right). \tag{12}$$

Assume, now, that the matching is pure (say, because the twist condition is satisfied). The envelope theorem then yields, wherever u is differentiable and $y = F(x)$ is matched with x,

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial s}{\partial x_i}(x, F(x)). \tag{13}$$

which gives the partials of u, and therefore defines u up to an additive constant.[10]

2.2 Matching with unequal dimensions

We now turn special attention to the case in which the dimensions $m \geq n$ of heterogeneity on the two sides of the market are unequal. In this case, one expects stable matchings where almost every man y is matched with positive probability to a continuum of potential wives x. Specifically, it is natural to expect that at equilibrium the subset $F^{-1}(y) \subset X \subset \mathbf{R}^m$ of partners which a man of type $y \in \mathrm{Dom}_0 Dv$ is indifferent to will generically have dimension $m - n$, or equivalently, codimension n.

[10]Note, incidentally, that these partial differential equations relating direct to indirect payoffs must be compatible, which generates restrictions on the matching function F; namely, assuming double differentiability:

$$\sum_k \frac{\partial^2 s}{\partial x_i \partial y_k} \frac{\partial F_k}{\partial x_j} = \sum_k \frac{\partial^2 s}{\partial x_j \partial y_k} \frac{\partial F_k}{\partial x_i} \tag{14}$$

$$\geq \frac{\partial^2 s}{\partial x_i \partial x_j}(x, F(x)) \tag{15}$$

where $y_k = F_k(x)$ and the partials of s are taken at $(x, F(x))$ and the inequality is from (6). In particular, in the case of multi to one dimensional matching, then $y = F(x_1, ..., x_m)$, and (14) becomes a system of partial differential equations that F must satisfy (which reduces to a single equation in case $m = 2 = n + 1$); together with the measure restrictions and the matrix inequality (15), this typically identifies the matching function F.

2.2.1 General results

Potential indifference sets. For any equilibrium matching γ and payoffs (u, v), we have already seen that γ-a.e. (x, y) produces equality of marginal direct and indirect payoffs for both husband and wife (5). In particular, all partner types $x \in X$ for husband $y \in \mathrm{Dom}_0 Dv$ lie in the same level set of the map $x \mapsto D_y s(x, y)$ taking her type to his marginal surplus. Knowing his marginal payoff $Dv(y)$ would determine this level set precisely; it depends on μ and ν as well as s. However, in the absence of this knowledge it is useful to define the potential indifference sets, which for given $y \in Y$ are merely the level sets of the map $x \in X \mapsto D_y s(x, y)$ taking her type to his marginal surplus. We can parameterize these level sets by (cotangent) vectors $k \in R^n$:

$$X(y, k) := \{x \in X \mid D_y s(x, y) = k\}, \tag{16}$$

or we can think of his type $y \in Y$ as inducing an equivalence relation between female types, under which x and $\bar{x} \in X$ are equivalent if and only if they provide him the same marginal surplus

$$D_y s(x, y) = D_y s(\bar{x}, y).$$

Under this equivalence relation, the equivalent classes take the form (16). We call these equivalence classes *potential indifference sets*, since they represent a set of partner types which $y \in \mathrm{Dom}_0 Dv$ has the potential to be indifferent between. The equivalence class containing a given female type $\bar{x} \in X$ will also be denoted by

$$L_{\bar{x}}(y) = X(y, D_y s(\bar{x}, y)) = \{x \in X \mid D_y s(x, y) = D_y s(\bar{x}, y)\}. \tag{17}$$

A key observation concerning potential indifference sets is the following proposition showing — for surpluses satisfying a local non-degeneracy condition as in [McAfee & McMillan (1988)] — that the potential indifference set consists either of isolated points, curves, surfaces, etc. (respectively) according to the difference between the dimensions of the female and male types: $n = m$, $m - 1, m - 2$, etc. (respectively).

Definition 5 (Surplus degeneracy) *Given $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$, we say $s \in C^2(X\times Y)$* degenerates *at $(x, y) \in X\times Y$ if rank$(D^2_{xy}s(\bar{x}, \bar{y})) < \min\{m, n\}$. Otherwise we say s is* non-degenerate *at $(\bar{x}, \bar{y})$.*

Proposition 6 (Structure of potential indifference sets) *Let $s \in C^2(X \times Y)$, where $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ with $m \geq n$. If s does not degenerate at $(\bar{x}, \bar{y}) \in X \times Y$, then near the female type $\bar{x}$, the potential indifference set $L_{\bar{x}}(\bar{y})$ of $\bar{y}$ passing through $\bar{x}$ coincides with the intersection of X with a C^1-smooth, codimension n, submanifold of $\mathbf{R}^m$.*

Proof. *Since $s \in C^2$, the surplus extends to a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ on which s continues to be non-degenerate (by lower semicontinuity of the rank). The set $\{x \in U \mid D_y s(x, y) = D_y s(\bar{x}, \bar{y})\}$ forms a codimension n submanifold of U, by the preimage theorem [Guillemin & Pollack (1974), §1.4]. More specifically, the rank condition implies that choosing a suitable orthonormal basis for $\mathbf{R}^m$ yields $\det[\frac{\partial^2 s}{\partial x^i \partial y^j}(\bar{x}, \bar{y})]_{1\leq i,j\leq n} \neq 0$. In these coordinates, the potential indifference set is locally parameterized as the inverse image under the C^1 map $x \in U \mapsto (D_y s(x, \bar{y}), x_{n+1}, \ldots, x_m)$ of the affine subspace $\{D_y s(\bar{x}, \bar{y})\} \times \mathbf{R}^{n-m}$. Taking U and V smaller if necessary, the inverse function theorem then shows $L_{\bar{x}}(\bar{y}) \cap U$ to be C^1.*

■

Remark 7 (Smoother surpluses) *For smooth surpluses s (with, say $r+1$ continuous derivatives) the same proof shows the potential indifference set to be correspondingly smoother (i.e. to be parametrized locally as a graph over R^{m-n} having r continuous derivatives).*

Although we have stated the proposition in local form, when s is globally non-degenerate it implies that each potential indifference set is the intersection of X with an $m - n$ dimensional submanifold of R^m. Indeed more is true: the potential indifference sets of $\bar{y}$ foliate the interior of X. On the other hand, this proposition says nothing about points $(\bar{x}, \bar{y})$ where s degenerates, which can happen throughout spt γ.

Potential versus actual indifference sets As argued above, the potential indifference sets (16) and (17) are determined by the surplus function $s(x, y)$ without reference to the populations μ and ν to be matched.

On the other hand, the indifference set actually realized by each $y \in Y$ depends on the relationship between μ, ν and s. This dependency is generally complicated, as illustrated by the following example.

Example 8 *Consider the surplus function:*

$$s(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_1 y_2$$

where $X \subset \mathbf{R}^3, Y \subset \mathbf{R}^2$. *The potential indifference sets are given, for any* $k \in \mathbf{R}^2$, *by:*

$$X(y, k) := \left\{ x \in X \mid \begin{array}{c} x_1 + x_3 y_2 = k_1 \\ \text{and} \\ x_2 + x_3 y_1 = k_2 \end{array} \right\}. \tag{18}$$

These are straight lines in $\mathbf{R}^3$, *parallel to the vector* $\begin{pmatrix} y_2 \\ y_1 \\ -1 \end{pmatrix}$. *Therefore, for any given* $y \in \mathbf{R}^2$, *we know that the set of spouses matched with* y *(the indifference set corresponding to husband* y*) will be contained in such a straight line. However, it is certainly not true that any such line (obtained for an arbitrary choice of* k*) will be an indifference set curve. For a given* y, *the exact equation of the indifference set corresponding to* y *is defined by the value of the specific vector* k *which is relevant for that particular* y *— and this depends on the measures* μ *and* ν.

Remark 9 *Under the same non-degeneracy assumption, [McAfee & McMillan (1988)] proposed a 'generalized single crossing property' on* s *which is equivalent to the assertion that each potential indifference set* $L_{\bar{x}}(y)$ *parallel the kernel of* $D^2_{xy}s : T_{\bar{x}}X \longrightarrow T_y Y$, *hence be an* affine *subspace. Although satisfied in the preceding example, their condition is extremely non-generic: even among twisted surpluses, the potential indifference sets are typically curved rather than flat.*

2.2.2 Multi-to-one dimensional matching ($n = 1$)

Iso-husband surfaces We now consider a specific class of models, largely unexplored in either the mathematics or economics literature, but which can often be solved explicitly with the techniques outlined below and developed more fully in [Chiappori, McCann & Pass (2017)]. These are *multi-to-one dimensional* models, in which agents on one side of the market (say wives) are bi-dimensional (or, potentially, higher dimensional) while agents on the other side (husbands) are one-dimensional. Thus, we are matching a distribution on $x = (x_1, ..., x_m) \in \mathbf{R}^m$ with another on $y \in \mathbf{R}$. The surplus s is then a function $s(x_1, ..., x_m, y)$ of $m+1$ real variables**.** In our analysis, a key role is played by the actual indifference sets; in line with the marriage market interpretation, we call these *iso-husband surfaces.* In practice, the iso-husband surface of a given husband y is defined as the submanifold of wives among which husband y turns out to be indifferent facing the given market conditions.

We first provide a result that directly generalizes the notion of assortative matching to a multidimensional setting:

Proposition 10 *Assume that the surplus is such that*

$$\frac{\partial^2 s}{\partial x_k \partial y} > 0 \quad \text{for } k = 1, ..., K$$

Consider two women $(x_1, ..., x_K, x_{K+1}, ...x_m)$, matched with husband y, and $(x'_1, ..., x'_K, x_{K+1}, ...x_m)$, matched with husband y'. If $x'_k \geq x_k$ for $k = 1, ..., K$ then $y' \geq y$.

Proof. *Assume not, then switching husbands would increase total surplus, a contradiction.* ■

In other words, the assortative matching argument can be generalized in the following way: if the second cross derivative is positive for female characteristics 1 to K, then among women *with identical characteristics* $K+1, ..., m$, those with higher values for the first K characteristics are matched with husband with a higher characteristic y. This property can actually be generalized under specific assumptions on the surplus functions; see for instance [Lindenlaub (2015)].

Next, the previous arguments, and particularly the relationship between potential and actual iso-husband surfaces, can sometimes be transposed to the multi-to-one context. In general, the situation is complicated by the fact that different types $y \neq y'$ need not agree on which types of women are 'higher' that others. However, each given type of man y has a clear order of preference among partners; one can in some cases exploit this fact to characterize the features of the stable matching.

Specifically, suppose s is non-degenerate (i.e. $D^2_{xy}s(\cdot, y)$ is non-vanishing). Then the potential indifference sets $X(y,k)$ are codimension 1 in $\mathbf{R}^m$; that is, they are curves in $\mathbf{R}^2$, surfaces in $\mathbf{R}^3$, and hypersurfaces in higher dimensions $m \geq 4$. As k moves through $\mathbf{R}$, these potential indifference sets sweep out more and more of the female types. For each $y \in Y$ there will be some choice of $k \in \mathbf{R}$ for which the number of women in $\{x \mid D_y s(x,y) \leq k\}$ exactly coincides with the number of men in $(-\infty, y]$ (assuming both are distributed absolutely continuously with respect to Lebesgue, or at least that μ concentrates no mass on hypersurfaces and ν has no atoms). In this case the potential indifference set $X(y,k)$ is said to split the population proportionately at y, making it a natural candidate for being the iso-husband set $F^{-1}(y)$ to be matched with y.[11]

Our goal, now, is to distinguish situations in which this expectation is born out and leads to a complete solution from those in which it does not. This happens if the potential iso-husband surfaces that divide the mass of μ in the same ratio as y divides ν may, when y is varied, fit together to form the level sets of a function. When they do, we say the model is *nested*, and in that case we show that the resulting function $F : X \longrightarrow Y$ produces a stable equilibrium match.

Constructing explicit solutions for nested data We now precisely characterize the nestedness property. This property is satisfied in a wide class of multi-to-one dimensional matching problems, that are illustrated in the theorem and examples presented below. However, except in the Spence-Mirrlees (with $m = n = 1$) and in the index and pseudo-index

[11] Since $k = s_y(x,y)$ can be recovered from any $x \in X(y,k)$ and y, we may equivalently say x splits the population proportionately at y, and vice versa. What is meant in either case is that the mass of women who would generate less marginal surplus for y than x does coincides with the mass of men of types lower than y.

cases (discussed below), this nestedness depends not only on the surplus s, but also on the measures μ and ν.

For each fixed $y \in Y \subseteq R$, our goal is to identify the iso-husband (or indifference) set $\{x \in X \mid F(x) = y\}$ of husband type y facing the given market conditions. When differentiability of v holds at y, the equality of his direct and indirect marginal payoffs implies that this is contained in one of the potential indifference sets $X(y,k)$ from (16). Proposition 6 indicates when this set will have codimension 1; it generally divides X into two pieces: the sublevel set

$$X_{\leq}(y,k) := \{x \in X \mid \frac{\partial s}{\partial y}(x,y) \leq k\}, \tag{19}$$

consisting of female types for whom y has marginal surplus less than he has on potential indifference set under discussion, and its complement $X_{>}(y,k) := X \setminus X_{\leq}(y,k)$. (We denote its strict variant by $X_{<}(y,k) := X_{\leq}(y,k) \setminus X(y,k)$.)

We now choose the unique level set *splitting the population proportionately* with y; that is, the $k = k(y)$ for which the mass $\mu[X_{\leq}(y,k)]$ of female types x below the potential indifference curve coincides with the ν mass of male types below y. We then hypothesize $y := F(x)$ for each x in the corresponding potential indifference curve $X(y,k)$.

The next definition specifies conditions under which the map $F : X \longrightarrow Y$ is well-defined; it precludes $X(y,k(y))$ from intersecting $X(\bar{y},k(\bar{y}))$ unless $y = \bar{y}$. In this case the resulting match $\gamma = (id \times F)_{\#}\mu$ turns out to optimize the Kantorovich problem (MK), as Theorem 12 shows. Thus nestedness is the natural generalization of the positive assortative matching results of [Mirrlees (1971)] [Becker (1973)] and [Spence (1973)] from the one-dimensional to the multi-to-one dimensional setting.

The precise definition is complicated slightly to allow for the possibility that μ vanishes on subregions of X:

Definition 11 (Nestedness) *Let $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}$ be connected open sets equipped with Borel probability measures μ and ν. Assume ν has no atoms and μ vanishes on each C^1 hypersurface. Use $s \in C^2(X \times Y)$ and $s_y = \frac{\partial s}{\partial y}$ to define $X_{\leq}$, $X_{<}$ etc., as in* (19)*. Assume moreover* *that s is non-degenerate, $|D_x s_y| \neq 0$, throughout $X \times Y$. Then for each $y \in \overline{Y}$ there is a maximal interval $K(y) = [k^-(y), k^+(y)] \neq \emptyset$ such that $\mu[X_\leq(y,k)] = \nu[(-\infty, y)]$. The model is said to be* nested *if both maps $y \in Y \mapsto X_\leq(y, k^\pm(y))$ are non-decreasing, and moreover that $\int_y^{y'} d\nu > 0$ implies*

$$X_\leq(y, k^+(y)) \subseteq X_<(y'^-(y')). \tag{20}$$

A crucial point is that, unlike the Spence-Mirrlees (or supermodularity) criterion, which depends only on s, the nestedness condition relates s to μ and ν. The intuition for this additional degree of complexity is simple: in a multidimensional context, there is no obvious ordering of the women's types, but generally a variety of possible orderings depending on population frequencies μ and ν. Nestedness essentially asserts that under the given market conditions, the men's indirect preferences enjoy some degree of compatibility, in the sense that for $\underline{y} < \bar{y}$, the potential indifference curves $X(\underline{y}, k(\underline{y}))$ and $X(\bar{y}, k(\bar{y}))$ of female types hypothesized on the basis of mass balance *do not cross each other.*

Under this hypothesis, one can show the following theorem. It states that when the potential indifference sets selected on the basis of proportionate splitting (i.e. mass balance) do not intersect each other, then they coincide with the iso-husband sets of the unique stable match.

Theorem 12 (Optimality of nested matchings) *Under the hypothesis of the previous definition: if the model is nested, then $k^+ = k^-$ holds ν-a.e. Setting $F(x) = y$ for each $x \in X(y, k^+(y))$ defines a stable match $F : X \longrightarrow Y$ [μ-a.e.]. Moreover, $\gamma = (id \times F)_\# \mu$ maximizes* (MK) *uniquely on $\Gamma(\mu, \nu)$. Finally, if* $\operatorname{spt} \nu$ *is connected then F extends continuously to X.*

Proof. A detailed proof can be found in [Chiappori, McCann & Pass (2017)]. The main intuition can be summarized as follows. Non-degeneracy implies $X(y,k) := X_\leq(y,k) \setminus X_<(y,k)$ is an $m-1$ dimensional C^1 submanifold of X orthogonal to $D_x s_y(x,y) \neq 0$. Since both μ and ν vanish on hypersurfaces, the function

$$h(y,k) := \mu[X_\leq(y,k)] - \nu[(-\infty, y)] \tag{21}$$

is continuous, and for each $y \in Y$ climbs monotonically from $-\nu[(-\infty, y)]$ to $1 - \nu[(-\infty, y)]$ with $k \in \mathbf{R}$. This proves the existence of $k^{\pm}(y)$ and confirms the zero set of $h(y,k)$ is closed. In fact, k^- is lower semicontinuous, k^+ is upper semicontinuous, and by the intermediate value theorem $[k^-(y), k^+(y)]$ is non-empty. The main strategy for the rest of the proof is to use $k^+(y)$ to construct a Lipschitz equilibrium payoff function v by solving $v'^+(y) = k^+(y)$ a.e. Together with u from (12), it can be shown that (u,v) minimizes the dual problem (MK$_*$) and $\gamma = (id \times F)_{\#}\mu$ maximizes the planners problem (MK). ∎

Remark 13 (Twisted v. nested) *Fix absolutely continuous distributions μ and ν of female and male types. Ignoring a $(\mu \otimes \nu)$-negligible set, we have already seen that equality* (5) *of the marginal direct and indirect payoffs of females and males is a necessary condition for stability of the pairing (x,y). Conversely, when the surplus is twisted, then the equalities*

$$\frac{\partial s}{\partial x_i}(x,y) = \frac{\partial u}{\partial x_i}(x) \qquad i = 1, \ldots, m \tag{22}$$

relating the woman's marginal payoffs alone are sufficient for the pairing between x and y to be stable. (Indeed, injectivity of $D_x s(x, \cdot)$ allows us to recover the matching function $y = F(x) = D_x s(x, \cdot)^{-1}(Du(x))$.) In contrast, for nested models the single(!) (since $n = 1$) equality

$$\frac{\partial s}{\partial y_i}(x,y) = \frac{\partial v}{\partial y_j}(y) \qquad j = 1, \ldots, n \tag{23}$$

relating the man's marginal payoffs becomes sufficient for stability of (x,y).[12]

Although either twist or nestedness alone would imply purity (hence uniqueness) of the stable match, they are complementary notions. Neither implies the other; however, (23) *consisting of fewer equations that* (22) *somehow suggests nestedness is the more specialized of the two notions, with correspondingly more powerful implications.*

From these results it is clear that nestedness, when present, has

[12]We state this condition for general n since it suggests a generalization of nestedness to arbitrary dimensions $m \geq n \geq 1$ explored in [McCann & Pass (2017)].

powerful implications. However, depending on the context, it may or may not be guaranteed. It is important to emphasize that, in general, nestedness is a property of the three-tuple (s, μ, ν). In particular, for most surplus functions, the model may or may not be nested depending on the measures under consideration. In a companion paper ([Chiappori, McCann & Pass (2017)]), we provide characterizations of nestedness based on a description of the motion of the iso-husband set in response to changes in the husband type. Here, we simply provide an example of a surplus function and two measures such that the model is nested for one measure but not for the other.

2.2.3 Index and pseudo-index models

Definition A special case of multi-to-one matching, which has been largely used in practical applications, is obtained when the surplus function s is weakly separable in one vector of characteristics. Assume, indeed, that there exist two functions I and σ, mapping $\mathbf{R}^n$ to $\mathbf{R}$ and $\mathbf{R}^{m+1}$ to $\mathbf{R}$ respectively, such that:

$$s(\mathbf{x}, \mathbf{y}) = \sigma(x, I(y)). \tag{24}$$

In words, the various male characteristics y affect the matching function only through some one dimensional index $I(y)$. It is important to understand why this assumption is restrictive. Start with its formal translation. If s is smooth, then the index form requires that s satisfies the following conditions:

$$\frac{\partial}{\partial x_h}\left(\frac{\partial s/\partial y_k}{\partial s/\partial y_l}\right) = 0 \quad \forall k, l, h. \tag{25}$$

These conditions express the fact that the marginal rate of substitution (MRS) $\sigma_{k,l}$ between y_k and y_l (which defines the slope of tangent to the corresponding iso-surplus curve) does not depend on x; indeed, (24) implies that:

$$\sigma_{k,l} = \frac{\partial s/\partial y_k}{\partial s/\partial y_l} = \frac{\partial I/\partial y_k}{\partial I/\partial y_l}.$$

Now, what are the practical implications of this form? The interpretation of the MRS between male characteristics is standard: it represents the variation in characteristic k that is needed to compensate some given, infinitesimal increase in characteristic l. Assume, for instance, that men are characterized by two traits - say, income and physical attractiveness, the latter being proxied by the person's Body Mass Index (BMI) as in COQ. Then the MRS indicates how much additional income would be needed to 'compensate' for an additional unit of BMI - the 'compensation' implying simply that the husband's global attractiveness remains unchanged. If the surplus is a smooth and strictly monotonic function of both characteristics, the existence, for any woman x, of such a compensation stems from the implicit function theorem. The crucial point, however, is that in general the MRS is woman-specific: potential wives with different characteristics will weight the two male traits differently (say, wealthier women may put relatively more weight on physical attractiveness). Condition (25) imposes, on the contrary, that the trade-off between male traits should be independent of female characteristics; in practice, men with different characteristics y and y' but the same index (i.e., $I(y) = I(y')$) must therefore be viewed as perfect substitutes on the matching market *by any potential spouse* x. Or consider the model by Heckman and Sedlacek (1985), where workers characterized by heterogeneous, multi-dimensional skills are allocated to sector-specific tasks. In general, the agent's productivity in any particular task is a smooth function of the vector of skills, and one can define a MRS between skills for that task; an index structure would require that this MRS be the same for all tasks.

The main practical interest of index models is that, whenever (24) is satisfied, the matching problems is de facto one-dimensional in y; technically, one can replace the space Y and the measure ν with $\tilde{Y} = \mathrm{Im} I \subset \mathbb{R}$ and the *push-forward* $\tilde{\nu} := I_{\#}\nu$ of ν through I defined as in (4). In particular, when the index property (24) is satisfied, then the matching problem boils down to a multi-to-one dimensional problem, of the type discussed in Section **2.2** and [**Chiappori, McCann & Pass (2017)**].[13]

[13] A practical difficulty is that, for most empirical applications, the in-

Pseudo-index models: The notion of index model can for some purposes be slightly relaxed. Specifically, we define a *pseudo-index* model by assuming that there exist three functions α, I and σ, mapping $\mathbf{R}^n$ to $\mathbf{R}$, $\mathbf{R}^n$ to $\mathbf{R}$ and $\mathbf{R}^{m+1}$ to $\mathbf{R}$ respectively, such that:

$$s(x,y) = \psi(y) + \sigma(x, I(y)). \tag{26}$$

Formally, the pseudo-index assumption boils down to a standard separability property; in particular, it implies that:

$$\frac{\partial^2 s/\partial y_k \partial x_h}{\partial^2 s/\partial y_l \partial x_h} \text{ is independent of } x \text{ for all } h,k,l, \tag{27}$$

Here, male characteristics y affect the matching function through *two* one-dimensional indices $\psi(y)$ and $I(y)$. The crucial point, however, is the following. Assume that the wife's marginal direct payoff $D_x\sigma(x,i)$ is injective with respect to the husband's index i; this will be the case, for instance, if $\partial\sigma/\partial x_k(x,i)$ is strictly monotonic in i for at least one k. Then the reduced surplus σ is twisted, so the stable matching on the reduced space $R^m \times R$ is unique and pure, meaning there exists a matching function $F : R^m \to R$ such that any woman x is matched with probability one to a man whose index is $I(y) = F(x)$. On the other hand, when $n > 1$ the original surplus s cannot be twisted since

$$y \in Y \mapsto D_x s(x,y) = D_x\sigma(x, I(y))$$

does not distinguish between men $y \neq y_0$ with the same index $I(y) = I(y_0)$. Indeed, all males with the same index are perfect substitutes, so the solution to the stable matching problem on the full space $R^m \times R^n$ will not generally be unique (nor pure).

As mentioned above, nestedness is a complex property that involves both the surplus function and the measures. There is, however, an exception: models which are index or pseudo-index in the x variable (while still unidimensional in the y variable) are generally nested irrespective of

dex I is not known ex ante and has to be empirically estimated. See [Chiappori, Oreffice & Quintana-Domeque (2012)] for a precise discussion.

the measures on the sets X and Y. To see why, assume that the surplus has the form:

$$s\left(x,y\right)=\phi\left(x\right)+\sigma\left(I\left(x\right),y\right). \tag{28}$$

Then the potential indifference sets of y and y' both coincide with level sets of $I(x)$. Therefore they cannot cross, since two different sets correspond to two different values of the index.

This intuition yields the following result:

Proposition 14 (Non-degenerate pseudo-index models are nested) *Fix connected $X \subset R^m$ and $Y \subset R$. If the surplus is pseudo-index and non-degenerate, then the model is nested for all measures on X and Y.*

Proof. Assuming $\sigma \in C^2$ and $I \in C^1$, non-degeneracy asserts $D_x s_y = \sigma_{Iy} DI \neq 0$ throughout the connected set $X \times Y$. Thus σ is either super- or sub-modular; we'll assume supermodularity

$$\frac{\partial^2 \sigma}{\partial I \partial y} > 0,$$

without loss of generality; the submodular case can be handled similarly.

Corollary 2 shows $\phi(x)$ to be irrelevant to the stability or instability of γ, so we may as well take $\phi = 0$. In this case $s(x,y)$ depends on x only through $I(x)$, so the problem of finding an s-stable matching of μ to ν on $\mathbf{R}^m \times \mathbf{R}$ reduces to the problem of finding a σ-stable matching of $I_{\#}\mu$ to ν on $\mathbf{R}^2$. Supermodularity of σ guarantees positive assortativity of σ-stable matchings, so the husband's type $y = H(I(x))$ will be a non-decreasing function H of his wife's index $I(x)$. Here we have used the fact that $I_{\#}\mu$ has no atoms, which follows from non-degeneracy $DI \neq 0$ of s and the requirement that μ vanish on all C^1 hypersurfaces in the definition of nestedness. Thus H pushes $I_{\#}\mu$ forward to ν, and $y \in Y$ implies

$$\int^y d\nu = \int^{H^{-1}(y)} d(I\#\mu) = \int_{\{x \in X | I(x) \leq H^{-1}(y)\}} d\mu \tag{29}$$

where the first equality follows from monotonicity of H, and $H^{-1}(y)$ is almost surely unambiguous because ν has no atoms either.

On the other hand

$$\frac{\partial \sigma}{\partial y}\left(I\left(x\right),y\right)=\frac{\partial s}{\partial y}(x,y) \tag{30}$$

shows y's marginal surplus depends on x only through her one-dimensional index $I(x)$ (again, independently of ϕ). His potential indifference sets are level sets $I = const$, hence cannot cross those of $\bar{y} \neq y$. More pedantically, comparing

$$X_{\leq}(y,k)=\{x\in X \mid I(x)\leq \sigma_y(\cdot,y)^{-1}(k)\}$$

to (29) shows we can take $k^{\pm}(y) = \sigma_y(H^{-1}(y),y)$ in the definition of nestedness. Hence $X_{\leq}(y,k^{\pm}(y)) = \{x \mid I(x) \leq H^{-1}(y)\}$ depends monotonically on y, as desired. ■

Finally, it should be noted that the converse is also true: if a non-degenerate surplus s is such that (s,μ,ν) is nested for all choices of absolutely continuous population densities μ and ν, then s has a pseudo-index structure (see [Chiappori, McCann & Pass (2017)]). This is a strong reason for the popularity of pseudo-index models: the construction described in subsection 2.2.2 can be applied irrespective of the measures under consideration. This convenience comes however at a price: pseudo-index models are much more restrictive than nested ones, in the sense that they generate stronger restrictions on observed matching patterns. These restriction are described in the next section.

3 Testability and identification: a 'pure theory' perspective

3.1 Testability: main issues

We now analyze the empirical content of multidimensional matching theory. Specifically, we consider two issues. One is testability: what restriction does theory impose on observable behavior? Equivalently, can the theory be falsified on existing data? The other issue relates to identifiability: to what extent is it possible, from the observation of actual behavior, to recover the underlying structure, namely the surplus function and possibly the distribution of unobservable characteristics?[14]

3.1.1 Data

It is useful, at that point, to clarify the exact meaning of 'existing data' (or 'actual behavior'). In what follows, we consider the simplest case, in which the econometrician only observes matching patterns ('who marries whom'); technically, therefore, we shall assume that the joint measure corresponding to the stable matching is (perfectly) known, while neither the surplus nor payoff functions are. We are thus considering an inverse problem: knowing the spaces X and Y and the measure γ, can we find a surplus s for which γ is stable?

Note, however, that the question should actually be rephrased to rule out degenerate solutions. Indeed, we have seen above that *any* measure is stable for an additively separable surplus. We should therefore consider the following problem: Given two spaces X, Y and some measure γ on $X \times Y$, is it always possible to find a surplus s such that γ is the *unique* stable matching of the matching problem (X, Y, s)?

This problem, however, raises a second issue, namely our ability to observe *all* relevant aspects of the matching game. From a theoretical perspective, the spaces X and Y describe all individual characteristics that enter the surplus function. In practice, however, available data sets contain (at best) a fraction of the relevant characteristics; many aspects, including tastes, talents, social skills and many dimensions of physical attractiveness, are simply not observed by the econometrician, although even casual empiricism strongly suggests they are likely to play an important role in the determination of actual matching patterns.

As often in applied microeconomics, empirical analysis typically captures these unobserved heterogeneity aspects through the introduction of an adequate stochastic structure. It follows that the properties of the model, in terms of testability as well as identifiability, depend on both the basic, mathematical framework and the specific stochastic components. In the present section, we consider the 'pure theory' case (where all relevant characteristics are observed); the 'applied' case, in which an

[14] For a recent investigation, see [Dupuy, Galichon & Sun (2016)].

explicit, stochastic structure is introduced, will be analyzed next.

A third question is whether available data relate to one or to several 'markets'. That is, are we able to observe different matchings, corresponding to the same surplus function but to different marginal distributions on the X and Y spaces? As argued by [Chiappori, Salanié & Weiss (2017)], the 'multimarket' approach is certainly more promising, particularly in terms of testability.[15]

Finally, in multi-to-one cases, the crucial notion, from an empirical perspective, is that of iso-husband surfaces. These surfaces can (in principle) be empirically identified; and their theoretical properties could in principle provide the most powerful empirical tests of matching theory. In what follows, we precisely investigate that claim.

3.1.2 Purity

In a 'pure theory' approach, the sets X and Y, together with the corresponding measures, summarize *all* relevant information of the game. We therefore consider the following problem: Given two spaces X, Y and some measure γ on $X \times Y$, is it always possible to find a surplus s such that γ is the *unique* stable matching of the matching problem (X, Y, s)?

A first point is that if we are willing to insist on pure matchings, then sufficient regularity makes the answer positive. Specifically, let us consider the case in which the support of the measure is born by the graph of some function F, and that F is non-degenerate (in the sense that the derivative of F has full rank over the entire space). Then one can always find a surplus for which γ is the unique stable matching: we just need to take $s(x, y) = -|F(x) - y|^2/2$. Indeed, γ obviously maximizes the primal, optimal transportation problem, which guarantees stability; moreover, the surplus satisfies the twist condition, which guarantees uniqueness. The corresponding payoffs are $u(x) = 0 = v(y)$. It follows that, in the pure (non-degenerate) case, matching theory is not testable from single market data.

[15] In addition, some empirical work rely on independent information on the surplus itself (for instance by analyzing demand or labor supply of married couples). We do not investigate this situation here; the interested reader may refer to [Chiappori, Costa Dias & Meghir (2017)] or the survey by [Chiappori & Salanié (2016)].

It should be noted that non-degeneracy is crucial for this result to hold. For one thing, if F is degenerate, the twist condition does not hold, and while γ is always stable for $s(x,y) = -|F(x) - y|^2/2$, it may not be the unique stable matching. Moreover, while any stable matching for a non-degenerate surplus concentrates on a set of dimension at most $\max(m,n)$, it is possible to find measures supported on sets of this dimension which are *not* stable for any C^2, *non-degenerate* surplus.[16] However, looking for necessary conditions that would be valid only for degenerate cases does not appear to be a particularly fruitful approach. Much more promising is the multimarket case, which we consider next.

3.1.3 The multi-market approach

We thus assume that we can observe various 'markets', indexed by $t = 1, ..., T$. In each market, the surplus function is the same function s; however, the marginal distributions are different in each case. Then stability generates testable conditions on the shape of the iso-husband surfaces in different markets. Specifically: *if two iso-husband surfaces, corresponding to two different markets, intersect (in the sense that man y is matched to woman x in both markets), then they coincide locally. Furthermore, in the nested case, they must coincide globally.*

Formally, let $X_t(y)$ denote the iso-husband surface of husband y in market t. Suppose that some couple (x,y) is matched with positive probability in two stable matchings t and t'; that is, $x \in X_t(y) \cap X_{t'}(y)$. Now, we know that $X_t(y)$ is included in some submanifold $\mathcal{S}_t(y)$ defined by an equation of the type:

$$D_y s(x,y) = k_t(y) \tag{31}$$

If $x \in X_t(y) \cap X_{t'}(y)$, then $k_t(y) = k_{t'}(y)$, implying that $\mathcal{S}_t(y) = \mathcal{S}_{t'}(y)$.

[16] To see this, consider the $m = n = 1$ case; let $X = Y = (0,1) \subseteq \mathbf{R}$. Nondegeneracy here simply means $\frac{\partial^2 s}{\partial x \partial y} \neq 0$, which implies either $\frac{\partial^2 s}{\partial x \partial y} > 0$ everywhere (so s is super-modular) or $\frac{\partial^2 s}{\partial x \partial y} < 0$ everywhere (so s is submodular). In these two cases, it is well known that stable matches concentrate on monotone increasing or decreasing sets, respectively. Therefore, any γ concentrating on a set of dimension $\max(m,n) = 1$ (for instance, a smooth curve), which is neither globally increasing nor decreasing (for example, the curve $(y = 4(x-1/2)^2)$), cannot be stable for any non-degenerate surplus.

Intuitively, if x is matched to y in markets t and t', then any nearby x' that is matched to y with positive probability in market t is also matched to y with positive probability in market t'. More precisely:

Proposition 15 (Corresponding iso-husband sets do not cross) *Assume $m > n$ and $s \in C^2\left(\overline{X} \times \overline{Y}\right)$ non-degenerate. Fix t, t'. For husband types $y \in Y^0$ outside a Lebesgue negligible*[17] *set $\Sigma_t \subset Y^0$ the wive types $X_t(y) \subset X$ matched with y are contained in a C^1 submanifold $\mathcal{S}_t(y) \subset \mathbf{R}^m$ of dimension $m - n$; $\mathcal{S}_t(y)$ is defined by* (31) *with k_t depending continuously on $y \in Y^0 \setminus \Sigma_t$. If $x \in X_t(y) \cap X_{t'}(y)$ for some $y \in Y^0 \setminus (\Sigma_t \cup \Sigma_{t'})$, then $\mathcal{S}_t(y) = \mathcal{S}_{t'}(y)$.*

Proof. *For market t, and let $u_t \in L^1(\mu)$ and $v_t \in L^1(\nu)$ denote the wives' and husbands' shares of the surplus from* (8)*. These are well-known to exist and may be taken to inherit semiconvexity from $\|s\|_{C^2} < \infty$; see e.g. [Chiappori, McCann & Nesheim (2010)]. Let Σ_t denote the set where differentiability of v fails; it is contained in a countable union of DC (a fortiori Lipschitz) hypersurfaces according to [Zajíček (1979)]; outside this set Dv_t is a continuous map. From* (5) *we see $y \in Y^0 \setminus \Sigma_t$ implies*

$$D_y s(x, y) = Dv_t(y) \tag{32}$$

for all $x \in X_t(y)$; the equation (31) *defining $\mathcal{S}_t(y)$ holds; moreover $k_t(y) = Dv_t(y)$ depends continuously on $y \in Y^0 \setminus \Sigma_t$. Proposition 6 asserts $X_t(y) \subset L_x(y) = X(y, Dv_t(y))$, the intersection of X with the C^1 smooth submanifold $\mathcal{S}_t(y)$ of dimension $m - n$ in $\mathbf{R}^m$. If $x \in X_{t'}(y)$ also and $y \in Y^0 \setminus (\Sigma_t \cup \Sigma_{t'})$, we conclude $k_{t'}(y) = k_t(y)$, hence $\mathcal{S}_t(y) = \mathcal{S}_{t'}(y)$.*
■

Remark 16 *Except on a Lebesgue negligible set $\Sigma_t'^0$, the semiconvexity asserted in the previous proof also implies that v_t has a second-order Taylor expansion (by Alexandrov's theorem).*

Proposition 15 describes a local property: assuming the distributions of men and women in both markets are absolutely continuous with

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[17]In fact, Σ_t is a countable union of DC hypersurfaces, where DC means each hypersurface is locally the graph of a difference of convex functions.

respect to Lebesgue, it follows almost surely that if the iso-husband surfaces of y in the two markets intersect at x, then they coincide nearby. This can be seen as a non-crossing condition: if two iso-husband surfaces, corresponding to two different markets, do not coincide (at least locally), then their intersection is almost surely void. It should be noted that the scope of this result is actually completely general; it applies in the full range $m \geq n \geq 1$, including when $n > 1$.

The local nature of the result is due to the possibility that the markets are not nested: while the set of women married to y with positive probability must belong to the manifold $\mathcal{S}(y)$, in non nested markets it may be a strict subset of $\mathcal{S}(y)$. On the other hand, when market t is nested the following corollary improves this to a global conclusion: either all of the wives or none of the wives $X_{t'}(y)$ paired with y in the (not necessary nested) market t' are also paired with y in the nested market t.

Corollary 17 (Iso-husband sets are disjoint in nested markets) *If market t is nested in Proposition 15 (so $n = 1$), then $y \in Y^0 \setminus \Sigma_t$ implies $X_t(y) = X \cap \mathcal{S}_t(y)$. If, in addition,* $\operatorname{spt}\mu \supset X$ *hence is connected, then Σ_t is empty.*

Proof. The equality $X_t(y) = X \cap \mathcal{S}_t(y)$ follows directly from Theorem 12. The additional claim follows from the first paragraph of the proof of Theorem 5.2(a) of [Chiappori, McCann & Pass (2017)], which shows the equation

$$\mu_t[X_{\leq}(y, k(y))] = \nu_t[(-\infty, y)]$$

admits a continuous solution $k : Y^0 \longrightarrow \mathbf{R}$ when $\operatorname{spt}\mu$ contains the domain X, which is connected from Definition 11. When the model is nested, Theorem 4.2 of the same paper identifies $k(y) = k_t(y) = v_t'(y)$ on Y^0, which shows we may take $v \in C^1(Y^0)$ and $\Sigma_t = \emptyset$. ■

Lastly, the condition expressed in Proposition 15 is necessary but not sufficient in general; under mild regularity conditions, additional restrictions can be derived. Specifically, consider t, t' and $y \in Y^0 \setminus (\Sigma_t' \cup \Sigma_{t'}')$ such that $x \in X_t(y) \cap X_{t'}(y) \neq \emptyset$; in other words, the optimal maps F_t and $F_{t'}$ are such that $F_t(x) = F_{t'}(x) = y$. Assuming that

$DF_i(x)$, $i = t, t'$, exist, the next proposition shows $DF_i(x)$ has full rank, and a (not necessarily unique) one-sided inverse $DF_i(x)^{-1}$; since $DF_i(x)$ is $n \times m$, $DF_i(x)^{-1}$ is $m \times n$, and $DF_i(x) \circ DF_i(x)^{-1}$ is the $n \times n$ identity. Moreover, it yields a testable positivity restriction on the product $DF_{t'}(x) \circ DF_t(x)^{-1}$, which represents the derivative of the correspondence between husbands in different markets.

Proposition 18 (Infinitesimal monotonicity of $F_{t'} \circ F_t^{-1}$) *Assume $s \in C^2\left(\overline{X} \times \overline{Y}\right)$ is twisted and non-degenerate. If F_t and $F_{t'}$ are both differentiable at $x \in X$ and $F_t(x) = F_{t'}(x) \in Y^0 \setminus (\Sigma_t' \cup \Sigma_{t'}')$, then $DF_i(x)$ have full-rank for $i = t, t'$, hence admit one-sided inverses. Moreover, all eigenvalues of the $n \times n$ matrix $DF_{t'}(x) \circ DF_t(x)^{-1}$ are positive. (Here the Σ_i' refer to the Lebesgue negligible subsets from Remark 16.)*

Proof. *Set $y = F_t(x)$, $A = D^2_{xy}s(x, y)$ and $M_i = DF_i(x)$, noting the the derivatives in question have been assumed to exist. First observe $y \in Y^0 \setminus (\Sigma_t' \cup \Sigma_{t'}')$ implies*

$$Dv_i(F_i(x)) - D_y s(x, F_i(x)) = 0$$

from (5)*, and the same identity extends to nearby points in X (by the differentiability assumed of F_i). Differentiation then yields*

$$[D^2 v_i(F_i(x)) - D^2_{yy}s(x, F_i(x))]M_i = A. \tag{33}$$

Non-degeneracy of s implies A has rank n, so the same must be true for both factors on the left-hand side. Thus M_i has a (non-unique) one-sided inverse. The preceding proposition shows both F_i (and $(D_y s(, y))$ are locally constant on $\mathcal{S}_t(y) = \mathcal{S}_{t'}(y)$, hence we may choose the ranges of M_t^{-1} and $M_{t'}^{-1}$ to coincide with any n-dimensional subspace intersecting the tangent space to $\mathcal{S}_t(y)$ at x transversally; the compositions $M_{t'}M_t^{-1}$ (and AM_i^{-1}) of interest are independent of this choice, and $M_{t'}^{-1}M_{t'}$ acts as the identity on the chosen subspace (which could, for example, be the subspace of directions normal to $T_x\mathcal{S}_i(y)$).

The second-order conditions coming from duality (6) *combine with* (33) *to show the quadratic form AM_i^{-1} is symmetric and non-negative* *definite on T_yY. Being full rank, it is positive definite. If λ is an eigenvalue, $M_{t'}M_t^{-1}v = \lambda v \in T_yY \setminus \{0\}$, then*

$$v'AM_t^{-1}v = v'AM_{t'}^{-1}M_{t'}M_t^{-1}v = \lambda v'AM_{t'}^{-1}v$$

But since $v'AM_tv$ and $v'AM_{t'}v$ are both positive, their ratio is too, showing $\lambda > 0$ as desired. ■

In summary, Proposition 15 states that the stable maps $F_{t'}$ and F_t corresponding to two markets t' and t must be compatible in levels, in the sense that if two level sets intersect then they coincide (locally, or in the nested case, globally). Proposition 18 adds that the maps must also have some compatibility in directions transversal to the level sets.

3.2 Testing the pseudo-index property

The previous test becomes much stronger in the case when the surplus depends on the female (multivariable) traits only through a pseudo-index (single-variable) structure (28). In that case, the equation defining the set $X_t(y)$ is of the form $I(x) = H_t^{-1}(y)$ for some monotone function H_t, as was shown in the proof of Proposition 14. Thus we obtain:

Corollary 19 *Let s be non-degenerate and pseudo-index on connected open sets $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}$, with $\sigma \in C^2$ and $I \in C^1$ in (28). Fix t and t'. Assume μ_t' concentrates no mass on C^1 hypersurfaces. Outside the countable set $\Sigma_t \subset Y$ of Proposition 15, if $y \in Y \setminus \Sigma_t$ then*

$$x, x' \in X_t(y) \Rightarrow \text{there exists } y' \text{ such that } x, x' \in X_{t'}(y').$$

Proof. *For $y \in Y \setminus \Sigma_t$, Proposition 15 yields $k_t(y)$ for which each $x \in X_t(y)$ satisfies*

$$\sigma_y(I(x), y) = k_t(y). \tag{34}$$

As in the proof of Proposition 14, the non-degeneracy of s yields $DI \neq 0$ and $\sigma(i, y)$ either strictly super- or strictly sub-modular. Thus $\sigma_y(i, y) \neq \sigma_y(i', y)$ unless $i = i'$, so we conclude $x, x' \in X_t(y)$ forces $I(x) = I(x')$.

On the other hand, s-stable matchings are sensitive only to the index $I(x)$ and not the type of each woman. So the above x and x' are *interchangeable from this point of view; they receive the same type of husband as long as the matching is pure. The stable match between* $I_{\#}\mu_{t'}$ *and* $\nu_{t'}$ *is positive or negative assortative, according to super- or sub-modularity of* σ*. As long as* $I_{\#}\mu_{t'}$ *has no atoms, it is pure (and given by some monotone function* $H_{t'}$*). But the possibility of atoms is ruled out by the implicit function theorem, using the facts that* $DI \neq 0$*, and* $\mu_{t'}$ *concentrates no mass on hypersurfaces. Purity of the s-stable match between* $\mu_{t'}$ *and* $\nu_{t'}$ *follows: it is given by the matching function* $x \in X \longrightarrow H_{t'}(I(x))$. ■

To summarize: in nested models, the main testable prediction was that if two women are matched to the same man in two different markets t and t', then *any* woman matched with that man in t is also matched with him in t'. In the pseudo-index case, we get a much stronger result, namely that *the iso-husband surfaces are the same in all markets* - although they may not be associated to the same husband. That is, if two women x and x' are matched with positive probability to the same husband y in one market (meaning that they are viewed as perfect substitutes for that particular market), then they are matched with positive probability to the same husband y' in any market, although y' is typically market-specific (in particular, y' typically differs from y). This property, which is readily testable, reflects the essence of the pseudo-index property; namely, the way any woman is perceived on the marriage market does not depend on the husband's identity or characteristics, so that if two women are viewed as perfect substitutes in one market, then this remains true for any market.

3.3 Surplus identification in the nested case

We finally consider the identification problem. Assume that we can observe iso-husband sets in a multiple market setting; what does it tell us about the surplus? We now give a precise answer to that question. As already noted, if we only observe matching patterns, then the surplus s can be identified at best up to an additive function of x and an additive function of y. That is, we can, at best, identify what [Chiappori, Salanié & Weiss (2017)] call the 'supermodular core' of the surplus function. Still, insofar as one is interested in matching patterns, the supermodular core contains all necessary information.

It is clear, however, that identifying s (up to a pair of additive functions) from the sole observation of matching patterns is not feasible without additional assumptions. To see why, remember that an arbitrary iso-husband set, with an equation of the form $y = F(x)$, lies in a level set

$$\frac{\partial s(x, F(x))}{\partial y} = k \tag{35}$$

for some constant $k = v'(y)$. A first conclusion, therefore, is that knowing the map F for a single pair (μ, ν) tells the direction (but not the magnitude) of $D_x s_y$ along the graph of F.

Not surprisingly, a multi-market perspective gives additional identification power. Namely, the supermodular core will be (locally) identified up to a mapping from $\mathbf{R}^2$ to $\mathbf{R}$. To see why, let us choose a fixed $\bar{y}$, and assume that we observe all the iso-husband curves corresponding to $\bar{y}$ for various distributions — in practice, thus, for different levels k. By (35), if s and $\bar{s}$ are two surplus functions generating the same family of iso-husband curves, then the functions $\partial s/\partial y(\cdot, \bar{y})$ and $\partial \bar{s}/\partial y(\cdot, \bar{y})$ have (locally) the same level sets. Now, the set of continuous functions with the same level sets as a given function is exactly the set of monotonic transforms of that function. In other words, a function $\partial s/\partial y(\cdot, y)$ has the same level sets as $\partial \bar{s}/\partial y(\cdot, y)$ if and only if:

$$\frac{\partial s(x, y)}{\partial y} = H\left(\frac{\partial \bar{s}(x, y)}{\partial y}, y\right) \tag{36}$$

for some H that is monotonic in its first argument. We conclude that if $\bar{s}$ is the surplus generating the given iso-husband sets, then another non-degenerate surplus s generates the same iso-husband sets if and only if there exists a function $H(z, y)$ with $H_z > 0$ such that:

$$s(x, y) = s(x, \bar{y}) + \int_{\bar{y}}^{y} H\left(\frac{\partial \bar{s}(x, t)}{\partial y}, t\right) dt.$$

implying that s is determined up to the function H (plus two additive functions of x and y respectively, as argued before).

4 Unobserved valuations and the normal-quadratic approach

4.1 Unobserved valuations

We now adopt a more directly applied perspective, by considering a multidimensional matching model in which some traits are unobservable (and therefore summarized by a stochastic vector). As a preliminary remark, note that there is virtually no hope to identify a model of this type from the sole observation of matching patterns from a non parametric perspective. The models we consider, therefore, are heavily parametric, although some of the main assumptions may be relaxed in a multi-market context.

4.1.1 An introductory example

As an introductory example, assume that while male characteristics $y \in Y \subset \mathbf{R}^n$ are observable, each woman is characterized by a pair (x, ε) where $x \in \mathbf{R}^m$ is a vector of observable traits and $\varepsilon \in \mathbf{R}^n$ a random vector independent of x, reflecting the person's unobservable preferences for a mate. Total surplus takes the separable form:

$$S\left(x, \varepsilon; y\right) = s(x, y) + y'\varepsilon$$

Intuitively, each component ε_i of ε can be interpreted as the woman's idiosyncratic valuation of the ith characteristic of a potential husband. Then first order conditions give:

$$Dv\left(y\right) = D_y s\left(x, y\right) + \varepsilon \tag{37}$$

Assume, now, that $s\left(x, y\right)$ is linear in y:

$$s\left(x, y\right) = y'\Psi\left(x\right)$$

for some mapping Ψ from $\mathbf{R}^m$ to $\mathbf{R}^n$. Then (37) becomes:

$$D_y v(y) = \Psi(x) + \varepsilon$$

If the mapping $\Phi : y \to D_y v(y)$, from $\mathbf{R}^n$ to $\mathbf{R}^n$, is invertible, one gets:

$$y = \Phi^{-1}(\Psi(y) + \varepsilon) \tag{38}$$

which is a multidimensional transformation model, and it is possible, under specific assumptions, to non parametrically identify the mappings Φ and Ψ (see [Chiappori, Komunjer & Kristensen (2015)]). For instance, if Ψ is linear and x, y and ε are normally distributed, then Φ is also linear, and (38) becomes a standard system of linear regressions that can be estimated using a Seemingly Unrelated Regressions (SUR) approach, as in [Chiappori, Oreffice & Quintana-Domeque (2012)].

Lastly, in the index case:

$$\Psi(x) = \psi(I(x))$$

then (38) implies that the conditional distribution of y given x only depends on $I(x)$, which can be directly tested either structurally or in reduced form. In the linear version, in particular, the regression matrix is of rank 1.

4.1.2 The general case: basic assumptions

The previous example can actually be generalized. In what follows, we consider a specific but extremely convenient stochastic structure, that directly extends the standard model of Choo and Siow (2006, from now on CS). The basic idea is reminiscent of the applied IO literature; it posits that individuals *on both sides of the market* differ by both their observable characteristics and their subjective valuation of the observable characteristics of the potential partners. Assume, for instance, that men and women differ by human capital and physical attractiveness, both of which are observable;[18] in addition, a particular person may

[18] Or can be proxied by observable traits, for instance income (or education) and BMI.

have his/her own valuation of each of the partner's traits, and these idiosyncratic preferences are not observed by the econometrician (although they are assumed observable by the agents).

This leads to the following set of assumptions:

(a) Each agent is characterized by a vector of traits, some of which are unobservable by the econometrician (and will therefore be considered as stochastic shocks). The observable traits of women and men are $x \in X \subset \mathbf{R}^{n_X}$ and $y \in Y \subset \mathbf{R}^{n_Y}$, respectively, where the sets X and Y are open and bounded. The unobservable traits are $\varepsilon \in \mathbf{R}^{n_\varepsilon}$ and $\eta \in \mathbf{R}^{n_\eta}$, respectively. Note that observable traits are assumed continuous.

(b) The surplus takes the separable form

$$S(x, \varepsilon; y, \eta) = s(x, y) + f(y, \varepsilon) + g(x, \eta)$$

where f and g are known (and s is to be identified). An important particular case, directly borrowed from the IO literature, obtains when both f and g are scalar products:

$$f(y, \varepsilon) = \sum_k y_k \varepsilon_k \quad \text{and } g(x, \eta) = \sum_l x_l \eta_l$$

(c) The marginals distributions $\mu_{x,\varepsilon}$ and $\nu_{y,\eta}$ are known. Note, in particular, that the unobservable characteristics may be correlated to the observable one. In that case, however, the correlation structure must (for the moment) be known a priori.[19]

Lastly, we need some technical assumptions; specifically, we assume that the model is regular, in the following sense:

(d) For each fixed x and y, the conditional probability $\mu_{\varepsilon|x}$ and $\nu_{\eta|y}$ are absolutely continuous with respect to Lebesgue measure. (Note that $\mu_{\varepsilon|x}$ is a probability measure on $\mathbb{R}^{n_\varepsilon}$, representing the distribution of unobservable types for the fixed observable type x).

[19] As in CS, an assumption of this type is necessary for the model to be identified in the single market case.

(e) The conditional probabilities $\gamma_{x|y}$ and $\gamma_{y|x}$ (which, by definition, are directly observable from available data) are absolutely continuous with respect to Lebesgue measure.

(f) For any fixed x, the function $f(y,\varepsilon)$ is bi-twisted; that is,

$$y \mapsto D_\varepsilon f(y,\varepsilon) \text{ and } \varepsilon \mapsto D_y f(y,\varepsilon)$$

are injective. Similarly, $g(x,\eta)$ is bi-twisted.

Although obviously specific, this form encompasses most approaches that have been considered so far in the literature. In particular, it directly generalizes CS, who consider the specific case where y_k (resp. x_l) is a category indicator and ε_k (η_l) is type 1 extreme value. Galichon and Salanié (2017) generalize CS by replacing the type 1 extreme value by any (known) distribution; still, they exclusively assume that y_k and x_l are category indicators, whereas in our context they can be any observable (discrete or continuous) variable.[20] We believe that the extension to continuous variables is particularly important. Note also that similar forms are classically used in empirical IO models, with a scalar product for f and g and specific distributional assumptions (typically normality) for the ε and η.[21]

We now show the following result:

Proposition 20 (Identification of deterministic contribution to surplus) *Under assumptions (a)-(f), in the single-market case, the surplus function $s(x,y)$ is identified up to an additive function of x and an additive function of y*

[20]Dupuy and Galichon (2014) and Bojilov and Galichon (2016) use a related but different approach, based on a continuous extension of logit models initially proposed by McFadden (1976); in their framework, each man of a given type only knows a random subset of the total population of women, and exclusively considers potential partners within this subset.

[21]An obvious difference with IO models is that in general, some parameters of the distributions (such as the covariance matrix) can be estimated from the data, which is not possible in our context, at least in the single market case. The difference is due to the fact that IO models typically observe prices, whereas we assume here that transfers are not observed. However, stronger identification results obtain in the multi-market case, as we shall see later on.

Proof. *Let $u(x,\varepsilon)$ and $v(y,\eta)$ be the (unobserved) payoff functions. For almost every fixed x, if (x,ε,y,η) belong to the support of the stable matching, we have:*

$$D_x u(x,\varepsilon) = D_x s(x,y) + D_x g(\eta,x). \tag{39}$$

and:

$$D_\varepsilon u(x,\varepsilon) = D_\varepsilon f(\varepsilon,y) \tag{40}$$

Now, let $\gamma_{y|x}$ and $\gamma_{\varepsilon|x}$ be the conditional probabilities of y and ε, for fixed x. Both are known ($\gamma_{y|x}$ is directly observable, while $\gamma_{\varepsilon|x} = \mu_{\varepsilon|x}$ is known from the marginal $\mu_{x,\varepsilon}$). For almost every fixed x, the conditional probability $\gamma_{y,\varepsilon|x}$ must be an optimal coupling between $\gamma_{y|x}$ and $\gamma_{\varepsilon|x}$, for the surplus function $f(\varepsilon,y)$ (note that for fixed x, η shows up in the surplus only in a separable way). As f is bi-twisted, $\gamma_{y,\varepsilon|x}$ is determined uniquely and is pure; let $\varepsilon := \epsilon(x,y)$ be the optimal map from $\gamma_{y|x}$ to $\gamma_{\varepsilon|x}$. The bi-twist in fact tells us this map is invertible; we write the inverse as $y = \theta(x,\varepsilon)$.

Note then that by (39) we get

$$D_\varepsilon u(x,\varepsilon) = D_\varepsilon f(\varepsilon,\theta(x,\varepsilon)). \tag{41}$$

Integrating over ε then determines (for a fixed x) $u(x,\cdot)$ up to a function of x. By a symmetric argument, we can determine $\eta = \eta(x,y)$, and so (39) gives us, for any $(x,y) \in spt(\gamma_{x,y})$

$$D_x s(x,y) = D_x u(x,\varepsilon(x,y)) - D_x g(\eta(x,y),x) \tag{42}$$

So $D_x s(x,y)$ is determined on the support of $\gamma_{x,y}$, up to a function of x only. If the support is all of $X \times Y$, then $s(x,y)$ is identified everywhere up to a function of the form $\phi(x) + \psi(y)$. ■

It is useful, at that point, to remember that the surplus cannot possibly be identified better then up to an additive function of x and an additive function of y; the identification result proved in Proposition 20 is thus the best one can hope for. In fact, the choice of $\phi(x)$ and $\psi(y)$

is essentially a normalization; a standard procedure is to normalize to zero the surplus of singles.

The idea of using optimal transportation methods in econometrics in not new; the interested reader is referred to a recent survey by Galichon (2016a). However, to the best of our knowledge, the approach just described is new for matching models, and provides a natural generalization of existing methods.

A related approach can be found in the line of research on identification of hedonic models, where agents match based on their preferences for a good they exchange rather than each other. In that context, the closest analogue of the present model is found in [Chernozhukov, Galichon, Henry & Pass (2015)]. A similar functional form is considered, in which the preference of an agent of observable type x and unobservable type ϵ for a good z has a known functional dependence on ϵ and an additively separable dependence on x and z to be identified. The econometrician observes the matching between x and z as well as the price $p(z)$ (which is a dual potential in the optimal transport problem between (x, ϵ) and z). Using the fact that for fixed x, the conditional marginals of good z (by observation) and unobserved type ϵ (by assumption) are known, one can recover the matching between them by stability. The marginal preferences for this problem, together with the known price then determine the unknown term in the original preference function.

On the other hand, in the present matching problem, we observe only matching patterns and not dual potentials (which now both correspond to utilities). This means that one must also use information involving the coupling between x and unobservable types on the other side of the market to determine marginal preferences $D_x s(x, y)$. This can be thought of as a continuous analogue of the argument of [Galichon & Salanié (2012)] in the discrete case.

A last remark will be useful for what follows. Start from equation:

$$y = \theta(x, \varepsilon). \tag{43}$$

Since the joint distribution of (x, ε) is known, the function θ (or equivalently the function ϵ) can in principle be estimated from the joint distribution of (x, y), which is empirically observable, using standard statistical tools (such as quantile regressions);[22] and the same is true of function η. This indicates that, using (41) and (42), the surplus function can be *constructively* recovered (up to an additive function of x and an additive function of y) from the joint distribution of observables. In the normal case considered below, (43) is actually linear, so the model can even be estimated by (a set of) ordinary least squares regressions - which basically recaptures the approach adopted by Chiappori, Oreffice and Quintana-Domeque (2012) in the index case.

4.1.3 The multi-market case

The identification result provided by Proposition 20 is much stronger that what we obtained from a 'pure theory' perspective in Subsection 3. There, by (36), the partial derivative of the surplus, $\partial s/\partial y$, was only identified (up to a function from $\mathbf{R}^2$ to $\mathbf{R}$) in the multimarket case. Here, the same derivative is identified up to a function from $\mathbf{R}$ to $\mathbf{R}$ ($\psi'(y)$ in the previous notations), which is moreover *additive*, even in the single market context. This additional identifying power is obviously due to the highly specific functional form we are using - a conclusion in line with previous findings in the empirical literature on matching,[23] and a direct generalization of CS.

In a multimarket context, not only do these results apply, but they typically generate additional, testable restrictions. For instance, considering several markets with different marginal distributions of observables, we may, following [Chiappori, Salanié & Weiss (2017)], assume that while the surplus function may differ across markets, its *supermodular core* (that these authors interpret as 'preferences for assortativeness') remains constant. In practice, this means that if s_t and $s_{t'}$ are the sur-

[22] In the case $m = n = 1$, for instance, one can readily check that

$$\epsilon(x, y) = F_{\varepsilon|x}^{-1}\left(F_{y|x}(x, y)\right)$$

where $F_{y|x}$ denote the conditional CDF of y given x, and $F_{\varepsilon|x}$ denote the conditional CDF of ε given x. The statistical problem then boils down to (non parametrically) estimating the conditional distribution $F_{y|x}$.

[23] See for instance the survey by [Chiappori & Salanié (2016)].

pluses corresponding to markets t and t' respectively, we must have:

$$D^2_{xy} s_t (x, y) = D^2_{xy} s_{t'} (x, y)$$

which implies that

$$s_t (x, y) = s_{t'} (x, y) + \phi_{tt'} (x) + \psi_{tt'} (y)$$

Since, by Proposition 20, each market allows us to identify the surplus up to two additive functions of x and y respectively (i.e., to exactly identify the matrix $D^2_{xy} s$ of second cross partials), this compatibility condition introduces a very strong testable condition that directly generalizes the properties of iso-husband surfaces derived earlier. Alternatively, one may, as in [Chiappori, Salanié & Weiss (2017)], exploit this increase in identification power and consider a more general model, including for instance a parametrization of the various random processes. While it is obviously difficult to provide general results, the analysis of specific examples should remain high on the agenda for future research. The next subsection provides a first step in this direction by considering the case where all distributions are normal and f and g are scalar products.

4.2 The Normal Quadratic model

4.2.1 The setting

We now analyze a particular case of the previous construct, which we believe could be very important from an empirical perspective. Specifically, we make the following assumptions:

- Women are characterized by a vector $(x, \varepsilon) \in \mathbf{R}^m \times \mathbf{R}^n$, where x is observable and ε is not; similarly, men are characterized by a vector $(y, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$, where y is observable and η is not.

- The vector x is normally distributed, with mean M_x and covariance matrix Σ_{xx}; similarly, the vector y is normally distributed, with mean M_y and covariance matrix Σ_{yy}.

- The components of the random variables are normal, independent of each other and of observables, and normally distributed with mean 0 and variance 1.

- The surplus generated by the matching of Mrs. (x, ε) and Mr. (y, η) takes the form:

$$S(x, \varepsilon; y, \eta) = s(x, y) + x'\eta + y'\varepsilon \tag{44}$$

Some comments can be made on this framework. First, the assumptions made on the random shocks are highly parametric; indeed, we assume normality, independence and homoskedasticity. Neither independence nor homoskedasticity are necessary for our analysis; what is crucial, however, is that the distribution of unobservables be *known a priori*. In that sense, our assumption are exactly reminiscent of CS. As we shall see later on, the requirement that distributions are known a priori can be relaxed in a multi-market context; then it may be possible to estimate a general covariance matrix for unobservable shocks. Secondly, we do not assume anything on the structure of the surplus function; our goal, indeed, is to *derive* the properties of s from the specific features of the joint distribution of x and y, which is observable.

4.2.2 Main result

Our main result is the following:

Theorem 21 *Assume that the optimal matching is such that (x, y) are joint normally distributed with mean and covariance matrix*

$$M = \begin{pmatrix} M_x \\ M_y \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{xx} \, \Sigma_{xy} \\ \Sigma_{yx} \, \Sigma_{yy} \end{pmatrix}$$

Then the surplus is quadratic:

$$D^2_{xy} s(x, y) = \Delta$$

where Δ is a $m \times n$ constant matrix. Moreover, Δ is fully determined *by M and Σ. Specifically:*

$$\Delta = \Sigma_{yy}^{-1'} \Sigma_{yx} \Sigma_{x|y}^{-1/2} + \Sigma_{y|x}^{-1/2} \Sigma_{yx} \Sigma_{xx}^{-1} \tag{45}$$

where $\Sigma_{x|y}$ (resp. $\Sigma_{y|x}$) is the covariance matrix of the (normal) conditional distribution of x given y (resp. y given x); that is:

$$\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{xy}'$$

and

$$\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{yx}'$$

Proof. *Recall that the conditional distribution $\gamma_{x|y}$ of x for a fixed y is also normal with mean $M_{x|y} = M_x + \Sigma_{xy} \Sigma_{yy}^{-1} (y - M_y)$ and covariance $\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{xy}'$, while when x is fixed the distribution $\gamma_{y|x}$ of y is normal with $M_{y|x} = M_y + \Sigma_{yx} \Sigma_{xx}^{-1} (x - M_x)$ and covariance $\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{yx}'$. Now, as before, for a fixed y, the matching $\gamma_{x|\eta,y}$ between $\gamma_{x|y}$ and $\gamma_{\eta|y}$ (normal with mean zero and covariance $\Sigma_\eta = I$), is stable. As $\gamma_{x|y}$ is normal, we get the optimal map taking the form:*

$$\eta(x) = \Sigma_{x|y}^{-1/2} (x - M_{x|y}) \tag{46}$$

Similarly, the matching between y and ε for a fixed x takes the form:

$$\varepsilon(y) = \Sigma_{y|x}^{-1/2} (y - M_{y|x}) \tag{47}$$

Now, we have

$$D_\eta v(y, \eta) = x = \Sigma_{x|y}^{1/2} \eta + M_{x|y} \tag{48}$$

so that (neglecting an additive function of y)

$$v(y, \eta) = \frac{1}{2} \eta' \Sigma_{x|y}^{1/2} \eta + \eta' M_{x|y}$$

Note that the only y dependence here is through the $M_{x|y}$ term, and so

$$D_y v(y, \eta) = \Sigma_{yy}^{-1'} \Sigma_{xy}' \eta = \Sigma_{yy}^{-1} \Sigma_{yx} \eta.$$

Therefore,

$$
\begin{aligned}
D_y s(x,y) &= D_y v(y,\eta) - \varepsilon \\
&= \Sigma_{yy}^{-1\prime}\Sigma_{yx}\eta - \Sigma_{y|x}^{-1/2}(y - M_{y|x}) \\
&= \Sigma_{yy}^{-1}\Sigma_{yx}\Sigma_{x|y}^{-1/2}(x - M_{x|y}) - \Sigma_{y|x}^{-1/2}\left(y - M_y - \Sigma_{yx}\Sigma_{xx}^{-1}(x - M_x)\right) \\
&= \Sigma_{yy}^{-1}\Sigma_{yx}\Sigma_{x|y}^{-1/2}(x - M_x - \Sigma_{xy}\Sigma_{yy}^{-1}(y - M_y)) \\
&\quad -\Sigma_{y|x}^{-1/2}\left(y - M_y - \Sigma_{yx}\Sigma_{xx}^{-1}(x - M_x)\right)
\end{aligned}
$$

Differentiating yields

$$D_{xy}^2 s(x,y) = \Delta = \Sigma_{yy}^{-1}\Sigma_{yx}\Sigma_{x|y}^{-1/2} + \Sigma_{y|x}^{-1/2}\Sigma_{yx}\Sigma_{xx}^{-1}$$

■

It is well known that if we assume normality of observable and unobservable characteristics and a quadratic surplus, then the distribution of observable characteristics over married couples is normal. Our result shows that the converse is also true; i.e., assuming normality of observable and unobservable characteristics, *joint* normality of couples' observable characteristics *requires* a quadratic surplus. Moreover, the surplus is then exactly identified from matching patterns, and one can recover a closed-form solution for the surplus. The proof exactly follows the path described in the previous subsection. A crucial remark is that, as in CS, the observation of a single joint distribution of observed characteristics (the 'single market' case) exactly identifies the surplus without generating any overidentifying restriction.

4.2.3 Estimation through OLS regressions

In most existing contributions, the empirical analysis of matching models typically relies on a direct estimation of the matching model (using for instance maximum likelihood or simulated moments estimators). While such an approach is still available in our context, one can equivalently adopt an alternative approach, initially introduced by [Chiappori, Oreffice & Quintana-Domeque (2012)] for the case of index models. The idea is to directly regress, over the population of couples, male observable characteristics over the characteristics of the spouse, and conversely. This method is easy to implement, and has been used in a variety of cases (see for instance Ong and Zhang 2016). As we shall see, our framework provides a direct and general justification for this approach, even when the index assumption is not satisfied. To see why, start with equation (48) above:

$$
\begin{aligned}
x &= \Sigma_{x|y}^{1/2}\eta + M_{x|y} \\
&= \Sigma_{xy}\Sigma_{yy}^{-1}y + M_x - \Sigma_{xy}\Sigma_{yy}^{-1}M_y + \Sigma_{x|y}^{1/2}\eta
\end{aligned}
\tag{49}
$$

where the second equation obtains by replacing $M_{x|y}$ with its value. Since, according to our assumptions, η is independent of y, this equation can be estimated by Ordinary Least Square (OLS):

$$
x = Ay + B + \tilde{\eta} \tag{50}
$$

with

$$
A = \Sigma_{xy}\Sigma_{yy}^{-1} \text{ and } cov\left(\tilde{\eta}\right) = \Sigma_{\tilde{\eta}} = \Sigma_{x|y}
$$

We conclude that, in equation (45), the first component of the matrix Δ can be recovered from the outcome of the OLS regression (50):

$$
\Sigma_{yy}^{-1}\Sigma_{yx}\Sigma_{x|y}^{-1/2} = A' cov\left(\tilde{\eta}\right)^{-1/2}
$$

The same result applies, mutatis mutandis, for the OLS regression of y on x; this identifies the second component of Δ. We can therefore conclude:

Proposition 22 *Under the joint normality assumption, the surplus is uniquely determined from the two OLS regressions of x over y and of y over x respectively.*

It is important to note that in our framework, the 'characteristic regressions' approach is justified in general; in particular, it does not require an index assumption (as considered by COQ). Yet, index models can readily be tested in this framework, as described next.

4.2.4 The index case

In the normal setting, the index assumption has a simple translation, as expressed by the following result:

Proposition 23 *Assume the surplus function has an index form in x:*

$$s(x,y) = \tilde{s}(I(x), y)$$

where I maps $\mathbf{R}^m$ to $\mathbf{R}$. If the optimal matching is such that (x,y) are joint normally distributed, then the surplus function is quadratic, and the corresponding Δ matrix is of rank 1. Equivalently, there exists two vectors

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

such that

$$\Delta = \alpha\beta'$$

so that:

$$s(x,y) = (x'\alpha)(\beta' y)$$

Proof. *Consider the characterization provided by equation (25); with a quadratic surplus, it becomes*

$$\frac{\partial}{\partial y_k}\left(\frac{\partial s/\partial x_t}{\partial s/\partial y_l}\right) = \frac{\partial}{\partial y_k}\left(\frac{\sum_r \delta_{rt} y_t}{\sum_r \delta_{rl} y_l}\right) = 0 \quad \forall y$$

which requires that there exists a scalar α_r and a vector β such that

$$\delta_{rt} = \alpha_r \beta_t$$

for all (r,t). ■

This result, in turn, has an immediate corollary:

Corollary 24 *In the normal framework, if the surplus function has an index form in one of the vectors of characteristics, it also has an index* *form in the other; namely:*

$$s(x,y) = I(x)\,J(y) \quad \textit{where } I(x) = x'\alpha \textit{ and } J(y) = \beta' y$$

With a quadratic surplus, the index assumption is highly restrictive; indeed, it imposes that the matrix Δ be of rank 1. If this is the case, then both x and y enter the surplus through a one-dimensional, linear index. Note also that this property can be readily tested from the characteristic regressions described in the previous subsection; this is exactly the test introduced by [Chiappori, Oreffice & Quintana-Domeque (2012)].

4.2.5 The multi-market context

The framework We now consider a direct extension of the model to a multi-market context. We therefore assume that the econometrician can observe matching data from several 'markets', indexed by $t \in \{1, ..., T\}$. In practice, markets may differ by period (as in [Chiappori, Salanié & Weiss (2017)]) or by geographic location. We make the following assumptions:

- Women in market t are characterized by a vector $(x_t, \varepsilon_t) \in \mathbf{R}^m \times \mathbf{R}^n$, where x_t is observable and ε_t is not; similarly, men in market t are characterized by a vector $(y_t, \eta_t) \in \mathbf{R}^n \times \mathbf{R}^m$, where y_t is observable and η_t is not.

- The vector x_t is normally distributed, with mean M_{xt} and covariance matrix Σ_{xxt}; similarly, the vector y_t is normally distributed, with mean M_{yt} and covariance matrix Σ_{yyt}.

- The random variables $\varepsilon_t, t = 1, ..., T$ are drawn from the same normal distribution, with mean 0 and unknown covariance matrices Σ_ε. Similarly, the random variables η_t are drawn from the same normal distribution, with mean 0 and unknown covariance matrices Σ_η. The vectors ε_t and η_s are independent from x_t and y_t, respectively.

- The surplus generated by the matching of Mrs. (x, ε) and Mr.

(y, η) takes the form:

$$S_t(x_t, \varepsilon_t; y_t, \eta_t) = s(x_t, y_t) + \phi_t(x_t) + \psi_t(y_t) + x_t' \eta_t + y_t' \varepsilon_t$$

Some remarks can be made about these assumptions. Essentially, we relax the iid assumption regarding the unobservable shocks; we now consider general (and unknown) covariance matrices. Time independence is a strong assumption, that we believe could be somewhat relaxed by the introduction of a specific dynamic structure, although we do not investigate this aspect here. On the other hand, independence between observables and unobservables is crucial for our results; relaxing it would require a set of instruments, an issue that we briefly discuss later on. Lastly, the surplus may entail arbitrary functions of either male or female observables; the main identifying assumption, here, is that the *supermodular core* $D_{xy}S_t$ does not depend on t. Again, this assumption could be relaxed, for instance by introducing a linear trend a la [Chiappori, Salanié & Weiss (2017)]. We also mention that the underlying goal here, using matching patterns in multiple markets to determine information about the distributions of unobserved characteristics, is the same as the goal in [Fox, Yang & Hsu (2018)], though in a different setting and under different assumptions.

The main result The previous result can be generalized as follows:

Theorem 25 *Assume that the optimal matching is such that (x_t, y_t) are joint normally distributed with mean and covariance matrix*

$$M_t = \begin{pmatrix} M_{xt} \\ M_{yt} \end{pmatrix}, \Sigma_t = \begin{pmatrix} \Sigma_{xxt} & \Sigma_{xyt} \\ \Sigma_{yxt} & \Sigma_{yyt} \end{pmatrix}$$

Then the surplus is quadratic:

$$D^2_{xy} S_t(x_t, y_t) = D^2_{xy} s(x_t, y_t) = \Delta$$

where Δ is a $m \times n$ constant matrix. Moreover, Δ is fully determined *by $M_t, \Sigma_\varepsilon, \Sigma_\eta$ and Σ_t. Specifically:*

$$D^2_{yx}s(x_t,y_t) = \Sigma_{yyt}^{-1\prime}\Sigma_{xyt}\Sigma_{x|yt}^{-1/2}\left(\Sigma_{x|yt}^{1/2}\Sigma_\eta\Sigma_{x|yt}^{1/2}\right)^{1/2}\Sigma_{x|yt}^{-1/2}+\Sigma_{y|xt}^{-1/2}\left(\Sigma_{y|xt}^{1/2}\Sigma_\varepsilon\Sigma_{y|xt}^{1/2}\right)^{1/2}\Sigma_{y|xt}^{-1/2}\Sigma_{yxt}\Sigma_{xxt}^{-1}$$

Proof. *The proof is a direct generalization of the previous one. The optimal matching $\gamma_{x|\eta,y}$ between $\gamma_{x|y}$ and $\gamma_{\eta|y}$ now takes the form:*

$$\eta(x) = \Sigma_{x|y}^{-1/2}\left(\Sigma_{x|y}^{1/2}\Sigma_\eta\Sigma_{x|y}^{1/2}\right)^{1/2}\Sigma_{x|y}^{-1/2}(x-M_{x|y}) \tag{51}$$

while the matching between y and ε for a fixed x becomes:

$$\varepsilon(y) = \Sigma_{y|x}^{-1/2}\left(\Sigma_{y|x}^{1/2}\Sigma_\varepsilon\Sigma_{y|x}^{1/2}\right)^{1/2}\Sigma_{y|x}^{-1/2}(y-M_{y|x}) \tag{52}$$

Again:

$$D_\eta v(y,\eta) = x = \Sigma_{x|y}^{1/2}\left(\Sigma_{x|y}^{1/2}\Sigma_\eta\Sigma_{x|y}^{1/2}\right)^{-1/2}\Sigma_{x|y}^{1/2}\eta + M_{x|y}$$

so that (neglecting an additive function of y)

$$v(y,\eta) = \frac{1}{2}\eta'\Sigma_{x|y}^{1/2}\left(\Sigma_{x|y}^{1/2}\Sigma_\eta\Sigma_{x|y}^{1/2}\right)^{-1/2}\Sigma_{x|y}^{1/2}\eta + \eta' M_{x|y}$$

Since the only y dependence is through the $M_{x|y}$ term, we have:

$$D_y v(y,\eta) = \Sigma_{yy}^{-1\prime}\Sigma_{xy}'\eta = \Sigma_{yy}^{-1}\Sigma_{yx}\eta.$$

Therefore,

$$\begin{aligned}
D_y s(x,y) &= D_y v(y,\eta) - \varepsilon \\
&= \Sigma_{yy}^{-1\prime}\Sigma_{xy}\eta - \Sigma_{y|x}^{-1/2}\left(\Sigma_{y|x}^{1/2}\Sigma_\varepsilon\Sigma_{y|x}^{1/2}\right)^{1/2}\Sigma_{y|x}^{-1/2}(y-M_{y|x}) \\
&= \Sigma_{yy}^{-1}\Sigma_{xy}\Sigma_{x|y}^{-1/2}\left(\Sigma_{x|y}^{1/2}\Sigma_\eta\Sigma_{x|y}^{1/2}\right)^{1/2}\Sigma_{x|y}^{-1/2}(x-M_{x|y}) \\
&\quad -\Sigma_{y|x}^{-1/2}\left(\Sigma_{y|x}^{1/2}\Sigma_\varepsilon\Sigma_{y|x}^{1/2}\right)^{1/2}\Sigma_{y|x}^{-1/2}\left(y-M_y-\Sigma_{yx}\Sigma_{xx}^{-1}(x-M_x)\right) \\
&= \Sigma_{yy}^{-1}\Sigma_{xy}\Sigma_{x|y}^{-1/2}\left(\Sigma_{x|y}^{1/2}\Sigma_\eta\Sigma_{x|y}^{1/2}\right)^{1/2}\Sigma_{x|y}^{-1/2}(x-M_x-\Sigma_{xy}\Sigma_{yy}^{-1}(y-M_y)) \\
&\quad -\Sigma_{y|x}^{-1/2}\left(\Sigma_{y|x}^{1/2}\Sigma_\varepsilon\Sigma_{y|x}^{1/2}\right)^{1/2}\Sigma_{y|x}^{-1/2}\left(y-M_y-\Sigma_{yx}\Sigma_{xx}^{-1}(x-M_x)\right)
\end{aligned}$$

and differentiating yields the announced result. ■

Identifying the covariance matrices In a single-market context, this general model is obviously under-identified: for any choice of Σ_ε and Σ_η, there exists a surplus whose supermodular core generates exactly the correlations observed in the data. Things are however different in a multi-market framework, since by assumption the various markets correspond to the same covariance matrices Σ_ε and Σ_η and the same supermodular core $D^2_{yx}s$. This property generates a set of additional restrictions, that may in general allow for full identification of the unknown covariance matrices.

Two remarks must however be made at that point. First, and quite obviously, identification obtains at best up to a multiplicative constant. Indeed, in equation (44), replacing S with λS, which is equivalent to multiplying s by λ and the covariance matrices of ε and η by λ^2, would generate the same stable matching. We therefore need a normalization (for instance, the variance of the first component of ε is set to 1).

Secondly, a general proof of identification is bound to be intricate. Indeed, equating the estimated supermodular cores over the various markets generates a system of equations of the general form:

$$AB^{-1}(B\Sigma_\eta B)^{1/2}B^{-1} + C^{-1}(C\Sigma_\varepsilon C)^{1/2}C^{-1}D \tag{53}$$

$$= A'B'^{-1}(B'\Sigma_\eta B')^{1/2}B'^{-1} + C'^{-1}(C'\Sigma_\varepsilon C')^{1/2}C'^{-1}D' \tag{54}$$

where A, B, C, D and A', B', C', D' are recovered from the observed data on two different markets. Any matrix equation of this form generates $m \times n$ equations, in the $\left(\frac{n(n+1)}{2} + \frac{m(m+1)}{2} - 1\right)$ unknown components of the covariance matrices Σ_ε and Σ_η. When

$$T > \frac{1}{mn}\left(\frac{n\,(n+1)}{2} + \frac{m\,(m+1)}{2} - 1\right) + 1$$

then we have more equations than unknowns; under a non degeneracy condition, the matrices are actually overidentified.[24] In particular, with

[24] For instance, we only need $T > 1$ when $m = n$, and $T > (n+1)/2$ when $m = 1$.

'enough' markets, different extensions (such as a linear trend in the supermodular core, as in Chiappori, Salanié and Weiss 2017) can then be considered.

In various special cases, however, we are able to obtain exact identification results.

The one dimensional case The first of these is the one dimensional setting, $m = n = 1$. Here we adopt the lower case notation: in market t, the covariance matrix for the observed matching between x and y is

$$\sigma_t = \begin{bmatrix} \sigma_{xxt} \, \sigma_{xyt} \\ \sigma_{yxt} \, \sigma_{yyt} \end{bmatrix}.$$

Proposition 26 *Assume $m = n = 1$, and we have data from $T = 2$ markets. Then the model is uniquely identified up to a normalization if and only if*

$$\frac{\sigma_{xy1}}{\sqrt{\sigma_{yy1} det(\sigma_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{yy2} det(\sigma_2)}}$$

and

$$\frac{\sigma_{xy1}}{\sqrt{\sigma_{xx1} det(\sigma_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{xx2} det(\sigma_2)}}$$

are not both zero and, if both are non-zero, have different signs.

Proof. *In the one dimensional case, the matrix equations in each market become scalar equations and simplify to*

$$s_{xy} = \frac{a_t}{b_t}\sqrt{\sigma_\eta} + \frac{d_t}{c_t}\sqrt{\sigma_\epsilon} \tag{55}$$

where

$$\frac{a_t}{b_t} = \frac{\sigma_{xyt}}{\sigma_{yyt}} \frac{1}{\sqrt{\sigma_{x|yt}}} = \frac{\sigma_{xyt}}{\sigma_{yyt}} \frac{1}{\sqrt{\sigma_{xxt} - \frac{\sigma_{xyt}^2}{\sigma_{yyt}}}} = \frac{\sigma_{xyt}}{\sqrt{\sigma_{yyt} det(\sigma_t)}}$$

and similarly,

$$\frac{d_t}{c_t} = \frac{\sigma_{xyt}}{\sqrt{\sigma_{xxt} det(\sigma_t)}}.$$

Now, assuming without loss of generality that

$$\frac{\sigma_{xy1}}{\sqrt{\sigma_{xx1}det(\sigma_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{xx2}det(\sigma_2)}} \neq 0$$

we choose the normalization $\sigma_\eta = 1$*, and equate the expressions for* s_{xy} *in the two markets to obtain*

$$\frac{\sigma_{xy2}}{\sqrt{\sigma_{yy2}det(\sigma_1)}} - \frac{\sigma_{xy1}}{\sqrt{\sigma_{yy1}det(\sigma_2)}} = [\frac{\sigma_{xy1}}{\sqrt{\sigma_{xx1}det(\sigma_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{xx2}det(\sigma_2)}}]\sqrt{\sigma_\epsilon},$$

or

$$\sqrt{\sigma_\epsilon} = \frac{\frac{\sigma_{xy2}}{\sqrt{\sigma_{yy2}det(\sigma_1)}} - \frac{\sigma_{xy1}}{\sqrt{\sigma_{yy1}det(\sigma_2)}}}{\frac{\sigma_{xy1}}{\sqrt{\sigma_{xx1}det(\sigma_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{xx2}det(\sigma_2)}}}.$$

Clearly $\sqrt{\sigma_\epsilon}$ *must be non-negative, which holds only under the conditions stated in the theorem. The supermodular core is then determined by* (55)*.*
■

This immediately implies the following, intuitive, easily testable prediction of the model.

Corollary 27 *Under the previous assumptions, if* x *and* y *are positively correlated in one market, they cannot be negatively correlated in another.*
Proof. *The conditions* $\sigma_{xy1} > 0$ *and* $\sigma_{xy2} < 0$ *mean that both*

$$\frac{\sigma_{xy1}}{\sqrt{\sigma_{yy1}det(\delta_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{yy2}det(\delta_2)}}$$

and

$$\frac{\sigma_{xy1}}{\sqrt{\sigma_{xx1}det(\delta_1)}} - \frac{\sigma_{xy2}}{\sqrt{\sigma_{xx2}det(\delta_2)}}$$

are positive. ■

One could also infer this more directly; the equation is of the form $s_{xy} = \sigma_{xyt}C_t$, where the constant C_t differs across markets but is always positive. Therefore, s_{xy} shares a sign with σ_{xyt}, and so the sign of the later must agree across all markets.

In fact, a generalization of this result holds without the normality assumption on the data; see Proposition 30 below.

The multi- to one-dimensional case The last result above has an analogue in the $m > n = 1$ case; note that in this setting the observed covariance matrices Σ_{xyt} are vectors in $\mathbf{R}^m$. Although the result below generically will not rule out the model unless $T > m$ (in which case we expect the model to be over identified) it can provide useful negative information in specific cases (for example, when the observed correlations $\Sigma_{xyt} = -\Sigma_{xyt'}$ point in opposite directions in two markets, or when the variables are uncorrelated in one, but not all, markets).

Proposition 28 *Assume $m > n = 1$. If we have data from T markets, such that at least one $\Sigma_{xyt} \neq 0$ and $\sum_{t=1}^{T} \alpha_t \Sigma_{xyt} = 0$, for some non-negative constants α_i, at least one of which is non-zero, then the model is over identified.*

Proof. *In each market, the equation can be written as*

$$D^2_{xy}s = P_t \Sigma_{xyt} + p_t \Sigma_{xyt}$$

where P_t is a (unknown) symmetric, positive definite $m \times m$ matrix and $p_t > 0$ a positive constant (both P_t and p_t differ across markets). We therefore have, for each t,

$$\Sigma_{xyt} \cdot D^2_{xy}s \geq p_t |\Sigma_{xyt}|^2$$

Therefore, $D^2_{xy}s \neq 0$, as at least one $\Sigma_{xyt} \neq 0$. It follows immediately that $\Sigma_{xyt} \neq 0$ for all t (as this would make $D^2_{xy}s = 0$); we then have

$$0 = D^2_{xy}s \cdot \sum_{t=1}^{T} \alpha_t \Sigma_{xyt} \geq \sum_{t=1}^{T} \alpha_t p_t |\Sigma_{xyt}|^2 > 0.$$

This contradiction means that we cannot have simultaneous solutions to the equations for each market. ■

We next consider the special case where $m > n = 1$ and the observed conditional variance matrix $\Sigma_{x|yt}$ is the same across all observed markets. Although this is a very restrictive assumption, we believe the analysis of the special case is still compelling and may provide a hint at the behaviour of more general solutions. In particular, as we show below, there are very strong testable restrictions, and somewhat counter-intuitively, identification is *never* possible with less than $T = m$ markets.

Under this assumption, with the normalization $\sigma_\epsilon = 1$, the equation in each market reduces to

$$\frac{1}{\sigma_{yyt}} M\Sigma_{xyt} + \frac{1}{\sqrt{\sigma_{y|xt}}}\ \Sigma_{xxt}^{-1}\Sigma_{xyt} = D_x s_y$$

where $M = \Sigma_{x|yt}^{-1/2}\left(\Sigma_{x|yt}^{1/2}\Sigma_\eta\Sigma_{x|yt}^{1/2}\right)^{1/2}\Sigma_{x|yt}^{-1/2}$ is a symmetric and positive definite matrix to be identified, which is the same across markets by assumption. Equating the left hand sides of these equations yields a system of $m-1$ equations of the form

$$MV_t = W_t, \text{ for } 2 \le t \le T \tag{56}$$

where $V_t = \frac{1}{\sigma_{yyt}}\Sigma_{xyt} - \frac{1}{\sigma_{yy1}}\Sigma_{xy1}$, and $W_t = \frac{1}{\sqrt{\sigma_{y|x1}}}\ \Sigma_{xx1}^{-1}\Sigma_{xy1} - \frac{1}{\sqrt{\sigma_{y|xt}}}\ \Sigma_{xxt}^{-1}\Sigma_{xyt}$, for $2 \le t \le T$, are known from the observed data.

Proposition 29 *System* (56) *has a unique, non-negative definite solution M if and only if the following hold:*

1. *The V_t span $\mathbb{R}^m$.*
2. *Whenever $\sum_{t=1}^T \alpha_t V_t = 0$, for scalars α_t, $\sum_{t=1}^T \alpha_t W_t = 0$,*
3. *The $T \times T$ matrix whose (t, t') entry is $V_t \cdot W_{t'}$ is symmetric and positive semi-definite.*

Proof. *The proof is standard linear algebra; conditions 1) and 2) ensure that there is a unique, $m \times m$ matrix M solving the system of equations. Conditions 3) then implies that that solution will be symmetric and non-negative definite, respectively.*

Finally, in the case when condition 1) fails but the others hold, there will be multiple solutions M to (56)*; we must rule out the possibility that there is a unique symmetric one. In this case, there is a non trivial vector U in the orthogonal complement of span$\{V_2, V_3, ..., V_T\}$. After a rotation, we may assume that $U = (0, 0, ..., 0, 1)$; therefore, the last* *component of each V_t must be 0. The equations and symmetry of M then tell us nothing about the last diagonal entry M_{mm} of M, and so the model is under identified.*

The first two conditions typically hold when $T = m+1$, in which case the V_t will generically form a basis for $\mathbf{R}^m$. The third condition, on the other hand, is much more restrictive; for generic data, it fails as soon as $T \geq 3$, in which case we have a system of two equations, and will typically not have $V_2 \cdot W_3 = V_3 \cdot W_2$. Therefore, for $T \geq 3$, the system is generically over identified. ■

A second remark is that identification is *only* possible when the number of markets at least $m+1$, even though in this case the number of variables Tm is strictly larger than the number of unknowns $\frac{m(m+1)}{2}$. This is because, as is evident from the proof above, knowing MV_t on a non-spanning set of vectors $\{V_t\}$ is not sufficient to uniquely determine M, even with the extra constraint that M must be symmetric.

A testable prediction without the normality assumption We now show that a generalization of Corollary 27 holds beyond the normal assumption on the data. In this setting, we assume again that $m = n = 1$ and the surplus function takes the form

$$S(x, y, \epsilon, \eta) = s(x, y) + y\epsilon + x\eta$$

with s to be indentified. We assume again that we have data from at least two markets, that observable and unobservable characteristics are independent in each market, and the unobservable distributions are the same across markets, but remove the normality assumption on the data. We assume instead only that the observed coupling $\gamma(x, y)$ is absolutely continuous.

Recall that $F_{x|y}(x, y)$ and $F_{y|x}(x, y)$ denote the conditional CDFs. Under our assumption $F_{\epsilon|x} = F_\epsilon$ and $F_{\eta|y} = F_\eta$ are independent of x and y, respectively, and constant throughout markets.

Proposition 30 *Fix a point (x, y) and two markets t and t'. Assume that γ^t and $\gamma^{t'}$ are absolutely continuous with positive densities at (x, y).*

It is not possible to have:

$$\frac{\partial F^t_{y|x}(x,y)}{\partial x} > 0,\ \frac{\partial F^t_{x|y}(x,y)}{\partial y} > 0,\ \frac{\partial F^{t'}_{y|x}(x,y)}{\partial x} < 0,\ \textit{and}\ \frac{\partial F^{t'}_{x|y}(x,y)}{\partial y} < 0,$$

These conditions are local measures of correlation between x and y. For instance, $F_{y|x}(x,y)$ tell us the proportion of wives who marry husband y whose type is less than x; we can interpret a decrease in this quantity as y increases as a local in y positive correlation between the variables.
Proof. In each market, equation (42) becomes

$$D_x s(x,y) = D_x u(x,\epsilon(x,y)) - \eta(x,y)$$

so that differentiating and using (41) gives

$$D^2_{xy}s(x,y) = D^2_{x\epsilon}u(x,\epsilon(x,y))D_y\epsilon(x,y) - D_y\eta(x,y) = D_x\theta(x,\epsilon(x,y))D_y\epsilon(x,y) - D_y\eta(x,y).$$

Therefore, with data from markets t and t', we must have, at each (x,y),

$$D_x\theta^t(x,\epsilon^t(x,y))D_y\epsilon^t(x,y) - D_y\eta^t(x,y) = D_x\theta^{t'}(x,\epsilon^{t'}(x,y))D_y\epsilon^{t'}(x,y) - D_y\eta^{t'}(x,y). \tag{57}$$

Now, each $\epsilon^t(x,y) = F^{-1}_\epsilon(F^t_{y|x}(x,y))$ is an optimal map from y to ϵ for a fixed x, with surplus $f(y,\epsilon) = y\epsilon$; it is therefore clearly increasing in y.

Differentiating the relationship $F^{-1}_{y|x}(x, F_{y|x}(x,y)) = y$ with respect to x, we have

$$\frac{\partial F^{-1}_{y|x}}{\partial x}(x, F_{y|x}(x,y)) + \frac{\partial F^{-1}_{y|x}}{\partial q}(x, F_{y|x}(x,y))\frac{\partial F_{y|x}(x,y)}{\partial x}(x,y) = 0$$

where we use q as a name for the second argument of $F^{-1}_{y|x}$. Since $\frac{\partial F^{-1}_{y|x}}{\partial q}(x,q) > 0$ everywhere, the sign of $\frac{\partial F^{-1}_{y|x}}{\partial x}(x, F_{y|x}(x,y))$ is opposite that of $\frac{\partial F_{y|x}(x,y)}{\partial x}(x,y)$. On the other hand, since $\eta(x,y) = F^{-1}_\eta(F_{x|y}(x,y))$ and $F^{-1}_\eta(\eta)$ is clearly increasing, the sign of $\frac{\partial \eta}{\partial y}$ is the same as that of $\frac{\partial F_{x|y}}{\partial y}$.

This means that, under our assumptions, $\theta^t(x,\epsilon) = (F^t_{y|x})^{-1}(x, F_\epsilon(\epsilon))$ has a negative derivative with respect to x while $\eta^t(x,y) = F^{-1}_\eta(F^t_{x|y}(x,y))$

has a positive derivative with respect to y.

The left hand side of (57) is therefore negative. By a similar argument, the right hand side is positive, which is impossible.

■

5 Conclusion

This paper provides a general characterization of multidimensional matching models, in terms of existence, uniqueness and qualitative properties of stable matches, as well as testability and identifiability of a stochastic version involving unobserved preferences. Of specific interest are situations in which the dimensions of heterogeneity on the two sides of the market are unequal. We explore the topology of the 'indifference sets' that arise in this setting, and provide conditions under which they can be expected to be smooth manifolds of dimension $m-n$. In particular, we investigate the set of 'multi-to-one dimensional matching problems', and we introduce a nestedness criterion under which the equilibrium match can be found more or less explicitly.

Lastly, we introduce an empirical specification aimed at capturing unobserved dimensions in the matching process. This formulation generalizes existing approaches, and particularly the seminal contribution of [Choo & Siow (2006)], to the case of continuous variables; it also reconciles the Choo and Siow methodology with an alternative approach, initially proposed by COQ for the specific case of index models. In the single market case, our empirical model is exactly identified under strong parametric assumptions. In a multi-market context, on the contrary, a more general version of the model, involving a general stochastic structure for unobservable preferences, is typically over-identified. One can only hope that these new insights will soon be taken to data.

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