# Structural results on optimal transportation plans

by

Brendan Pass

A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

# Abstract

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2011

In this thesis we prove several results on the structure of solutions to optimal transportation problems.

The second chapter represents joint work with Robert McCann and Micah Warren; the main result is that, under a non-degeneracy condition on the cost function, the optimal is concentrated on a $n$-dimensional Lipschitz submanifold of the product space. As a consequence, we provide a simple, new proof that the optimal map satisfies a Jacobian equation almost everywhere. In the third chapter, we prove an analogous result for the multi-marginal optimal transportation problem; in this context, the dimension of the support of the solution depends on the signatures of a $2^{m-1}$ vertex convex polytope of semi-Riemannian metrics on the product space, induce by the cost function. In the fourth chapter, we identify sufficient conditions under which the solution to the multi-marginal problem is concentrated on the graph of a function over one of the marginals. In the fifth chapter, we investigate the regularity of the optimal map when the dimensions of the two spaces fail to coincide. We prove that a regularity theory can be developed only for very special cost functions, in which case a quotient construction can be used to reduce the problem to an optimal transport problem between spaces of equal dimension. The final chapter applies the results of chapter 5 to the principal-agent problem in mathematical economics when the space of types and the space of available goods differ. When the dimension of the space of types exceeds the dimension of the space of goods, we show if the problem can be formulated as a maximization over a convex set, a quotient procedure can reduce the problem to one where the two dimensions coincide. Analogous conditions are investigated when the dimension of the space of goods exceeds that of the space of types.

# Dedication

To Cristen

## Acknowledgements

I am deeply indebted to my thesis advisor, Robert McCann, for his patient guidance and support. Working with Robert has been both enjoyable and extremely profitable for me and I owe much of my development as a mathematician to him. This thesis could certainly not have been completed without his help.

I gratefully acknowledge the help of my thesis committee members, Jim Colliander and Bob Jerrard, as well as my external examiner Wilfrid Gangbo, all of whom provided insightful suggestions and feedback on this work at various stages of its completion. Stimulating discussions and useful remarks were also provided by Martial Agueh, Luigi Ambrosio, Almut Burchard, Guillaume Carlier, Man-Duen Choi, Chandler Davis, Young-Heon Kim, Paul Lee, Lars Stole and Micah Warren. The first chapter of this thesis represents joint work with Robert McCann and Micah Warren.

It is a pleasure to thank the staff in the Department of Mathematics at the University of Toronto, and in particular Ida Bulat, for helping make every aspect of my PhD run as smoothly as possible. I am also grateful to all the friends and family members who lent their support during the course of my degree.

Finally, I acknowledge that a refined version of Chapter 3 has been accepted for publication by *Calculus of Variations and Partial Differential Equations*, DOI : 10.1007/s00526-011-0421-z.

# Contents

# Chapter 1

# Introduction

## 1.1 Background on optimal transportation

The optimal transportation problem asks what is the most efficient way to transform one distribution of mass to another relative to a given cost function. The problem was originally posed by Monge in 1781 [59]. In 1942, Kantorovich proposed a relaxed version of the problem [41]; roughly speaking, he allowed a piece of mass to be split between two or more target points. Since then, these problems have been studied extensively by many authors and have found applications in such diverse fields as geometry, fluid mechanics, statistics, economics, shape recognition, inequalities and meteorology.

Much of this thesis focuses on a multi-marginal generalization of the above; how do we align $m$ distributions of mass with maximal efficiency, again relative to a prescribed cost function. Precisely, given Borel probability measures $\mu_i$ on smooth manifolds $M_i$ of respective dimensions $n_i$, for $i = 1, 2..., m$ and a continuous cost function $c : M_1 \times M_2 \times .... \times M_m \to \mathbb{R}$, the multi-marginal version of Monge's optimal transportation problem is to minimize:

$$C(G_2, G_3, ..., G_m) := \int_{M_1} c(x_1, G_2(x_1), G_3(x_1), ..., G_m(x_1)) d\mu_1 \qquad (\mathbf{M})$$

among all $(m-1)$-tuples of measurable maps $(G_2, G_3, ..., G_m)$, where $G_i : M_1 \to M_i$

pushes $\mu_1$ forward to $\mu_i$, $G_{\#}\mu_1 = \mu_i$, for all $i = 2, 3, ..., m$. The Kantorovich formulation of the multi-marginal optimal transportation problem is to minimize

$$C(\mu) = \int_{M_1 \times M_2 ... \times M_m} c(x_1, x_2, ..., x_m) d\mu \tag{K}$$

among all measures $\mu$ on $M_1 \times M_2 ... \times M_m$ which project to the $\mu_i$ under the canonical projections; that is, for any Borel subset $A \subset M_i$,

$$\mu(M_1 \times M_2 \times .... \times M_{i-1} \times A \times M_{i+1} .... \times M_m) = \mu_i(A).$$

For any $(m-1)$-tuple $(G_2, G_3, ..., G_m)$ such that $G_{i\#}\mu_1 = \mu_i$ for all $i = 2, 3, ..., m$, we can define the measure $\mu = (Id, G_2, G_3, ..G_M)_{\#}\mu_1$ on $M_1 \times M_2 \times ... \times M_m$, where $Id : M_1 \to M_1$ is the identity map. Then $\mu$ projects to $\mu_i$ for all $i$ and $C(G_2, G_3, ..., G_m) = C(\mu)$; therefore, **K** can be interpreted as a relaxed version of **M**. Roughly speaking, the difference between the two formulations is that in **M** almost every point $x_1 \in M_1$ is coupled with exactly one point $x_i \in M_i$ for each $i = 2, 3, ..., m$, whereas in **K** an element of mass at $x_i$ is allowed to be *split* between two or more target points in $M_i$ for $i = 2, 3, ..., m$. When $m = 2$, these are precisely the Monge and Kantorovich formulations of the classical optimal transportation problem.

Under mild conditions, a minimizer $\mu$ for **K** will exist. Over the past two decades, a great deal of research has been devoted to understanding the structure of these solutions. When $m = 2$, under a regularity condition on $\mu_1$ and a twist condition on $c$, which we will define subsequently, Levin showed that this solution is concentrated on the graph of a function over $x_1$, building on results of Gangbo [35], Gangbo and McCann [36] and Caffarelli [14]. It is then straightforward to show that this function solves **M** and to establish uniqueness results for both **M** and **K**. More recently, in the case where $n_1 = n_2$, understanding the regularity, or smoothness, of the optimal map, has grown into an active and exciting area of research, due to a major breakthrough by Ma, Trudinger and Wang [52]. They identified a fourth order differential condition on $c$ (called **(A3s)** in the literature) which implies the smoothness of the optimizer, provided the marginals $\mu$ and

$\nu$ are smooth. Subsequent investigations by Trudinger and Wang [71, 70] revealed that these results actually hold under a slight weakening of this condition, called **(A3w)**, encompassing earlier results of Caffarelli [16][15][17], Urbas [72] and Delanoe [25, 26] when $c$ is the distance squared on either $\mathbb{R}^n$ or certain Riemannian manifolds, and Wang for another special cost function [73]. Loeper [49] then verified that **(A3w)** is in fact necessary for the solution to be continuous for arbitrary smooth marginals $\mu$ and $\nu$. Loeper also proved that, under **(A3s)**, the optimizer is Holder continuous even for rougher marginals; this result was subsequently improved by Liu [48], who found a sharp Holder exponent. Since then, many interesting results about the regularity of optimal transportation have been established [43][44][50][51][32][34][33][29][30].

A striking development in the theory of optimal transportation over the last 15 years has been its interplay with geometry. Recently, the insight that intrinsic properties of the solution $\mu$, such as the regularity of Monge solutions, should not depend on the coordinates used to represent the spaces has been very fruitful. The natural conclusion is that understanding these properties is related to tensors, or coordinate independent quantities. The relevant tensors encode information about the way that the cost function and the manifolds interact. For example, Kim and McCann [43] introduced a pseudo-Riemannian form on the product space, derived from the mixed second order partial derivatives of the cost, whose sectional curvature is related to the regularity of Monge solutions; they also noted that smooth solutions must be timelike for this form.

Whereas the two marginal problem is relatively well understood, results concerning the structure of these optimal measures have thus far been elusive for $m > 2$. Much of the progress to date has been in the special case where the $M_i$'s are all Euclidean domains of common dimension $n$ and the cost function is given by $c(x_1, x_2, ..., x_m) = \sum_{i \neq j} |x_i - x_j|^2$, or equivalently $c(x_1, x_2, ..., x_m) = -|(\sum_i x_i)|^2$. When $n = 3$, partial results for this cost were obtained by Olkin and Rachev [62], Knott and Smith [45] and Rüschendorf and Uckelmann [66], before Gangbo and Święch proved that for a general $m$, under a mild regularity condition on the first marginal, there is unique solution to the Kantorovich problem and it is concentrated on the graph of a function over $x_1$, hence inducing a solution to a Monge type problem [37]; an alternate proof of Gangbo and Święch's theorem was subsequently found by Rüschendorf and Uckelmann [67]. This result was then extended by Heinich to cost functions of the form $c(x_1, x_2, ..., x_m) = h(\sum_i x_i)$ where $h$ is strictly concave [39] and, in the case when the domains $M_i$ are all 1-dimensional, by Carlier [19] to cost functions satisfying a strict 2-monotonicity condition. More recently, Carlier and Nazaret [21] studied the related problem of maximizing the determinant (or its absolute value) of the matrix whose columns are the elements $x_1, x_2, ..., x_n \in \mathbb{R}^n$; unlike the results in [37],[39] and [19], the solution in this problem may not be concentrated on the graph of a function over one of the $x_i$'s and may not be unique. The proofs of many of these results exploit a duality theorem, proved in the multi-marginal setting by Kellerer [42]. Although this theorem holds for general cost functions, it alone says little about the structure of the optimal measure; the proofs of each of the aforementioned results rely heavily on the special forms of the cost.

The final chapter of this thesis focuses on the application of optimal transportation to the principal-agent problem in economics. Problems of this type frequently in a variety of different contexts in mathematical economic theory. The following formulation can be found in Wilson [74], Armstrong [7] and Rochet and Chone [64]. A monopolist wants to sell goods to a distribution of buyers. Knowing only the preference $b(x, y)$ that a buyer of type $x \in X$ has for a good of type $y \in Y$, the density $d\mu(x)$ of the buyer types and the cost $c(y)$ to produce the good $y$, the monopolist must decide which goods to produce and how much to charge for them in order to maximize her profits.

When the distribution of buyer types $X$ and the available goods $Y$ are either discrete or 1-dimensional, this problem is well understood [69][58][61][8]. However, it is typically more realistic to distinguish between both consumer types and goods by more than one characteristic. An illuminating illustration of this is outlined by Figalli, Kim and

McCann [31]: consumers buying automobiles may differ by, for instance, their income and the length of their daily commute, while the vehicles themselves may vary according to their fuel efficiency, safety, comfort and engine power, for example. It is desirable, then, to study models where the respective dimensions $n_1$ and $n_2$ of $X$ and $Y$ are greater than 1 [53][63][65]. This *multi-dimensional* screening problem is much more difficult and relatively little is known about it; for a review and an extensive list of references, see the book by Basov [10].

When $n_1 = n_2$ and the preference function $b(x,y) := x \cdot f(y)$ is linear in types, Rochet and Chone [64]developed an algorithm for studying this problem. A key element in their analysis is that, in this case, the problem may be formulated mathematically as an optimization problem over the set of convex functions, which is itself a convex set. They were then able to deduce the existence and uniqueness of an optimal pricing strategy, as well as several interesting economic characteristics of it. Basov then analyzed the case where $b$ is linear in types but $n_1 \neq n_2$ [9]. When $n_1 < n_2$, he was able to essentially reduce the $n_2$-dimensional space $Y$ to an $n_1$-dimensional space of *artificial* goods and then apply the machinery of Rochet and Chone. When $n_1 > n_2$, no such reduction is possible in general. Under additional hypotheses, however, he showed that the solution actually coincides with the solution to a similar problem where both spaces are $n_1$-dimensional.

For more general preference functions, Carlier, using tools from the theory of optimal transportation, was able to formulate the problem as the maximization of a functional $P$ over a certain set of functions $U_{b,\phi}$ (a subset of the so called $b$-convex functions, which will be defined below) [18]. He was then able to assert the existence of a solution to this problem; that is, the existence of an optimal pricing schedule; an equivalent result is also proven in [60]. However, for general functions $b$, the set of $b$-convex functions may not be convex and so characterizing the solution using either computational or theoretical tools is an extremely imposing task. Very little progress had been made in this direction until recently, when Figalli, Kim and McCann [31] found necessary and sufficient conditions on $b$ for $U_{b,\phi}$ to be convex, assuming $n_1 = n_2$. Assuming in addition that the cost $c$ is $b$ convex, they then demonstrated that the functional $P$ is concave and from here were able to prove uniqueness of the solution and demonstrate that some of the interesting economic features observed by Rochet and Chone persist in this setting. Surprisingly, the tools they use are also adapted from an optimal transportation context; their necessary and sufficient condition is derived from a condition developed by Ma, Trudinger and Wang [52], governing the regularity of optimal maps.

## 1.2 Overview of Results

This thesis consists of 6 chapters, including the introduction. The second chapter represents joint work with Robert McCann and Micah Warren and focuses on the case two marginal problem when $n_1 = n_2 := n$. We study what can be said about the solution under a certain non-degeneracy condition on the cost function, which was originally introduced in an economic context by McAfee and McMillan [53] and later rediscovered by Ma, Trudinger and Wang [52]; in the terminology of Ma, Trudinger and Wang, it is also known as the **(A2)** condition. The main result is that, under this non-degeneracy condition, the optimal measure concentrates on an $n$-dimensional, Lipschitz submanifold of $M_1 \times M_2$ (see Theorem 2.0.2).

The proof of this theorem is based on an idea of Minty [57], which was also used by Alberti and Ambrosio to show that the graph of any monotone function $T : \mathbb{R}^n \to \mathbb{R}^n$ is contained in a Lipschitz graph over the diagonal $\Delta = \{u = \frac{x+y}{\sqrt{2}} : (x, y) \in \mathbb{R}^n \times \mathbb{R}^n\}$ [4].

The non-degeneracy condition can be viewed as a linearized version of the twist condition, which asserts that the mapping $y \in M^- \longmapsto D_x c(x, y)$ is injective. Under suitable regularity conditions on the marginals, Levin [46] showed that the twist condition ensures that the solution to the Kantorovich problem is concentrated on the graph of a function and is therefore unique; see also Gangbo [35].

In one dimension, non-degeneracy implies twistedness, as was noted by many authors, including Spence [69] and Mirrlees [58], in the economics literature; see also [56]. In higher dimensions, this is no longer true; the non-degeneracy condition will imply that the map $y \in M^- \longmapsto D_x c(x, y)$ is injective locally but not necessarily globally. Non-degeneracy was a hypothesis in the smoothness proof in [52], but does not seem to have received much attention in higher dimensions before then. While our result demonstrates that the non-degeneracy condition is enough to ensure that solutions still have certain regularity properties, we will show by example that the uniqueness result that follows from twistedness can fail for non-degenerate costs which are not twisted. The twist condition is asymmetric in $x$ and $y$; that is, there are cost functions for which the map $y \in M^- \longmapsto D_x c(x, y)$ is injective but $x \in M^+ \longmapsto D_y c(x, y)$ is not. However, since $(D^2_{xy} c)^T = D^2_{yx} c$ the non-degeneracy condition is certainly symmetric in $x$ and $y$. In view of this, it is not surprising that the twist condition can only be used to show solutions are concentrated on the graphs of functions of $y$ over $x$ whereas the non-degeneracy condition implies solution are concentrated on $n$-dimensional submanifolds, a result that does not favour either variable over the other.

Smooth optimal maps solve certain Monge-Ampère type equations. Typically, an optimal map will be differentiable almost everywhere, but may not be smooth. It has proven useful to know when non-smooth optimal maps solve the corresponding equations almost everywhere. Formally, the link between optimal transportation and these equations was observed by Brenier [12], then Gangbo and McCann [36], and they were studied in detail by Ma, Trudinger and Wang [52]. An important step in showing that an optimal map solves a Monge-Ampère type equation is first showing that it solves the Jacobian — or change of variables — equation. An injective Lipschitz function satisfies the change of variables formula almost everywhere, so some sort of Lipschitz rectifiability for the graphs of optimal maps is a useful tool in resolving this question. As an application of Theorem 2.0.2, we provide a simple proof that optimal maps satisfy the change of variables formula almost everywhere.

This work is related to another interesting line of research. A measure $\mu$ on the product $M^+ \times M^-$ is called *simplicial* if it is extremal among the convex set of all measures which share its marginals. There are a number of results describing simplicial measures and their supports [28][47][11][40][3]. One consequence is that the support of simplicial measures are in some sense small; in particular, the support of a simplicial measure on $[0, 1] \times [0, 1]$ must have two-dimensional Lebesgue measure zero [47][40]. Although any measure supported on the graph of a function is simplicial, it is known that there exist functions whose graphs have Hausdorff measure $2 - \epsilon$, for any $\epsilon > 0$ [1]. For any cost, the Kantorovich functional is linear and is hence minimized by some simplicial measure. Conversely, any simplicial measure is the solution to a Kantorovich problem for some continuous cost function, and so by the remarks above there are continuous cost functions whose optimizers are supported on sets of Hausdorff dimension $2 - \epsilon$. On the other hand, an immediate consequence of our result is that the support of optimizers of Kantorovich problems with non-degenerate $C^2$ costs have Hausdorff dimension at most $n$, ie, at most one in this case.

The result of Ma, Trudinger and Wang proving smoothness of the optimal map under certain conditions immediately implies that the support of the optimizer has Hausdorff dimension $n$; however, the proof of this result requires that the marginals be $C^2$ smooth. Under the same assumptions on the cost functions but weaker regularity conditions on the marginals, Loeper [49] and Liu [48] have demonstrated that the optimal map is Hölder continuous for some Hölder constant $0 < \alpha < 1$. It is worth noting that there are examples of functions on $\mathbb{R}^n$ [1] which are Hölder continuous with exponent $\alpha$ but whose graphs have Hausdorff dimension $n + 1 - \alpha$, so the latter results do not imply that the Hausdorff dimension of the optimizer must be $n$.

The third chapter applies similar techniques to the multi-marginal problem. Precisely, we establish an upper bound on $\dim(spt(\mu))$. This bound depends on the cost function;

however, it will always be greater than or equal to the largest of the $n_i$'s. In the case when the $n_i$'s are equal to some common value $n$, we identify conditions on $c$ that ensure our bound will be $n$ and we show by example that when these conditions are violated, the solution may be supported on a higher dimensional submanifold and may not be unique. In fact, the costs in these examples satisfy naive multi-marginal extensions of both the twist and non-degeneracy conditions; given the aforementioned results in the two marginal case, we found it surprising that higher dimensional solutions can exist for twisted, non-degenerate costs. On the other hand, if the support of at least one of the measures $\mu_i$ has Hausdorff dimension $n$, the remarks above imply that $spt(\mu)$ must be at least $n$ dimensional; therefore, in cases where our upper bound is $n$, the support is exactly $n$-dimensional, in which case we show it is actually $n$-rectifiable.

Unlike the results of Gangbo and Święch, Heinich and Carlier, this contribution does not rely on a dual formulation of the Kantorovich problem; instead, our method uses an intuitive $c$-monotonicity condition to establish a geometrical framework for the problem. The question about the dimension of $spt(\mu)$ should certainly have a coordinate independent answer. Indeed, inspired partially by Kim and McCann, our condition is related to a family of semi-Riemannian metrics[1]; heuristically, $spt(\mu)$ must be timelike for these metrics and so their signatures control its dimension. From this perspective, the major difference from the $m = 2$ case is that with two marginals, the metric of Kim and McCann always has signature $(n, n)$. In the multi-marginal case, there is an entire convex family of relevant metrics, generated by $2^{m-1} - 1$ extreme points, and their signatures may vary depending on the cost.

Like the results in chapter 2 and in contrast to the results of Gangbo and Święch [37], Heinich [39], and Carlier [19], the results in chapter 3 only concern the local structure of the optimizer $\mu$ and cannot be easily used to assert uniqueness of $\mu$ or the existence of a

[1]For the purposes of this paper, the term semi-Riemannian metric will refer to a symmetric, covariant 2-tensor (which is not necessarily non-degenerate). The term pseudo-Riemannian metric will be reserved for semi-Riemannian metrics which are also non-degenerate.

solution to **M**. On the other hand, we do explicitly exhibit fairly innocuous looking cost functions which have high dimensional and non-unique solutions and so it is apparent that these questions cannot be resolved in the affirmative without imposing stronger conditions on $c$.

Question about Monge solutions and uniqueness are addressed in the fourth chapter. We identify general conditions on $c$ under which both **K** and **M** admit unique solutions, generalizing the results of Gangbo and Swiech [37] and Heinich [39]. With one exception, the conditions we impose will look similar to standard conditions which arise when studying the two marginal problem. Our lone novel hypothesis is that a certain covariant 2-tensor on the product space $M_2 \times M_2 \times ... \times M_{m-1}$ should be negative definite. Whereas the question about the dimension of the support of a solution $\mu$ to **K** is purely local, showing that $\mu$ gives rise to a solution to **M** is a global issue: for almost all $x_1 \in M_1$ we must show that there is exactly one $(x_2, x_3, ..., x_m) \in M_2 \times M_3 \times, ..., M_m$ which get coupled to $x_1$ by $\mu$. Our tensor here is designed to capture this global aspect of the problem.

The fifth chapter focuses on the regularity theory of optimal maps when $m = 2$ but $n_1 \neq n_2$. A serious obstacle arises immediately; the regularity theory of Ma, Trudinger, and Wang requires invertibility of the matrix of mixed second order partials $(\frac{\partial^2 c}{\partial x^i \partial y^j})_{ij}$, and its inverse appears explicitly in their formulations of **(A3w)** and **(A3s)**. When $m$ and $n$ fail to coincide, however, $(\frac{\partial^2 c}{\partial x^i \partial y^j})_{ij}$ clearly cannot be invertible. Alternate formulations of the **(A3w)** and **(A3s)** that do not explicitly use this invertibility are known; however, they rely instead on local surjectivity of the map $y \mapsto D_x c(x, y)$, which cannot hold in our setting either.

Nonetheless, there is a certain class of costs for which our problem can easily be solved using the results from the equal dimensional setting. Suppose

$$c(x, y) = b(Q(x), y), \tag{1.1}$$

where $Q : X \to Z$ is smooth and $Z$ is a smooth manifold of dimension $n_2$. In this case, it is not hard to show that the optimal map takes every point in each level set of $Q$ to a common $y$ and studying its regularity amounts to studying an optimal transportation problem on the $n_2$-dimensional spaces $Z$ and $Y$. We will show that costs of this form are essentially the only costs on $X \times Y$ for which we can hope for regularity results for arbitrary smooth marginals $\mu$ and $\nu$. Indeed, for the quadratic cost on Euclidean domains, the regularity theory of Caffarelli requires convexity of the target $Y$ [16][15] and, for general costs, it became apparent in the work of Ma, Trudinger and Wang [52] that continuity of the optimizer cannot hold for arbitrary smooth marginals unless $Y$ satisfies an appropriate, generalized notion of convexity. Due to its dependence on the cost function, this condition is referred to as $c$-convexity; when $n_1 > n_2$, we will show that $c$-convexity necessarily fails *unless* the cost function is of the form alluded to above.

Given the preceding discussion, it is apparent that for cost functions that are not of the special form (1.1), there are smooth marginals for which the optimal map is discontinuous. However, as the condition (1.1) is so restrictive, it is natural to ask about regularity for costs which are not of this form; any result in this direction will require stronger conditions on the marginals than smoothness. In the final section of chapter 5, we address this problem when $n_1 = 2$ and $n_2 = 1$.

In the sixth and final chapter, we turn our attention to the principal-agent problem. Although the result of Figalli, Kim and McCann [31] represents major progress on this problem, it is limited in that they had to assume that the spaces of types and products were of the same dimension. There are many interesting and relevant economic models in which these spaces have different dimensions, as in outlined in, for example, Basov [10]. Our primary goal here is to study how the results in [31] extend to the case when $n_1 \neq n_2$; in particular, we want to determine under what conditions the set of $b$-convex functions is convex for general values of $n_1$ and $n_2$. Our first contribution is to establish a necessary condition for the convexity of this set. This condition, known as $b$-convexity of $Y$, was a hypothesis in [31]; prior to that, to the best of my knowledge, it had not been explored in the principal-agent context, although it is well known in the optimal transportation literature since the work of Ma, Trudinger and Wang [52].

We then study separately the cases $n_1 > n_2$ and $n_1 < n_2$. The analysis here parallels the work in chapter 5 on the regularity of optimal transportation between spaces whose dimensions differ. When $n_1 > n_2$, we show that the $b$-convexity of $Y$ implies that the dimensions cannot differ in a meaningful way. That is, although $b$ may *appear* to depend on an $n_1$ dimensional variable, there is a natural disintegration of $X$ into smooth sub-manifolds of dimension $n_1 - n_2$ such that, no matter how the monopolist sets her prices, types in the same sub-manifold *always* choose the same good. Therefore, types in the same set are indistinguishable, and rather than working in an $n_1$ dimensional space, we may as well identify the types in a single sub-manifold and work instead in the resulting $n_2$ dimensional quotient space.

When $n_2 > n_1$, consumers' marginal utilities cannot uniquely determine which product they buy, making the problem largely intractable. In this case, given a price schedule, a certain buyer's surplus may be maximized by many different goods, making him indifferent between those goods. The monopolist's profits will be very different, however, depending on which good the buyer chooses. A naive possible solution would be to only produce from the indifference set the good which maximizes the monopolist's profit; however, in doing this a good may be excluded which would maximize her profit from another buyer. It turns out that the $b$-convexity on $X$ (which was also an assumption in [31]) precludes this from happening; under this condition, we can again reduce the problem to one where the two spaces share the same dimension. A special case of this result where $b(x, y) = x \cdot v(y)$ for a function $v : Y \mapsto \mathbb{R}^{n_1}$ was established by Basov [9].

# Chapter 2

# Rectifiability when $m = 2$ and $n_1 = n_2$.

This chapter represents joint work with Robert McCann and Micah Warren. We focus on the two marginal problem when the dimensions $n_1 = n_2 := n$ are equal and study the local structure of the solution $\mu$, assuming a non-degeneracy condition on $c$, which we define below. For simplicity, throughout this chapter we will denote variables in $M_1$ and $M_2$ by $x$ and $y$, respectively, rather than $x_1$ and $x_2$.

In what follows, $D^2_{xy}c(x_0, y_0)$ will denote the $n$ by $n$ matrix of mixed second order partial derivatives of the function $c$ at the point $(x_0, y_0) \in M_1 \times M_2$; its $(i, j)$th entry is $\frac{d^2c}{dx^i dy^j}(x_0, y_0)$.

**Definition 2.0.1.** *Assume $c \in C^2(M_1 \times M_2)$. We say that $c$ is non-degenerate at a point $(x_0, y_0) \in M_1 \times M_2$ if $D^2_{xy}c(x_0, y_0)$ is nonsingular; that is if $det(D^2_{xy}c(x_0, y_0)) \neq 0$.*

For a probability measure $\mu$ on $M_1 \times M_2$ we will denote by $\mathrm{spt}(\mu)$ the support of $\mu$; that is, the smallest closed set $S \subseteq M_1 \times M_2$ such that $\mu(S) = 1$.

Our main result is:

**Theorem 2.0.2.** *Suppose $c \in C^2(M_1 \times M_2)$ and $\mu_1$ and $\mu_2$ are compactly supported; let $\mu$ be a solution of the Kantorovich problem. Suppose $(x_0, y_0) \in spt(\mu)$ and $c$ is non-*

*degenerate at* $(x_0, y_0)$. *Then there is a neighbourhood* $N$ *of* $(x_0, y_0)$ *such that* $N \cap spt(\mu)$ *is contained in an* $n$*-dimensional Lipschitz submanifold. In particular, if* $D^2_{xy}c$ *is nonsingular everywhere,* $spt(\mu)$ *is contained in an* $n$*-dimensional Lipschitz submanifold.*

In the first section, we prove Theorem 2.0.2, while section 2.2 is devoted to discussion and examples. In the final section we use Theorem 2.0.2 to provide a simple proof that optimal maps satisfy a presrcribed Jacobian equation almost everywhere.

## 2.1 Lipschitz rectifiability of optimal transportation plans

We now prove Theorem 2.0.2. Note that $\mu$ minimizes the Kantorovich functional if and only if it maximizes the corresponding functional for $b(x, y) = -c(x, y)$. To simplify the computation, we consider $\mu$ that maximizes $b$.

Our proof relies on the *b-monotonicity* of the supports of optimal measures:

**Definition 2.1.1.** *A subset* $S$ *of* $M_1 \times M_2$ *is* $b$*-monotone if all* $(x_0, y_0), (x_1, y_1) \in S$ *satisfy* $b(x_0, y_0) + b(x_1, y_1) \geq b(x_0, y_1) + b(x_1, y_0)$.

It is well known that the support of any optimizer is $b$-monotone [68], provided that the cost is continuous and the marginals are compactly supported. The reason for this is intuitively clear; if $b(x_0, y_0) + b(x_1, y_1) < b(x_0, y_1) + b(x_1, y_0)$ then we could move some mass from $(x_0, y_0)$ and $(x_1, y_1)$ to $(x_0, y_1)$ and $(x_1, y_0)$ without changing the marginals of $\mu$ and thus increase the integral of $b$.

The strategy of our proof is to change coordinates so that locally $b(x, y) = x \cdot y$, modulo a small perturbation. We then switch to diagonal coordinates $u = x + y, v = x - y$ and show that the monotonicity condition becomes a Lipschitz condition for $v$ as a function of $u$. This trick dates back to Minty who used it to study monotone operators on Hilbert spaces [57]; more recently, Alberti and Ambrosio used it to investigate the fine properties of monotone functions on $\mathbb{R}^n$ [4].

We are now ready to prove Theorem 2.0.2:

*Proof.* Choose $(x_0, y_0)$ in the support of $\mu$. Fix local coordinates for $M_2$ in a neighbourhood of $y_0$ and set $A := D^2_{xy}b(x_0, y_0)$. Then make the local change of coordinates $y \to Ay$. In these new coordinates, we have $D^2_{xy}b(x_0, y_0) = I$. We then have $b(x, y) = x \cdot y + G(x, y)$, where $D^2_{xy}G \to 0$ as $(x, y) \to (x_0, y_0)$. Set $u\sqrt{2} = x + y$ and $v\sqrt{2} = y - x$. Given $\epsilon > 0$, choose a convex neighbourhood $N$ of $(x_0, y_0)$ such that $||D^2_{xy}G|| \leq \epsilon$ on $N$. We will show that $\mu \cap N$ is contained in a Lipschitz graph of $v$ over $u$; hence, $u$ and $v$ serve as local coordinates for our submanifold. Take $(x, y)$ and $(x', y') \in N \cap spt\mu$. Then, by $b$-monotonicity, we have $b(x, y) + b(x', y') \geq b(x, y') + b(x', y)$, hence

$$\begin{aligned} &x \cdot y + G(x, y) + x' \cdot y' + G(x', y') \\ \geq\ &x \cdot y' + G(x, y') + x' \cdot y + G(x', y). \end{aligned}$$

Setting $\Delta x = x' - x$, $\Delta y = y' - y$, $\Delta u = u' - u$, $\Delta v = v' - v$, and rewriting yields

$$(\Delta x) \cdot (\Delta y) + (\Delta x) \cdot \int_0^1 \int_0^1 D^2_{xy}G[x + s\Delta x, y + t\Delta y](\Delta y)dsdt \geq 0 \qquad (2.1)$$

which simplifies to: $\Delta x \cdot \Delta y \geq -\epsilon|\Delta x||\Delta y|$.

Observe that $\Delta y\sqrt{2} = \Delta u + \Delta v$ and $\Delta x\sqrt{2} = \Delta u - \Delta v$. Now,

$$\begin{aligned} |\Delta u|^2 - |\Delta v|^2 &= 2(\Delta x) \cdot (\Delta y) \\ &\geq -2\epsilon|\Delta x||\Delta y| \\ &= -\epsilon|\Delta u - \Delta v||\Delta u + \Delta v| \\ &\geq -\epsilon[|\Delta u|^2 + |\Delta v|^2] \end{aligned}$$

The last inequality follows by squaring the absolute values of each side and expanding the first term. Rearranging yields $(1 + \epsilon)|\Delta u|^2 \geq (1 - \epsilon)|\Delta v|^2$, the desired result.

Note that $v$ may not be everywhere defined; that is, for certain values of $u$ there may be no corresponding $v$ in $spt(\mu)$. However, the function $v(u)$ can be extended by Kirzbraun's theorem and hence we can conclude that $spt(\mu)$ is contained in the graph of a Lipschitz function of $v$ over $u$.

□

**Remark 2.1.1.** *Note that the only property of optimal transportation plans used in the proof is b-monotonicity, so we have actually proven that any b-monotone subset of $M_1 \times M_2$ is contained in an $n$-dimensional Lipschitz submanifold, provided b is non-degenerate.*

## 2.2 Discussion and examples

For twisted costs, one can show that $spt(\mu)$ is concentrated on the graph of a function, provided the marginal $\mu_1$ does not charge sets whose dimension is less than or equal to $n-1$[35] [46] [52] [3] [55] [36][1]; however, this can fail if $\mu_1$ charges small sets. On the other hand, notice that our proof did not require any regularity hypotheses on the marginals.

In the example below, we exhibit a non-degenerate cost which is not twisted. We use this example to illustrate how, in this setting, solutions may be supported on submanifolds which are are not necessarily graphs. In addition, we show that these solutions may not be unique. We can view this example as expressing an optimal transportation problem on a right circular cylinder via its universal cover, which is $\mathbb{R}^2$. The non-twistedness of the cost and non-uniqueness of the solution arise because different points in the universal cover correspond to the same point in the cylinder and are therefore indistinguishable by our cost function. In fact, if we expressed the problem on the cylinder, we would have a twisted cost function and therefore a unique solution.

**Example 2.2.1.** *Let $M_1 = M_2 = \mathbb{R}^2$ and $c(x,y) = e^{x^1+y^1}cos(x^2 - y^2) + \frac{e^{2x^1}}{2} + \frac{e^{2y^1}}{2}$. Then $D_x c(x,y) = (e^{x^1+y^1}cos(x^2 - y^2) + e^{2x^1}, -e^{x^1+y^1}sin(x^2 - y^2))$, so $y \in M_2 \longmapsto D_x c(x,y)$ is*

[1]In fact, this condition on the regularity of $\mu_1$ has recently been sharpened [38].

*not injective and c is not twisted. However, note that* $D^2_{xy}c(x,y) =$

$$\begin{bmatrix} e^{x^1+y^1}cos(x^2-y^2) & e^{x^1+y^1}sin(x^2-y^2) \\ -e^{x^1+y^1}sin(x^2-y^2) & e^{x^1+y^1}cos(x^2-y^2) \end{bmatrix}$$

*Therefore,* $detD^2_{xy}c(x,y) = e^{2(x^1+y^1)} > 0$ *for all* $(x,y)$*, so c is non-degenerate. Optimal measures for c, then, must be supported on 2-dimensional Lipschitz submanifolds, but we will now exhibit an optimal measure whose support is not contained in the graph of a function.*

*Now let M be the union of the three graphs:*

$$G_1 : y^1 = x^1, y^2 = x^2 + \pi \tag{2.2}$$

$$G_2 : y^1 = x^1, y^2 = x^2 + 3\pi \tag{2.3}$$

$$G_3 : y^1 = x^1, y^2 = x^2 + 5\pi \tag{2.4}$$

*Clearly, M is a smooth 2-d submanifold but not a graph. However,* $c(x,y) \geq -e^{x^1+y^1} + \frac{e^{2x^1}}{2} + \frac{e^{2y^1}}{2} \geq \frac{(e^{x^1}-e^{y^1})^2}{2}$ *and we have equality on M. Therefore, any probability measure whose support is concentrated on M is optimal for its marginals.*

*We now show that optimal measures supported on M may not be unique. Let*

$$S = \{((x^1,x^2),(y^1,y^2))|0 \leq x^1 \leq 1, 0 \leq x^2 \leq 4\pi\}$$

*Note that*

$$M \cap S = (G_1 \cap S) \cup (G_2 \cap S) \cup (G_3 \cap S).$$

*consists of 3, flat 2-d regions. Let* $\mu$ *be uniform measure on these regions. Now, let* $\overline{\mu_1}$ *be uniform measure on the the first half of* $G_1 \cap S$*; that is, on*

$$G_1 \cap \{((x^1,x^2),(y^1,y^2))|0 \leq x^1 \leq 1, 0 \leq x^2 \leq 2\pi\}.$$

*Let* $\overline{\mu_3}$ *be uniform measure on the the second half of* $G_3 \cap S$*, or*

$$G_3 \cap \{((x^1,x^2),(y^1,y^2))|0 \leq x^1 \leq 1, 2\pi \leq x^2 \leq 4\pi\}.$$

*Take $\overline{\mu_2}$ to be twice uniform measure on $G_2 \cap S$ and set $\overline{\mu} = \overline{\mu_1} + \overline{\mu_2} + \overline{\mu_3}$. Then $\mu$ and $\overline{\mu}$ share the same marginals and are both optimal measures. Furthermore, any convex combination $t\mu + (1-t)\overline{\mu}$ will also share the same marginals and will be optimal as well.*

The next example is similar in that the cost function is non-degenerate but not twisted. However, this cost *would* be twisted if we exchanged the roles of $x$ and $y$. This demonstrates that, unlike non-degeneracy, the twist condition is not symmetric in $x$ and $y$. For this cost function, solutions will be unique as long as the *second* marginal does not charge small sets.

**Example 2.2.2.** *Let $M_1 = M_2 = \mathbb{R}^2$ and*

$$c(x,y) = -(x^1 \cos(y^1) + x^2 \sin(y^1))e^{y^2} + \frac{e^{2y^2}}{2} + \frac{(x^1)^2 + (x^2)^2}{2}.$$

*Note that $detD^2_{xy}c(x,y) = -e^{2y^2} < 0$, so $c$ is non-degenerate. However, $D_x c(x,y) = (-cos(y^1)e^{y^2} + x^1, -sin(y^1)e^{y^2} + x^2)$, so $y \in M_2 \longmapsto D_x c(x,y)$ is not injective and $c$ is not twisted. On the other hand, $D_y c(x,y) = ((x^1 sin(y^1) + x^2 cos(y^1))e^{y^2}, -(x^1 cos(y^1) + x^2 sin(y^1))e^{y^2} + e^{2y^2})$ and so $x \in M_1 \longmapsto D_y c(x,y)$ is injective. This implies that solutions are supported on graphs of $x$ over $y$ but that these graphs are not necessarily invertible. In fact, $c(x,y) \geq \frac{(((x^1)^2+(x^2)^2)^{\frac{1}{2}} - e^{y^2})^2}{2} \geq 0$, where equality holds if and only if $cos(y^1) = \frac{x^1}{((x^1)^2+(x^2)^2)^{\frac{1}{2}}}$, $sin(y^1) = \frac{x^2}{((x^1)^2+(x^2)^2)^{\frac{1}{2}}}$, and $((x^1)^2 + (x^2)^2)^{\frac{1}{2}} = e^{y^2}$. This set of equality is a non-invertible graph of $x$ over $y$; any measure whose support is contained in this graph is optimal for its marginals. Note that as any minimizer for this problem must be supported on this graph, the solution is unique [3].*

**Remark 2.2.3.** *For twisted costs with regular marginals, any solution is concentrated on the graph of a particular function [52]. It is not hard to show that at most one measure with prescribed marginals can be supported on such a graph; hence, uniqueness of the optimizer follows immediately.*

*While our result asserts that for non-degenerate costs the solution concentrates on some $n$-dimensional Lipschitz submanifold, the proof says little more about the subman-*

*ifold itself. In contrast to the twisted setting, then, our result cannot be used to deduce a uniqueness argument. Furthermore, as Example 5.3.4 shows, even if we do know the support of the optimizer explicitly, solutions may not be unique if this support is not concentrated on the graph of a function.*

Theorem 2.0.2 also says something about problems where $D^2_{xy}c$ is allowed to be singular, but where the gradient of its determinant is non-zero at the singular points. In this case, the implicit function theorem implies that the set where $D^2_{xy}c$ is singular has Hausdorff dimension $2n-1$. Theorem 2.0.2 is valid wherever $D^2_{xy}c$ is nonsingular, so that the optimal measure is concentrated on the union of a smooth $2n-1$ dimensional set and an $n$ dimensional Lipschitz submanifold. For example, when $n = 1$, this shows that the support of the optimal measure is 1-dimensional.

## 2.3 A Jacobian equation

We now provide a simple proof that an optimal map satisfies a prescribed Jacobian equation almost everywhere. This result was originally proven for the quadratic cost in $\mathbb{R}^n$ by McCann [54], and for the quadratic cost on a Riemannian manifold by Cordero-Erasquin, McCann and Schmuckenschläger [24]. Cordero-Erasquin generalized this approach to deal with strictly convex costs on $\mathbb{R}^n$ [23]; see also [2]. It was observed by Ambrosio, Gigli and Savare that this can be deduced from results in [5] and [6] when the optimal map is approximately differentiable, which is true even for some non-smooth costs. Our method works only when the cost is $C^2$ and non-degenerate, but has the advantage of a simpler proof, relying only on the area/coarea formula for Lipschitz functions.

For a Jacobian equation to make sense, the solution must be concentrated on the graph of a function, and that function must be differentiable in some sense, at least almost everywhere. A twisted cost suffices to ensure the first condition. The second follows from the smoothness and non-degeneracy of $c$. Recall that for a twisted cost the optimal map has the form $T(x) = c\text{-}exp_x(Du(x))$; as $c\text{-}exp_x(\cdot)$ is the inverse of $y \longmapsto D_xc(x,y)$, its differentabiliy follows from the non-degeneracy of $c$ and the inverse function theorem. The almost everywhere differentiability of $Du(x)$ (or, equivalently, the almost everywhere twice differentiability of $u$) follows from $C^2$ smoothness of $c$; $u$ takes the form $u(x) = \inf_y(c(x,y) - v(y))$ for some function $v(y)$ and is hence semi-concave [36]. In the present context, we need only the weaker condition that the optimal map is continuous almost everywhere; its differentiability will follow from Theorem 2.0.2.

**Proposition 2.3.1.** *Assume that the cost is non-degenerate and that an optimizer $\mu$ is supported on the graph of some function $T : dom(T) \rightarrow M_2$ which is injective and continuous when restricted to a set $dom(T) \subseteq M_1$ of full Lebesgue measure. Suppose that the marginals are absolutely continuous with respect to volume; set $d\mu_1 = f^+(x)dx$ and $d\mu_2 = f^-(y)dy$. Then, for almost every $x$, $f^+(x) = |detDT(x)|f^-(T(x))$.*

*Proof.* Choose a point $x$ where $T$ is continuous and a neighbourhood $U^-$ of $T(x)$ such that for $U^+ = T^{-1}(U^-)$, the part of the optimal graph contained in $U^+ \times U^-$ lies in a Lipschitz graph $v = G(u)$ over the diagonal $\Delta = \{u = \frac{x+y}{\sqrt{2}} : (x,y) \in U^+ \times U^-\}$, after a change of coordinates. Now $x = \frac{u+v}{\sqrt{2}}$ and $y = \frac{u-v}{\sqrt{2}}$, so the optimal measure is supported on the graph of the Lipschitz function $(x,y) = (F^+(u), F^-(u)) := (\frac{u+G(u)}{\sqrt{2}}, \frac{u-G(u)}{\sqrt{2}})$. By projecting onto the diagonal, we obtain a measure $\nu$ on $\Delta$ that pushes forward to $\mu_1|_{U^+}$ and $\mu_2|_{U^-}$ under the Lipschitz mappings $F^+$ and $F^-$, respectively. Now, as $F^+$ is Lipschitz, the image of any zero volume set must also have zero volume; as $\mu_1|_{U^+}$ is absolutely continuous with respect to Lebesgue, $\nu$ must be as well; we will write $\nu = h(u)du$. Now, for almost every $x \in U^+$ there is a unique $y = T(x)$ such that $(x,y) \in \text{spt}(\mu)$ and hence a unique $u = \frac{x+y}{\sqrt{2}}$ on the diagonal such that $x = F^+(u)$. It follows that the map $F^+$ is one to one almost everywhere and so for every set $A \subseteq \Delta$ we have $\int_A h(u)du = \int_{F^+(A)} f^+(x)dx$. But the right hand side is $\int_A f^+(F^+(u))|\text{det}DF^+(u)|du$ by the area formula; as $A$ was arbitrary, this means $h(u) = f^+(F^+(u))|\text{det}DF^+(u)|$ almost everywhere. Similarly, $h(u) = f^-(F^-(u))|\det DF^-(u)|$ almost everywhere, hence

$$f^+(F^+(u))|\det DF^+(u)| = f^-(F^-(u))|\det DF^-(u)|$$

almost everywhere. As the image under $F^+$ of a negligible set must itself be negligible, we have

$$f^+(x)|\det DF^+((F^+)^{-1}(x))| = f^-(F^-((F^+)^{-1}(x)))|\det DF^-((F^+)^{-1}(x))| \tag{2.5}$$

for almost all $x$. Note that as $F^+$ is one to one almost everywhere and $F^+(\{u \in \Delta : \det DF^+(u) = 0\})$ has measure zero by the area formula, $(F^+)^{-1}$ is differentiable almost everywhere. As $T \circ F^+ = F^-$, it follows that $T$ is differentiable almost everywhere and

$$\det DT(F^+(u))\det DF^+(u) = \det DF^-(u)$$

whenever $F^+$ and $F^-$ are differentiable at $u$ and $T$ is differentiable at $F^+(u)$. Hence,

$$\det DT(x)\det DF^+((F^+)^{-1}(x)) = \det DF^-((F^+)^{-1}(x)) \tag{2.6}$$

for all $x$ such that $T$ is differentiable at $x$ and $F^+$ and $F^-$ are differentiable at $(F^+)^{-1}(x)$. $T$ is differentiable for almost every $x$ , $F^+$ and $F^-$ are differentiable for almost every $u$ and $F^+$ is Lipschitz; it follows that the above holds almost everywhere. Now, combining (6) and (7) we obtain $f^+(x) = |\det DT(x)|f^-(T(x))$ for almost every $x$.

□

**Remark 2.3.1.** *Note that the preceding proposition does not require that continuity of $T$ extend outside dom$(T)$. Thus it applies to $T = Du$, for example, where $u$ is an arbitrary convex function and dom$(T)$ is its domain of differentiability.*

# Chapter 3

# Quantified rectifiability for multi-marginal problems

In this chapter, we prove an upper bound on the Hausdorff dimension of $spt(\mu)$ without any restriction on $m$.

For a general $m$, there is an immediate lower bound on the Hausdorff dimension of $spt(\mu)$; as $spt(\mu)$ projects to $spt(\mu_i)$ for all $i$, $\dim(spt(\mu)) \geq \max_i(\dim(spt(\mu_i)))$. In the present chapter, we establish an upper bound on $\dim(spt(\mu))$. This bound depends on the cost function; however, it will always be greater than the largest of the $n_i$'s. In the case when the $n_i$'s are equal to some common value $n$, we identify conditions on $c$ that ensure our bound will be $n$ and we show by example that when these conditions are violated, the solution may be supported on a higher dimensional submanifold and may not be unique. In fact, the costs in these examples satisfy naive multi-marginal extensions of both the twist and non-degeneracy conditions; given Theorem 2.0.2 and the results in [46][35][36] and [14] outlined in the introduction, we found it surprising that higher dimensional solutions can exist for twisted, non-degenerate costs. On the other hand, if the support of at least one of the measures $\mu_i$ has Hausdorff dimension $n$, the remarks above imply that $spt(\mu)$ must be at least $n$ dimensional; therefore, in cases where our upper bound is $n$, the support is exactly $n$-dimensional, in which case we show it is actually $n$-rectifiable.

The chapter is organized as follows: in section 3.1, we state and prove our main result. In section 3.2 we apply this result to several example cost functions. These include the costs studied in [37][39] and [21] and we discuss how they fit into our framework. In section 3.3, we discuss conditions that ensure the relevant metrics have only $n$ timelike directions, which will ensure $spt(\mu)$ is at most $n$-dimensional. In section 3.4, we discuss some applications of our main result to the two marginal problem and in the final section we take a closer look at the case when the marginals all have one dimensional support.

## 3.1 Dimension of the support

Before stating our main result, we must introduce some notation. Suppose that $c \in C^2(M_1 \times M_2 \times ... \times M_m)$. Consider the set $P$ of all partitions of the set $\{1,2,3,...,m\}$ into 2 disjoint, nonempty subsets; note that $P$ has $2^{m-1}-1$ elements. For any partition $p \in P$, label the corresponding subsets $p_+$ and $p_-$; thus, $p_+ \cup p_- = \{1,2,3,...,m\}$ and $p_+ \cap p_-$ is empty. For each $p \in P$, define the following symmetric, bi-linear form on $M_1 \times M_2 ... \times M_m$

$$g_p = \sum_{j \in p_+, k \in p_-} \frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}} (dx_j^{\alpha_j} \otimes dx_k^{\alpha_k} + dx_k^{\alpha_k} \otimes dx_j^{\alpha_j}) \tag{3.1}$$

where, in accordance with the Einstein summation convention, summation on the $\alpha_k$ and $\alpha_j$ is implicit. Here, the index $\alpha_k$ ranges from 1 through $n_k$ and represents local coordinates on $M_k$. Explicitly, given vectors $v = \bigoplus_{j=1}^m v_j^{\alpha_j} \frac{\partial}{\partial x_j^{\alpha_j}}$ and $w = \bigoplus_{j=1}^m w_j^{\alpha_j} \frac{\partial}{\partial x_j^{\alpha_j}}$ we have

$$g_p(v,w) = \sum_{j \in p_+, k \in p_-} \frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}} (v_j^{\alpha_j} w_k^{\alpha_k} + v_k^{\alpha_k} w_j^{\alpha_j})$$

Further details on this notation can be found in Appendix A.

**Definition 3.1.1.** *We will say that a subset $S$ of $M_1 \times M_2 \times ... \times M_m$ is c-monotone with respect to a partition $p$ if for all $y = (y_1, y_2, ..., y_m)$ and $\tilde{y} = (\tilde{y_1}, \tilde{y_2}, ..., \tilde{y_m})$ in $S$ we have*

$$c(y) + c(\tilde{y}) \leq c(z) + c(\tilde{z}),$$

*where*

$$z_i = y_i \text{ and } \tilde{z}_i = \tilde{y}_i, \text{ if } i \in p_+,$$

$$z_i = \tilde{y}_i \text{ and } \tilde{z}_i = y_i, \text{ if } i \in p_-,$$

The following lemma, which is well known when $m = 2$, provides the link between $c$-monotonicity and optimal transportation.

**Lemma 3.1.2.** *Suppose $\mu$ is an optimizer and $C(\mu) < \infty$. Then the support of $\mu$ is c-monotone with respect to every partition $p \in P$.*

*Proof.* Define $M_{p_+} = \otimes_{i \in p_+} M_i$ and $M_{p_-} = \otimes_{i \in p_-} M_i$. Note that we can identify $M_1 \times M_2 \times ... \times M_m$ with $M_{p_+} \times M_{p_-}$ and let $\mu_{p_+}$ and $\mu_{p_-}$ be the projections of $\mu$ onto $M_{p_+}$ and $M_{p_-}$ respectively. Consider the two marginal problem

$$\inf \int_{M_{p_+} \times M_{p_-}} c(x_1, x_2, ..., x_m) d\lambda,$$

where the infinum is taken over all measures $\lambda$ whose projections onto $M_{p_+}$ and $M_{p_-}$ are $\mu_{p_+}$ and $\mu_{p_-}$, respectively. Then $\mu$ is optimal for this problem and, as $c$ is continuous, the result follows from $c$-monotonicity for two marginal problems; see for example [68]. □

We will say a vector $v \in T_{(x_1,x_2,...,x_m)} M_1 \times M_2 \times ... \times M_m$ is spacelike (respectively timelike or lightlike) for a semi-Riemannian metric $g$ if $g(v, v) \geq 0$ (respectively $g(v, v) \leq 0$ or $g(v, v) = 0$). We will say a subspace $V \subseteq T_{(x_1,x_2,...,x_m)} M_1 \times M_2 \times ... \times M_m$ is spacelike (respectively timelike or lightlike) for $g$ if every non-zero $v \in V$ is spacelike (respectively timelike or lightlike) for $g$. We will say $V$ is strictly spacelike (respectively strictly timelike) for $g$ if no nonzero $v \in V$ is timelike (respectively spacelike). We will say a submanifold of $T_{(x_1,x_2,...,x_m)}M_1 \times M_2 \times ... \times M_m$ is spacelike (respectively timelike, lightlike, strictly spacelike or strictly timelike) at $(x_1, x_2, ..., x_m)$ if its tangent space at $(x_1, x_2, ..., x_m)$ is spacelike (respectively timelike, lightlike, strictly spacelike or strictly timelike).

We are now ready to state our main result:

**Theorem 3.1.3.** *Let $g$ be a convex combination of the $g_p$'s defined in equation (3.1); that is $g = \sum_{p \in P} t_p g_p$ where $t_p \geq 0$ for all $p \in P$ and $\sum_{p \in P} t_p = 1$. Suppose $\mu$ is an optimizer and $C(\mu) < \infty$; choose a point $(x_1, x_2, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$. Let $N = \sum_{i=1}^{m} n_i$. Suppose the $(+, -, 0)$ signature of $g$ at $(x_1, x_2, ..., x_m)$ is $(q_+, q_-, N - q_+ - q_-)$ (ie, the corresponding matrix has $q_+$ positive eigenvalues, $q_-$ negative eigenvalues and a zero eigenvalue with multiplicity $N - q_+ - q_-$). Then there is a neighbourhood $O$ of $(x_1, x_2, ..., x_m)$ such that the intersection of the support of $\mu$ with $O$ is contained in a Lipschitz submanifold of dimension $N - q_+$. Wherever the support is differentiable, it is timelike for $g$.*

Before we prove Theorem 3.1.3, a few remarks are in order. The theorem roughly says that the dimension of *spt*$(\mu)$ is controlled by the signature of *any* convex combinations of the $g_p$'s; as these metrics may have very different signatures for different choices of the $t_p$'s, we are free to pick the one with the fewest timelike directions to give us the best upper bound on the dimension of *spt*$(\mu)$ for a particular cost. When $m = 2$, there is only one partition in $P$ and consequently there is only one relevant metric, $\frac{\partial^2 c}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}(dx_1^{\alpha_1} \otimes dx_2^{\alpha_2} + dx_2^{\alpha_2} \otimes dx_1^{\alpha_1})$ in local coordinates. The matrix corresponding to this metric is the block matrix studied by Kim and McCann [43]:

$$G = \begin{bmatrix} 0 & D^2_{x_1 x_2} c \\ D^2_{x_2 x_1} c & 0 \end{bmatrix}.$$

Here $D^2_{x_j x_k} c$ is the $n_j$ by $n_k$ matrix whose $(\alpha_j, \alpha_k)$th entry is $\frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}}$.

For $m > 2$, in the remainder of this paper we will focus primarily on the special case when $t_p = \frac{1}{2^{m-1}-1}$ for all $p \in P$. To distinguish it from the metrics obtained by other convex combinations of the $g_p$'s, we will denote the corresponding metric by $\overline{g}$. Note that the matrix of $\overline{g}$ in local coordinates is the block matrix given by

$$\overline{G} = \frac{2^{m-2}}{2^{m-1}-1} \begin{bmatrix} 0 & D^2_{x_1x_2}c & D^2_{x_1x_3}c & ... & D^2_{x_1x_m}c \\ D^2_{x_2x_1}c & 0 & D^2_{x_2x_3}c & ... & D^2_{x_2x_m}c \\ D^2_{x_3x_1}c & D^2_{x_3x_2}c & 0 & ... & D^2_{x_3x_m}c \\ ... & ... & ... & ... & ..., \\ D^2_{x_mx_1}c & D^2_{x_mx_2}c & D^2_{x_mx_3}c & ... & 0 \end{bmatrix}.$$

Let us note, however, that other choices of the $t_p$'s can give new and useful information. For example, suppose we take $t_p$ to be 1 for a particular $p$ and 0 for all others. As in the proof of Lemma 2.2, we can identify $M_1 \times M_2 ... \times M_m = M_{p_+} \times M_{p_-}$ , where $M_{p_\pm} = \otimes_{j \in p_\pm} M_j$ and $c(x_1, x_2, ..., x_m) = c(x_{p_+}, x_{p_-})$ where $x_{p_\pm} \in M_{p_\pm}$. In this case, $G$ will take the form:

$$G = \begin{bmatrix} 0 & D^2_{x_{p_+}x_{p_-}}c \\ D^2_{x_{p_-}x_{p_+}}c & 0 \end{bmatrix}.$$

The signature of this $g$ is $(r, r, N-2r)$ where $r$ is the rank of the matrix $D^2_{x_{p_+}x_{p_-}}c$. Letting $n_{p_\pm} = \sum_{j \in p_\pm} n_j$ be the dimension of $M_{p_\pm}$, we will have $r \leq \min(n_{p_+}, n_{p_-})$. If it is possible to choose a partition so that $n_{p_+} = n_{p_-} = \frac{N}{2}$ and $D^2_{x_{p_+}x_{p_-}}c$ has full rank, we can conclude that $spt(\mu)$ is at most $\frac{N}{2}$ dimensional. As we will see later, the number of timelike directions for $\overline{g}$ may be very large and so this bound may in fact be better.

Our proof is an adaptation of our argument in chapter 2. When $m = 2$, after choosing appropriate coordinates, we rotated the coordinate system and showed that $c$-monotonicity implied that the solution was concentrated on a Lipschitz graph over the diagonal, a trick dating back to Minty [57]. When passing to the multi-marginal setting, however, it is not immediately clear how to choose coordinates that make an analogous rotation possible; unlike in the two marginal case, it is not possible in general to choose coordinates around a point $(x_1, x_2, ..., x_m)$ such that $D^2_{x_i x_j} c(x_1, x_2, ..., x_m) = I$ for all $i \neq j$. The key to resolving this difficulty is the observation that Minty's trick amounts to diagonalizing the pseudo-metric of Kim and McCann and that this approach generalizes to $m \geq 3$.

*Proof.* Choose a point $x = (x_1, x_2, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$ . Choose local coordinates around $x_i$ on each $M_i$ and set $A_{ij} = D^2_{x_i x_j} c(x_1, x_2, ..., x_m)$. For any $\epsilon > 0$, there is a neighbourhood $O$ of $(x_1, x_2, ..., x_m)$ which is convex in these coordinates such that for all $(y_1, y_2, ..., y_m) \in O$ we have $||A_{ij} - D^2_{x_i x_j} c(y_1, y_2, ..., y_m)|| \leq \epsilon$, for all $i \neq j$.

Let $G$ be the matrix of $g$ at $x$ in our chosen coordinates. There exists some invertible $N$ by $N$ matrix $U$ such that

$$UGU^T = H := \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the diagonal $I$, $-I$ and $0$ blocks have sizes determined by the signature of $g$.

Define new coordinates in $O$ by $u := Uy$, where $y = (y_1, y_2, ..., y_m)$ and let $u = (u_1, u_2, u_3)$ be the obvious decomposition. We will show that the optimizer is locally contained in a Lipschitz graph in these coordinates.

Choose $y = (y_1, y_2, ..., y_m)$ and $\tilde{y} = (\tilde{y_1}, \tilde{y_2}, ..., \tilde{y_m})$ in the intersection of $spt(\mu)$ and $O$. Set $\Delta y = y - \tilde{y}$. Set $z = (z_1, z_2, ...z_m)$ where

$$z_i = \begin{cases} y_i & \text{if } i \in p_+, \\ \tilde{y_i} & \text{if } i \in p_-. \end{cases}$$

Similarly, set $\tilde{z} = (\tilde{z_1}, \tilde{z_2}, ..., \tilde{z_m})$ where

$$\tilde{z_i} = \begin{cases} y_i & \text{if } i \in p_-, \\ \tilde{y_i} & \text{if } i \in p_+. \end{cases}$$

Lemma 3.1.2 then implies

$$c(y) + c(\tilde{y}) \leq c(z) + c(\tilde{z})$$

or

$$\int_0^1 \int_0^1 \sum_{j \in p_+, i \in p_-} (\Delta y_i)^T D^2_{x_i x_j} c(y(s,t)) \Delta y_j dt ds \leq 0,$$

where

$$y_i(s,t) = \begin{cases} y_i + s(\Delta y_i) & \text{if } i \in p_+, \\ y_i + t(\Delta y_i) & \text{if } i \in p_-. \end{cases}$$

This implies that

$$\sum_{j \in p_+, i \in p_-} (\Delta y_i)^T A_{ij} \Delta y_j \leq \epsilon \sum_{j \in p_+, i \in p_-} |\Delta y_i||\Delta y_j|.$$

Hence,

$$\sum_{p \in P} t_p \sum_{j \in p_+, i \in p_-} (\Delta y_i)^T A_{ij} \Delta y_j \leq \epsilon \sum_{p \in P} t_p \sum_{j \in p_+, i \in p_-} |\Delta y_i||\Delta y_j|.$$

But this means

$$(\Delta y)^T G \Delta y \leq \epsilon \sum_{p \in P} t_p \sum_{j \in p_+, i \in p_-} |\Delta y_i||\Delta y_j|. \tag{3.2}$$

With $\Delta u = U\Delta y$ and $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ being the obvious decomposition, this becomes:

$$\begin{aligned} |\Delta u_1|^2 - |\Delta u_2|^2 = (\Delta u)^T H \Delta u = (\Delta y)^T G \Delta y \\ \leq \epsilon \sum_{p \in P} t_p \sum_{j \in p_+, i \in p_-} |\Delta y_i||\Delta y_j| \\ \leq \epsilon m^2 ||U^{-1}||^2 \sum_i^3 |\Delta u_i|^2, \end{aligned}$$

where the last line follows because for each $i$ and $j$ we have

$$\begin{aligned} |\Delta y_i||\Delta y_j| &\leq |\Delta y|^2 \\ &\leq ||U^{-1}||^2 |\Delta u|^2 \\ &= ||U^{-1}||^2 \sum_{i=1}^3 |\Delta u_i|^2. \end{aligned}$$

Choosing $\epsilon$ sufficiently small, we have

$$|\Delta u_1|^2 - |\Delta u_2|^2 \leq \frac{1}{2}\sum_{i}^{3} |\Delta u_i|^2.$$

Rearranging yields

$$\frac{1}{2}|\Delta u_1|^2 \leq \frac{3}{2}|\Delta u_2|^2 + \frac{1}{2}|\Delta u_3|^2.$$

Together with Kirzbraun's theorem, the above inequality implies that the support of $\mu$ is locally contained in a Lipschitz graph of $u_1$ over $u_2$ and $u_3$.

If $spt(\mu)$ is differentiable at $x$, the non-spacelike implication follows from taking $y = x$ in (3.2), then noting that we can take $\epsilon \to 0$ as $\tilde{y} \to x$.

□

## 3.2 Examples

In this section we apply Theorem 3.1.3 to several cost functions. Throughout this section, we restrict our attention to the special semi-Riemannian metric $\overline{g}$ defined in the last section.

### 3.2.1 Functions of the sum: $c(x_1, x_2, ..., x_m) = h(\sum_{i=1}^m x_i)$

We first consider the case where $M_i = \mathbb{R}^n$ for all $i$ and that $c(x_1, x_2, ..., x_m) = h(\sum_{i=1}^m x_i)$.

**Proposition 3.2.1.1.** *Suppose $M_i = \mathbb{R}^n$ for all $i$ and that $c(x_1, x_2, ..., x_m) = h(\sum_{i=1}^m x_i)$. Denote the signature of $D^2h$ by $(q_+, q_-, n - q_+ - q_-)$; then the signature of $\overline{g}$ is $\big(q_+ + (m-1)(q_-), q_- + (m-1)q_+, m(n - q_+ - q_-)\big)$.*

*Proof.* Up to a positive, multiplicative constant, the matrix corresponding to $\overline{g}$ is

$$\overline{G} = \begin{bmatrix} 0 & D^2h & D^2h & ... & D^2h \\ D^2h & 0 & D^2h & ... & D^2h \\ D^2h & D^2h & 0 & ... & D^2h \\ ... & ... & ... & ... & ..., \\ D^2h & D^2h & D^2h & ... & 0 \end{bmatrix}.$$

If $v$ is an eigenvector of $D^2h$ with eigenvalue $\lambda$, then

$$[v, v, v, v...v, v]^T$$

is an eigenvector of $\overline{G}$ with eigenvalue $(m-1)\lambda$ and

$$[v, -v, 0, 0, ..., 0]^T, [v, 0, -v, 0, ..., 0]^T, ..........................., [v, 0, 0, 0, ..., -v]^T$$

are linearly independent eigenvectors with eigenvalue $-\lambda$. The result now follows easily. □

**Remark 3.2.1.2.** *When $D^2h$ is negative definite (corresponding to a uniformly concave $h$), the signature of $\overline{g}$ reduces to $((m-1)n, n, 0)$; combined with Theorem 3.1.3, this implies that the support of any opimal measure $\mu$ is contained in an $n$-dimensional submanifold. This is consistent with the results of Gangbo and Święch[37] and Heinich[39], who show that if the first marginal assigns measure zero to every set of Hausdorff dimension $n-1$, then spt$(\mu)$ is contained in the graph of a function over $x_1$.*

On the other hand, when $D^2h$ is not negative definite, the signature of $\overline{g}$ has more than $n$ timelike directions. In this case, Theorem 3.1.3 does not preclude optimal measures with higher dimensional supports. The next two results verify that this can in fact occur.

First we consider the extreme case, where $h$ is uniformly convex; the signature of $\overline{g}$ is then $(n, (m-1)n, 0)$.

**Proposition 3.2.1.3.** *Suppose* $c(x_1, x_2, ..., x_m) = h(\sum_{i=1}^m x_i)$, *with* $D^2h > 0$. *Then any measure supported on the* $n(m-1)$*- dimensional surface*

$$S = \{(x_1, x_2, ..., x_m) | \sum_{i=1}^{m} x_i = y\},$$

*where* $y \in \mathbb{R}^n$ *is any constant, is optimal for its marginals.*

It should be noted that when $m = 2$, this surface is $n$ dimensional.

*Proof.* Adding a function of the form $\sum_{i=1}^m u_i(x_i)$ to the cost $c$ shifts the functional $C(\mu)$ by an amount $\sum_{i=1}^m \int_{M_i} u_i(x_i) d\mu_i$ for each $\mu$ but does not change its minimizers. In particular, minimizing the cost $c$ is equivalent to minimizing

$$c'(x_1, x_2, ..., x_m) := c(x_1, x_2, ..., x_m) - \sum_{i=1}^{m} x_i \cdot Dh(y) = f(\sum_{i=1}^{m} x_i),$$

where $f(z) := h(z) - z \cdot Dh(y)$. Then $f$ is a strictly convex function whose gradient vanishes at $z = y$; it follows that $y$ is the unique minimum of $f$. Hence, $c'(x_1, x_2, ..., x_m) \leq f(y)$ with equality only when $\sum_{i=1}^m x_i = y$. It follows that any measure supported on $S$ is optimal for its marginals.

□

We now turn to the intermediate case where $h$ has both concave and convex directions. We show that there exist optimal measures whose supports have the maximal dimension allowed by Theorem 3.1.3.

**Proposition 3.2.1.4.** *Let* $c(x_1, x_2, ..., x_m) = h(\sum_{i=1}^m x_i)$, *where the signature of* $D^2h$ *is* $(q, n-q, 0)$. *Then there exist optimal measures whose support has dimension* $(n - q + q(m-1))$.

*Proof.* At a fixed point $p$, we can add an affine function of $(x_1 + x_2 + ... + x_m)$ so that $Dh(p) = 0$ and choose variables so that

$$D^2h(p) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

where the top left hand corner block is $q$ by $q$ and the bottom left hand corner block is $n-q$ by $n-q$. Then define the $q$-dimensional variables $y_i = (x_i^1, x_i^2, ..., x_i^q)$ and the $n-q$ dimensional variables $z_i = (x_i^{q+1}, x_i^{q+2}, ..., x_i^n)$, so that $h(\sum_{i=1}^m x_i) = h(\sum_{i=1}^m y_i, \sum_{i=1}^m z_i)$. Now, near $p$, the implicit function theorem implies that for fixed $z_i, i = 1, 2, ..., m$ there is a unique $K = K(\sum_{i=1}^m z_i))$, such that

$$D_y h(K(\sum_{i=1}^m z_i), \sum_{i=1}^m z_i) = 0$$

and $K$ is smooth as a function of $\sum_{i=1}^m z_i$. As $h$ is convex in it's first slot near $p$,

$$h(K(\sum_{i=1}^m z_i), \sum_{i=1}^m z_i) \leq h(\sum_{i=1}^m y_i, \sum_{i=1}^m z_i)$$

for all nearby $y_i$. Now, if we $f(\sum_{i=1}^m z_i) = h(K(\sum_{i=1}^m z_i), \sum_{i=1}^m z_i)$ then $f$ is a concave function of $\sum_{i=1}^m z_i$. If we consider an optimal transportation problem for the $z_i$ with cost $f$, the solution must be concentrated on a Lipschitz $n-k$ dimensional submanifold. Choose an $n-q$ dimensional set $S$ which supports an optimizer for this problem; by considering a dual problem as in Gangbo and Święch [37], we can find functions $u_i(z_i)$ such that $f(\sum_{i=1}^m z_i) - \sum_{i=1}^m u_i(z_i) \geq 0$ with equality if and only if $(z_1, z_2, ...z_m) \in S$. Therefore,

$$h(\sum_{i=1}^m y_i, \sum_{i=1}^m z_i)) - \sum_{i=1}^m u_i(z_i) \geq h(K(\sum_{i=1}^m z_i), \sum_{i=1}^m z_i) - \sum_{i=1}^m u_i(z_i) \geq 0$$

and we have equality only when $(z_1, z_2, ...z_m) \in S$ and $\sum_{i=1}^m y_i = K(\sum_{i=1}^m z_i)$, which is a $n-q+(m-1)q$ dimensional set. It follows that this set is the support of an optimizer for appropriate marginals. □

Finally, we show that when the dimension of $spt(\mu)$ is larger than $n$, the solution may not be unique.

**Proposition 3.2.1.5.** *Set $m = 4$ and $c(x, y, z, w) = h(x + y + z + w)$ for $h$ strictly convex. Suppose all four marginals $\mu_i$ are Lebesgue measure on the unit cube $I^n$ in $\mathbb{R}^n$. Then the optimal measure is not unique.*

*Proof.* Let $S_1$ be the surface $y = -w + (1, 1, 1, ..., 1)$, $z = -x + (1, 1, 1, ..., 1)$ and take $\mu$ be uniform measure on the intersection of $S_1$ with $I^n \times I^n \times I^n \times I^n$. This projects to $\mu_i$ for $i = 1, 2, 3$ and 4 and by the argument in Proposition 3.2.1.3, it must be optimal. Now, if we take $S_2$ to be the surface $y = -x + (1, 1, 1, ..., 1)$, $z = -w + (1, 1, 1, ..., 1)$ and $\overline{\mu}$ to be uniform measure on the intersection of $S_2$ with $I^n \times I^n \times I^n \times I^n$, we obtain a second optimal measure. □

It is worth noting that this cost is twisted: the maps $x_i \mapsto D_{x_j} c(x_1, x_2, ..x_m)$ are injective for all $i \neq j$ where $x_k$ is held fixed for all $k \neq i$. In the two marginal case, the twist condition and mild regularity on the $\mu_1$ suffices to imply the uniqueness of the solution $\mu$ [46]; this example demonstrates that this is no longer true for $m \geq 3$.

### 3.2.2 Hedonic pricing costs

Our next example has an economic motivation. Chiappori, McCann and Nesheim [22] and Carlier and Ekeland [20] introduced a hedonic pricing model based on a multi-marginal optimal transportation problem with cost functions of the form

$$c(x_1, x_2, ..., x_m) = \inf_{y \in Y} \sum_{i=1}^{m} f_i(x_i, y)$$

Combined with Theorem 3.1.3, the following result demonstrates that, assuming all the dimensions $n_i = n$ are equal, the support of the opimizer is at most $n$-dimensional.

**Proposition 3.2.3.** *Suppose $n_i = n$ for all $i$ and let $c(x_1, x_2, ..., x_m) = \inf_{y \in Y} \sum_{i=1}^{m} f_i(x_i, y)$, where $y$ belongs to a $C^2$, $n$-dimensional manifold $Y$. Assume the following conditions:*

1. *For all $i$, $f_i$ is $C^2$ and the $n \times n$ off-diagonal block $D^2_{x_i y} f_i$ of mixed, second order partial derivatives is everywhere non-singular.*

2. *For each $(x_1, x_2, ..., x_m)$ the infinum is attained by a unique $y(x_1, x_2, ..., x_m) \in Y$.*

3. *The sum $\sum_{i=1}^{m} D^2_{yy} f_i(x_i, y(x_1, x_2, ..., x_m))$ of $n \times n$ diagonal blocks is non-singular.*

*Then the signature of $\overline{g}$ is $((m-1)n, n, 0)$.*

*Proof.* Fixing $(x_1, x_2, ..., x_m)$, we can choose coordinates so that

$$D^2_{x_i y} f_i(x_i, y(x_1, x_2, ..., x_m)) = I$$

for all $i$. Now, $\sum_{i=1}^m D_y f_i(x_i, y(x_1, x_2, ..., x_m)) = 0$. Set $M = \sum_{i=1}^m D^2_{yy} f_i(x_i, y(x_1, x_2, ..., x_m))$ and note that as $M$ is non-singular by assumption we must have $M > 0$.. The implicit function theorem now implies that $y$ is differentiable with respect to each $x_j$ and:

$$\sum_{i=1}^m D^2_{yy} f_i(x_i, y(x_1, x_2, ..., x_m)) D_{x_j} y(x_1, x_2, ..., x_m) + D^2_{yx_j} f_j(x_i, y(x_1, x_2, ..., x_m)) = 0.$$

So $D_{x_j} y(x_1, x_2, ..., x_m) = -M^{-1}$. Now, as $c(x_1, x_2, ..., x_m) \leq \sum_{i=1}^m f_i(x_i, y))$ with equality when $y = y(x_1, x_2, ..., x_m)$ we have

$$D_{x_i} c(x_1, x_2, ..., x_m) = D_{x_i} f(x_i, y(x_1, x_2, ..., x_m)).$$

Differentiating with respect to $x_j$ yields

$$D^2_{x_i x_j} c(x_1, x_2, ..., x_m) = D_{x_i y} f(x_i, y(x_1, x_2, ..., x_m)) D_{x_j} y(x_1, x_2, ..., x_m) = -M^{-1}$$

for all $i \neq j$. The result now follows by the same argument as in Proposition 3.2.1. □

### 3.2.4 The determinant cost function

Here we consider a problem studied by Carlier and Nazaret in [21], where the cost function is $-1$ times the determinant; ie, for $x_1, x_2, , ..., x_n \in \mathbb{R}^n$, $c(x_1, x_2, ..., x_n)$ is $-1$ times the determinant of the $n$ by $n$ matrix whose $ith$ column is the vector $x_i$. When $n = 3$, they exhibit a specific example where the solution has 4-dimensional support; specifically, it's support is the set

$$S = \{(x_1, x_2, x_3) : |x_1| = |x_2| = |x_3| \text{ and } (x_1, x_2, x_3) \text{ forms a direct, orthogonal basis for } \mathbb{R}^3\}.$$

Although the signature of $\overline{g}$ varies for this cost, we show that on $S$ it is $(5, 4, 0)$.

**Proposition 3.2.5.** *Assume* $c(x_1, x_2, x_3) = -det(x_1 x_2 x_3)$ *and suppose* $(x_1, x_2, x_3)$ *forms a direct, orthogonal basis for* $\mathbb{R}^3$*. Then the signature of* $\overline{g}$ *is* $(5, 4, 0)$*.*

*Proof.* Choose $(x_1, x_2, x_3)$ in the support; after applying a rotation we may assume $x_1 = (|x_1|, 0, 0), x_2 = (0, |x_1|, 0)$ and $x_3 = (0, 0, |x_1|)$. A straightforward calculation then yields:

$$\overline{G} = |x_1| \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are 5 eigenvectors with eigenvalue 1:

$$[010100000]^T, [001000100]^T, [000001010]^T, [1000\text{-}10000]^T, [10000000\text{-}1]^T.$$

There are 3 eigenvectors with eigenvalue -1:

$$[010\text{-}100000]^T, [001000\text{-}100]^T, [0000010\text{-}10]^T.$$

Finally, there is a single eigenvector with eignenvalue -2:

$$[100010001]^T$$

□

## 3.3 The Signature of $g$

This section is devoted to developing some results about the signature of the semi-metric $g = \sum_{p \in P} t_p g_p$ at some point $x = (x_1, x_2, ..., x_m)$. Studying the signature at a point reduces to understanding the matrix

$$g \to G = \begin{bmatrix} 0 & G_{12} & G_{23} & ... & G_{1m} \\ G_{21} & 0 & G_{23} & ... & G_{2m} \\ G_{31} & G_{32} & 0 & ... & G_{3m} \\ ... & ... & ... & ... & ..., \\ G_{m1} & G_{m2} & G_{m3} & ... & 0 \end{bmatrix}. \tag{3.3}$$

Here, for $i \neq j, G_{ij} = a_{ij} D^2_{x_i x_j} c$ where $a_{ij} = \sum t_p$ and the sum is over all partitions $p \in P$ that separate $i$ and $j$; that is, $i \in p_+$ and $j \in p_-$ or $i \in p_-$ and $j \in p_+$. Although $G$ is an $N \times N$ matrix, where $N = \sum_{i=1}^m n_i$, its signature can often be computed from lower dimensional data, because of its special form. To illustrate this point, suppose momentarily that the $n_i's$ are all equal to some common $n$ and $G_{ij}$'s are non-singular. In this case, when $m = 2$ the signature of $G$ will always be $(n, n, 0)$ and, as we will see, when $m = 3$ it is enough to calculate the signature of an appropriate $n \times n$ matrix.

One observation about the signature of the matrix $G$ is immediate; as $G$ has zero blocks on the diagonal, it is possible to construct a lightlike subspace of dimension $n_{max} = \max_i\{n_i\}$. This in turn implies that the number of spacelike directions can be no greater than $N - n_{max}$; otherwise, it would be possible to construct a spacelike subspace of dimension $N - n_{max} + 1$, which would have to intersect non trivially with the null subspace. Therefore, the best possible bound on the dimension of $spt(\mu)$ that Theorem 3.1.3 can provide is $n_{max}$. This result is not too surprising. We have already noted that for suitable marginals, the Hausdorff dimension of $spt(\mu)$ must be at least $n_{max}$; the discussion above verifies that this is consistent with Theorem 3.1.3.

The first proposition gives an upper and lower bound for the number of timelike directions.

**Proposition 3.3.1.** *Let $G$ be as in equation (3.3) and suppose $rank(G_{ij}) = r$ for some $i \neq j$. Then the number of positive eigenvalues and the number of negative eigenvalues* *of $G$ are both at least $r$.*

In particular, if $n_i = n$ for all $i$ and $G_{ij}$ is invertible for some $i \neq j$, Theorem 3.1.3 implies that the support of any optimizer $\mu$ is at most $(m-1)n$ dimensional.

*Proof.* On the subspace $T_{x_i}M_i \times T_{x_j}M_j$ $G$ restricts to

$$\begin{bmatrix} 0 & G_{ij} \\ G_{ij} & 0 \end{bmatrix}.N$$

Note that $(v,u)$ is a null vector if and only if $u$ is in the null space of $G_{ij}$ and $v$ is in the nullspace of $G_{ji}$. As these spaces are respectively $n_i - r$ and $n_j - r$ dimensional, the nullspace of this matrix is $n_i + n_j - 2r)$ dimensional.

As has been noted by Kim and McCann [43], the nonzero eigenvalues of this matrix come in pairs of the form $\lambda, -\lambda$, with corresponding eigenvectors $(v,u)$ and $(v,-u)$, respectively, where we take $\lambda \geq 0$. Therefore, there are $\frac{1}{2}(n_i + n_j - (n_i + n_j - 2r)) = r$ positive eigenvalues and as many negative ones.

We can now construct a $r$ dimensional timelike subspace for $g$. If $q_+ < r$, then we could construct a non-timelike subspace of dimension $N - q_+ > N - r$ (for example, take the space spanned by all negative and null eigenvalues of $G$). These two spaces would have to intersect non-trivially as their dimensions add to more than $N$, which is a contradiction. An analagous argument applies to $q_-$.

□

Next, we describe the signature in the $m = 3$ case:

**Lemma 3.3.2.** *Suppose $m = 3$, for all $i$ and that $G_{23}$ in equation (3.3) is invertible. Set $A = G_{12}(G_{32})^{-1}G_{31}$; suppose $A + A^T$ has signature $(r_+, r_-, n_1 - r_+ - r_-)$. Then $G$ has signature $(q_+, q_-, \sum_{i=1}^{3} n_i - q_+ - q_-) = (n_2 + r_-, n_2 + r_+, n_1 - r_+ - r_-)$.*

*Proof.* Note that the invertibility of $G_{23}$ implies that $n_2 = n_3$. Consider the subspace

$$S = \{(0,p,q) : p \in T_{x_2}M_2, q \in T_{x_3}M_3\}.$$

By Proposition 3.3.1 we can find an orthonormal basis for this subspace consisting of $n_2$ spacelike and $n_2$ timelike directions. To determine the signature of $g$ then, it suffices to consider the restriction of $g$ to the orthogonal complement (relative to $g$) $S^{\perp}$ of $S$; any orthonormal basis of $S^{\perp}$ can be concatenated with a basis for $S$ to form an orthonormal basis for $T_{x_1}M_1 \times T_{x_2}M_2 \times T_{x_3}M_3$.

A simple calculation yields that $S^{\perp} = \{(v, -A^T v, -Av) : v \in T_{x_1}M_1\}$ and

$$(v, -A^T v, -Av)^T G(v, -A^T v, -Av)) = -(A + A^T)(v, v),$$

which yields the desired result. □

In particular, when $n_i = n$ for $i = 1, 2, 3$ and $A$ is negative definite, $g$ has signature $(2n, n, 0)$ and the support of any minimizer has dimension at most $n$.

A brief remark about Lemma 3.3.2 is in order. We mentioned in section 2 that, while there is only one interesting pseudo metric when $m = 2$, there is an entire family of metrics in the $m \geq 3$ setting which may give new information about the behaviour of $spt(\mu)$. However, when $m = 3$, $n_i = n$ for all $i$, $D^2_{x_i x_j}c$ is non-singular for all $i \neq j$, and the coefficients $a_{ij}$ are all non zero, the signature of $G$ is determined entirely by $A = G_{12}(G_{32})^{-1}G_{31} = \frac{a_{12}a_{31}}{a_{32}} D^2_{x_1x_2}c(D^2_{x_3x_2}c)^{-1}D^2_{x_3x_1}c$. Choosing a different $g$ simply changes the $a_{ij}$'s, which does not effect the signature of $A + A^T$. If one of the $a_{ij}$'s is zero, it is easy to check that the signature of $g$ must be $(n, n, n)$; this yields a bound of $2n$ on the dimension of $spt(\mu)$ which is no better than the bound obtained when all the $a_{ij}$'s are non-zero. Thus, the only information about the dimension of $spt(\mu)$ which can be provided by Theorem 3.1.3 is encoded in the bi-linear form $D^2_{x_1x_2}c(D^2_{x_3x_2}c)^{-1}D^2_{x_3x_1}c(x)$ on $T_{x_1}M_1 \times T_{x_1}M_1$.

When $m > 3$, Lemma 3.3.2 easily yields the following necessary condition for the signature of $G$ to be $((m-1)n, n, 0)$:

**Corollary 3.3.3.** *Suppose $n_i = n$ for all $i$ and the signature of $G$ is $((m-1)n, n, 0)$.*

*Then*

$$D^2_{x_ix_j}c(D^2_{x_kx_j}c)^{-1}D^2_{x_kx_i}c < 0$$

*for all distinct $i, j$ and $k$.*

*Proof.* Note that the $G_{ij}$'s must be invertible (and hence $D^2_{x_ix_j}c$ must be invertible and $a_{ij} > 0$) ; otherwise, the argument in Proposition 3.3.1 implies the existence of a non-spacelike subspace of $T_{x_i}M_i \times T_{x_j}M_j$ whose dimension is greater than $n$. The signature of $G$ ensures the existence of a $(m-1)n$ dimensional spacelike subspace, however, and so these two spaces would have to intersect non-trivially, a contradiction.

Similarly, if $D^2_{x_ix_j}c(D^2_{x_kx_j}c)^{-1}D^2_{x_kx_i}c$ was not negative definite, we could use Lemma 3.3.2 to construct a non-timelike subspace of $T_{x_i}M_i \times T_{x_j}M_j \times T_{x_k}M_k$ of dimension greater that $n$; this, in turn, would have to intersect our $(m-1)n$ dimensional timelike subspace, which is again a contradiction. $\square$

The method in the proof of 3.3.2 can be extended to give us a method to explicitly calculate the signature of $G$ for larger $m$ when a certain set of matrices are invertible.

Let $\tilde{G}$ be the lower right hand corner $\sum_{i=2}^m n_i \times \sum_{i=2}^m n_i$ block of $G$ and $G_1$ be the upper right hand corner $n_1 \times \sum_{i=2}^m n_i$ block of $G$; that is,

$$G = \begin{bmatrix} 0 & G_1 \\ G_1^T & \tilde{G} \end{bmatrix}. \tag{3.4}$$

**Lemma 3.3.4.** *Suppose $\tilde{G}$ in equation (3.4) has signature $(q, \sum_{i=2}^m n_i - q, 0)$ Let $\tilde{G}^{-1}$ be inverse of $\tilde{G}$ and consider the symmetric $n_1 \times n_1$ matrix $G_1\tilde{G}^{-1}G_1^T$. Suppose this matrix has signature $(r_+, r_-, n_1 - r_+ - r_-)$. Then the signature of $G$ in equation (3.3) is $(q + r_-, \sum_{i=2}^m n_i - q + r_+, n_1 - r_+ - r_-)$.*

For an algorithm to calculate the signature in the general case, start with the lower right hand two by two block, which has signature $(n, n, 0)$. Use Lemma 3.3.4, or equivalently Lemma 3.3.2 to find the signature of the lower right hand three by three block.

Then use Lemma 3.3.4 again to determine the signature of the lower right hand four by four block and so on. After $m-1$ applications of Lemma 3.3.4 we obtain the signature of $G$.

## 3.4 Applications to the two marginal problem

We showed in chapter 2 that any solution to the two marginal problem was supported on an $n$-dimensional Lipschitz submanifold, provided the marginals both live on smooth $n$-dimensional manifolds and the cost is non-degenerate; that is, $D^2_{x_1x_2}c(x_1,x_2)$ seen as a map from $T_{x_1}M_1$ to $T^*_{x_2}M_2$ is injective. Kim and McCann noted that in this case, the signature of $\overline{g}$ is $(n,n,0)$ [43], so Theorem 3.1.3 immediately implies this result. In fact, our analysis here is applicable to a larger class of two marginal problems, as in Theorem 3.1.3 we assumed neither non-degeneracy nor equality of the dimensions $n_1$ and $n_2$. If $r$ is the rank of the map $D^2_{x_1x_2}c(x_1,x_2)$, then the signature of $\overline{g}$ at $(x_1,x_2)$ is $(r,r,n_1+n_2-2r)$ and so Theorem 3.1.3 yields the following corollary.

**Corollary 1.** *Let $m=2$ and $r = rank(D^2_{x_ix_j}c)$ at some point $(x_1,x_2)$. Then, near $(x_1,x_2)$, the support of any optimizer is contained in a Lipschitz manifold of dimension $n_1+n_2-r$*

It is worth noting that, even when $n_1=n_2$, the topology of many important manifolds prohibits the non-degeneracy condition from holding everywhere. Suppose, for example, that $M_1=M_2=S^1$, the unit circle. Then periodicity in $x_1$ of $\frac{\partial c}{\partial x_2}(x_1,x_2)$ implies

$$\int_{S^1}\frac{\partial^2 c}{\partial x_1\partial x_2}(x_1,x_2)dx_1 = 0.$$

It follows that for every $x_2$ there is at least one $x_1$ such that $\frac{\partial^2 c}{\partial x_1\partial x_2}(x_1,x_2)=0$. In chapter 2, we noted that under certain conditions the set where non-degeneracy fails is at most $(2n-1)$-dimensional, which yields an immediate upper bound on the dimension of $spt(\mu)$.

Corollary 3.4 yields an improved bound; $spt(\mu)$ is at most $2n - r$ dimensional. A global lower bound on $r$ immediately yields an upper bound for the dimension of $spt(\mu)$.

Next we consider a two marginal problem where the dimensions of the spaces fail to coincide; this type of problem has received very little attention in the literature. Suppose $n_2 \leq n_1$. If $D^2_{x_1x_2}c$ has full rank, ie, if $r = n_2$ then this reduces to $(n_2, n_2, n_1 - n_2)$ and the solution may have as many as $n_1$ dimensions (in fact, if the support of the first marginal has Hausdorff dimension $n_1$, then the Hausdorff dimension of $spt(\mu)$ must be exactly $n_1$). This result has a nice heuristic explanation. To solve the problem, one would first solve its dual problem, yielding two potential functions $u_1(x_1)$ and $u_2(x_2)$, and the solutions lies in the set where the first order condition $Du_2(x_2) = D_{x_2}c(x_1, x_2)$ is satisfied. For a fixed $x_2$, this is a level set of the function $x_1 \mapsto D_{x_2}c(x_1, x_2)$, which is generically $n_1 - n_2$ dimensional. Fixing $x_2$ and moving along this level set corresponds exactly to moving along the null directions of $\overline{g}$. On the other hand, as $x_2$ varies, $x_1$ must vary in such a way so that the resulting tangent vectors are timelike. Hence, the solution may contain all the lightlike directions of $\overline{g}$, which correspond to fixing $x_2$ and varying $x_1$, plus $n_2$ timelike directions, which correspond to varying $x_2$ and with it $x_1$.

## 3.5 The 1-dimensional case: coordinate independence and a new proof of Carlier's result

In [19], Carlier studied a multi-marginal problem where all the measures were supported on the real line and proved that under a 2-monotonicity condition on the cost, the solution must be one dimensional. To the best of our knowledge, this is the only result about the multi-marginal problem proved to date that deals with a general class of cost functions. The purpose of this section is to expose the relationship between 2-monotonicity and the geometric framework developed in this paper. We will find an invariant form of this condition and provide a new and simpler proof of Carlier's result.

We begin with a definition:

**Definition 3.5.1.** *We say $c : \mathbb{R}^m \to \mathbb{R}$ is $i, j$ strictly 2-monotone with sign $\pm 1$ and write $sgn(c)_{ij} = \pm 1$ if for all $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$ and $s, t > 0$ we have*

$$\pm[c(x) + c(x + te_i + se_j)] < \pm[c(x + te_i) + c(x + se_j)]$$

*where $(e_1, e_2, ...e_m)$ is the canonical basis for $\mathbb{R}^m$.*

In this notation, Carlier's 2-monotonicity condition is that $sgn(c)_{ij} = -1$ for all $i \neq j$. This is not invariant under smooth changes of coordinates, however; the change of coordinates $x_i \mapsto -x_i$ takes a cost with $sgn(c)_{ij} = -1$ and transforms it to one with $sgn(c)_{ij} = 1$. However, it is easy to check that the following condition is coordinate independent.

**Definition 3.5.2.** *We say $c$ is compatible if, for all distinct $i, j, k$ we have*

$$\frac{sgn(c)_{ij} sgn(c)_{jk}}{sgn(c)_{ik}} = -1.$$

It is also easy to check that $c$ is compatible if and only if there exist smooth changes of coordinates $x_i \mapsto y_i = f_i(x_i)$ for $i = 1, 2, ..., m$ which transform $c$ to a 2-monotone cost. Combined with Carlier's result, this observation implies that compatibility is sufficient to ensure that the support of any optimizer is 1-dimensional.

If the cost is $C^2$, the condition $\frac{d^2c}{dx_i dx_j} < 0$ is sufficient to ensure $sgn(c)_{ij} = -1$; likewise, $\frac{d^2c}{dx_i dx_j}(\frac{d^2c}{dx_k dx_j})^{-1}\frac{d^2c}{dx_i dx_k} < 0$ ensures that $c$ is compatible. We can think of the condition on the threefold products $D^2_{x_1x_2}c(D^2_{x_3x_2}c)^{-1}D^2_{x_3x_1}c$ in Lemma 3.3.2 as a multi-dimensional, coordinate independent version of Carlier's condition. Corollary 3.3.3 demonstrates that this condition is necessary for $\overline{g}$ to have signature $((m-1)n, n, 0)$ and, when $m = 3$, Lemma 3.3.2 shows that it is also sufficient. For $m > 3$, however, it is not sufficient even in one dimension. As a counterexample, consider the cost function

$$c(x_1, x_2, x_3, x_4) = -x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - 5x_3x_4.$$

For this cost,

$$\overline{G} = -\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 5 \\ 1 & 0 & 5 & 0 \end{bmatrix},$$

which has signature $(2, 2, 0)$

Thus, Theorem 3.1.3 implies neither Carlier's result nor the generalization above , at least if we restrict our attention to the special metric $\overline{g}$. Below, we reconcile this by providing a new proof of Carlier's result, with the slightly stronger assumption $\frac{d^2c}{dx_i dx_j} < 0$ in place of 2-monotonicity.

We call a set $S \subseteq \mathbb{R}^2$ non-decreasing if $(x - \overline{x})(y - \overline{y}) \geq 0$ whenever $(x, y), (\overline{x}, \overline{y}) \subseteq S$. The crux of Carlier's argument is the following result:

**Theorem 3.5.3.** *Suppose $\frac{d^2c}{dx_i dx_j} < 0$ for all $i \neq j$ . Then the projections of the support of the optimizer onto the planes spanned by $x_1$ and $x_j$ are non-decreasing subsets for all $j$.*

In view of the preceding remarks, this implies that when the cost has negative threefold products $\frac{d^2c}{dx_i dx_j}(\frac{d^2c}{dx_k dx_j})^{-1}\frac{d^2c}{dx_i dx_k}$, the support is 1-dimensional.

Carlier's proof relies heavily on duality. He shows that he can reduce the problem to a series of two marginal problems with costs derived from the solution to the dual problem. He then shows that these cost inherit monotonicity from $c$ and hence their solutions are concentrated on monotone sets. We provide a simple proof that uses only the $c$-monotonicity of the support. In addition, our proof does not require any compactness assumptions on the supports of the measures. However, after establishing this result, it is not hard to show that, if the first measure is nonatomic, the support is concentrated on the graph of a function over $x_1$.

Morally, our proof applies the non-spacelike conclusion of Theorem 3.1.3 to a well chosen semi-metric; however, because we don't know a priori that the optimizer is smooth we will prove the theorem directly from $c$-monotonicity.

*Proof.* Suppose $(x_1, ..., x_m)$ and $(y_1, ..., y_m)$ belong to the support of the optimizer. We want to show $(x_1 - y_1)(x_i - y_i) \geq 0$ for all $i$. If not, we may assume without loss of generality that for some $2 \leq k \leq m$ we have $(x_1 - y_1)(x_i - y_i) \geq 0$ for all $i < k$ and $(x_1 - y_1)(x_i - y_i) < 0$ for $i \geq k$. Hence, $(x_j - y_j)(x_i - y_i) \leq 0$ for all $j < k$ and $i \geq k$. By $c$-monotonicity, we have

$$c(x_1, ..., x_m) + c(y_1, ..., y_m) \leq c(y_1, ..., y_{k-1}, x_k, ..., x_m) + c(x_1, ..., x_{k-1}, y_k, ..., y_m).$$

Hence,

$$\sum_{i=1}^{k-1} \sum_{j=k}^{m} (x_i - y_i)(x_j - y_j) \int_0^1 \int_0^1 \frac{d^2 c}{dx_i dx_j}(y_1(t), y_2(t), ..., y_{k-1}(t), y_k(s), ..., y_m(s)) dt ds \leq 0$$

where $y_i(t) = y_i + t(x_i - y_i)$ for $i = 1, 2, ..k - 1$ and $y_j(s) = y_j + s(x_j - y_j)$ for $j = k, k + 1, ..., m$. But, as $\frac{d^2}{dx_i dx_j} c(y_1(t), y_2(t), ..., y_k - 1(t), y_k(s), ..., y_m(s)) < 0$, and $(x_i - y_i)(x_j - y_j) \leq 0$ for all $i < k$ and $j \geq k$, every term in the sum is nonnegative. As $(x_1 - y_1)(x_j - y_j) < 0$ for $j \geq k$, the sum must be positive, a contradiction.

□

# Chapter 4

# Monge solutions and uniqueness for $m \geq 3$

Our aim in this chapter is to establish necessary conditions on $c$ under which **M** admits a solution; this amounts to showing that the solution $\mu$ to **K** is concentrated on the graph of a function over $x_1$. We will then demonstrate that, under these conditions, the solutions to **M** and **K** are both unique.

In the first section we formulate the conditions we will need. In section 4.2 we state and prove our main result and in the third section we exhibit several examples of cost functions which satisfy the criteria of our main theorem.

Throughout this chaper, we will assume the dimensions $n_i$ are all equal and denote their common value by $n$.

## 4.1 Preliminaries and definitions

We will assume that each $M_i$ can be smoothly embedded in some larger manifold in which its closure $\overline{M_i}$ is compact and that the cost $c \in C^2(\overline{M_1} \times \overline{M_2} \times ... \times \overline{M_m})$. In addition, we will assume that $M_i$ is a Riemannian manifold for $i = 2, 3, ..m-1$ and that any two points can be joined by a smooth, length minimizing geodesic[1], although no such assumptions will be needed on $M_1$ or $M_m$. The requirement of a Riemannian structure is related to the global nature of $\mathbf{M}$ that we alluded to in the introduction; a Riemannian metric gives us a natural way to connect any pair of points, namely geodesics.

We will denote by $D_{x_i}c(x_1, x_2, ..., x_m)$ the differential of $c$ with respect to $x_i$. For $i \neq j$, we recall the bi-linear form $D^2_{x_ix_j}c(x_1, x_2, ..., x_m)$ on $T_{x_i}M_i \times T_{x_j}M_j$, originally introduced in [43] and employed in the previous chapter; in local coordinates, it is defined by

$$D^2_{x_ix_j}c\langle\frac{\partial}{\partial x_i^{\alpha_i}}, \frac{\partial}{\partial x_j^{\alpha_j}}\rangle = \frac{\partial^2 c}{\partial x_i^{\alpha_i}\partial x_j^{\alpha_j}}.$$

As $M_i$ is Riemannian for $i = 2, ..., m-1$, Hessians or unmixed, second order partial derivatives with respect to these coordinates make sense and we will denote them by $Hess_{x_i}c(x_1, x_2, ..., x_m)$; note, however, that no Riemannian structure is necessary to ensure the tensoriality of the mixed second order partials $D^2_{x_ix_j}c(x_1, x_2, ..., x_m)$, as was observed in [43].

The dual problem to $\mathbf{K}$ is to maximize

$$\sum_{i=1}^{m}\int_{M_i} u_i(x_i)d\mu_i \qquad (\mathbf{D})$$

among all $m$-tuples $(u_1, u_2, ..., u_m)$ of functions $u_i \in L^1(\mu_i)$ for which $\sum_{i=1}^m u_i(x_i) \leq c(x_1, ..., x_m)$ for all $(x_1, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$.

There is a special class of functions satisfying the constraint in $\mathbf{D}$ that will be of particular interest to us:

**Definition 4.1.1.** *We say that an m-tuple of functions $(u_1, u_2, ..u_m)$ is c-conjugate if for all $i$*

$$u_i(x_i) = \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big(c(x_1, x_2, ..., x_m) - \sum_{j \neq i} u_j(x_j)\Big)$$

[1]Note that we do *not* assume $M_i$ is complete, however, as we do not wish to exclude, for example, bounded, convex domains in $\mathbb{R}^n$.

Whenever $(u_1, u_2, ..u_m)$ is $c$-conjugate, the $u_i$ are semi-concave and hence have super differentials $\overline{\partial} u_i(x_i)$ at each point $x_i \in M_i$. By compactness, for each $x_i \in M_i$ we can find $x_j \in \overline{M_j}$ for all $j \neq i$ such that $u(x_i) = c(x_1, x_2, ..., x_m) - \sum_{j \neq i} u_j(x_j)$; furthermore, as long as $|u_i(x_i)| < \infty$ for at least one $x_i$, $u_i$ is locally Lipschitz [54].

The following theorem makes explicit the link between the Kantorovich problem and its dual.

**Theorem 4.1.1.** *There exists a solution $\mu$ to the Kantorovich problem and a c-conjugate solution $(u_1, u_2, ..., u_m)$ to its dual. Furthermore, the maximum value in $\boldsymbol{D}$ coincides with the minimum value in $\boldsymbol{K}$. Finally, for any solution $\mu$ to $\boldsymbol{K}$, any c-conjugate solution $(u_1, u_2, ..., u_m)$ to $\boldsymbol{D}$ and any $(x_1, ..., x_m) \in spt(\mu)$ we have $\sum_{i=1}^{m} u_i(x_i) = c(x_1, ..., x_m)$.*

This result is well known in the two marginal case; for $m \geq 3$, the existence of solutions to **K** and **D** as well as the equality of their extremal values was proved in [42]. The remaining conclusions were proved for a special cost by Gangbo and Święch [37] and for a general, continuous cost when each $M_i = \mathbb{R}^n$ by Carlier and Nazaret [21]. The same proof applies for more general spaces $M_i$; we reproduce it below in the interest of completeness.

*Proof.* As mentioned above, a proof of the existence of solutions $\mu$ to **K** and $(v_1, v_2, ..., v_m)$ to **D** as well as the equality:

$$\sum_{i=1}^{m} \int_{M_i} v_i(x_i) d\mu_i = \int_{M_1 \times M_2 .... \times M_m} c(x_1, x_2, x_3, ..., x_m) d\mu \tag{4.1}$$

can be found in [42]. We use a convexification trick, also found in [37] and [21], to build a $c$-conjugate solution to **D**.

Define

$$u_1(x_1) = \inf_{\substack{x_j \in M_j \\ j \geq 2}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j=2}^{m} v_j(x_j) \Big)$$

and $u_i$ inductively by

$$u_i(x_i) = \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j=1}^{i-1} u_j(x_j) - \sum_{j=i+1}^{m} v_j(x_j) \Big)$$

As

$$u_m(x_m) = \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j=1}^{m-1} u_j(x_j) \Big),$$

we immediately obtain

$$u_i(x_i) \leq \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j \neq i} u_j(x_j) \Big). \tag{4.2}$$

The definition of $u_{i-1}$ implies that for all $(x_1, x_2, ...x_m)$

$$v_i(x_i) \leq c(x_1, x_2, ..., x_m) - \sum_{j=1}^{i-1} u_j(x_j) - \sum_{j=i+1}^{m} v_j(x_j)$$

Therefore, $v_i(x_i) \leq u_i(x_i)$. It then follows that

$$\begin{aligned} u_i(x_i) &= \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j=1}^{i-1} u_j(x_j) - \sum_{j=i+1}^{m} v_j(x_j) \Big) \\ &\geq \inf_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j \neq i} u_j(x_j) \Big), \end{aligned}$$

which, together with (4.2), implies that $(u_1, u_2, ..., u_m)$ is $c$-conjugate. Now, we have

$$\begin{aligned} \sum_{i=1}^{m} \int_{M_i} v_i(x_i) d\mu_i &\leq \sum_{i=1}^{m} \int_{M_i} u_i(x_i) d\mu_i \\ &= \sum_{i=1}^{m} \int_{M_1 \times M_2 .... \times M_m} u_i(x_i) d\mu \\ &\leq \int_{M_1 \times M_2 .... \times M_m} c(x_1, x_2, x_3, ..., x_m) d\mu \end{aligned}$$

and so by (4.1) we must have

$$\sum_{i=1}^{m} \int_{M_i} u_i(x_i) d\mu_i = \sum_{i=1}^{m} \int_{M_1 \times M_2 .... \times M_m} u_i(x_i) d\mu = \int_{M_1 \times M_2 .... \times M_m} c(x_1, x_2, x_3, ..., x_m) d\mu$$

But because $\sum_{i=1}^{m} u_i(x_i) \leq c(x_1, x_2, x_3, ..., x_m)$, we must have equality $\mu$ almost everywhere. Continuity then implies equality holds on $spt(\mu)$. □

As a corollary to the duality theorem, we now prove a uniqueness result for the solution to **D**. When $m = 2$, this result, under the weak conditions on $c$ stated below, is due to Chiappori, McCann and Nesheim [22]; for certain special, multi-marginal costs, it was proven by Gangbo and Święch [37] and Carlier and Nazaret [21]. Although this result is tangential to the main goals of this chapter, we prove it here to emphasize that, whereas uniqueness in **K** requires certain structure conditions on the cost, uniqueness in **D** depends only on the differentiability of $c$.

**Corollary 4.1.1.** *Suppose the domains $M_i$ are all connected, that $c$ is continuously differentiable and that each $\mu_i$ is absolutely continuous with respect to local coordinates with a strictly positive density. If $(v_1, v_2, ..., v_m)$ and $(\overline{v_1}, \overline{v_2}, ..., \overline{v_m})$ solve **D**, then there exist constants $t_i$ for $i = 1, 2.,,, m$ such that $\sum_{i=1}^{m} t_i = 0$ and $v_i = \overline{v_i} + t_i$, $\mu_i$ almost everywhere, for all $i$.*

*Proof.* Using the convexification trick in the proof of Theorem 4.1.1, we can find $c$-conjugate solutions $(u_1, u_2, ..., u_m)$ and $(\overline{u_1}, \overline{u_2}, ..., \overline{u_m})$ to **D** such that $v_i(x_i) \leq u_i(x_i)$ and $\overline{v_i}(x_i) \leq \overline{u_i}(x_i)$ for all $x_i \in M_i$. Now, as

$$\sum_{i=1}^{m} \int_{M_i} v_i(x_i) d\mu_i = \sum_{i=1}^{m} \int_{M_i} u_i(x_i) d\mu_i$$

we must have $v_i = u_i$, $\mu_i$ almost everywhere. Similarly, $\overline{v_i} = \overline{u_i}$, $\mu_i$ almost everywhere. Now, choose $x_i \in M_i$ where $u_i$ and $\overline{u_i}$ are differentiable. Then there exists $x_j$ for all $j \neq i$ such that

$$(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1} ..., x_m) \in spt(\mu);$$

Theorem 4.1.1 then yields

$$u_i(x_i) - c(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1} ..., x_m) = -\sum_{j \neq i} u_j(x_j).$$

Because

$$u_i(z_i) - c(x_1, x_2, ..., x_{i-1}, z_i, x_{i+1} ..., x_m) \leq -\sum_{j \neq i} u_j(x_j)$$

for all other $z_i \in M_i$ we must have

$$Du_i(x_i) = D_{x_i}c(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}..., x_m).$$

Similarly,

$$D\overline{u_i}(x_i) = D_{x_i}c(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}..., x_m),$$

hence $Du_i(x_i) = D\overline{u_i}(x_i)$. As this equality holds for almost all $x_i$ we conclude $u_i(x_i) = \overline{u_i}(x_i) + t_i$ for some constant $t_i$. Choosing any $(x_1, x_2, ..., x_m) \in \mathrm{spt}(\mu)$ and noting that

$$\sum_{i=1}^{m} u_i(x_i) = c(x_1, x_2, ..., x_m) = \sum_{i=1}^{m} \overline{u_i}(x_i),$$

we obtain $\sum_{i=1}^{m} t_i = 0$. □

The next two definitions are straightforward generalizations of concepts borrowed from the two marginal setting.

**Definition 4.1.2.** *For $i \neq j$, we say that $c$ is $(i, j)$-twisted if the map $x_j \in M_j \mapsto D_{x_i}c(x_1, x_2, ..., x_m) \in T^*_{x_i}M_i$ is injective, for all fixed $x_k$, $k \neq j$.*

**Definition 4.1.3.** *We say that $c$ is $(i, j)$-non-degenerate if $D^2_{x_i x_j}c(x_1, x_2, ..., x_m)$, considered as a map from $T_{x_j}M_j$ to $T^*_{x_i}M_i$, is injective for all $(x_1, x_2, ..., x_m)$.*

In local coordinates, non-degeneracy simply means that the corresponding matrix of mixed, second order partial derivatives has a non-zero determinant. When this condition holds, the inverse map $T^*_{x_i}M_i \to T_{x_j}M_j$ will be denoted by $(D^2_{x_i x_j}c)^{-1}(x_1, x_2, ..., x_m)$.

When $m = 2$, the non-degeneracy condition is not needed to ensure the existence of an optimal map (although it plays an important role in studying the regularity of that map). On the other hand, the twist condition plays an essential role in showing that Monge's problem has a solution; it ensures that a first order, differential condition arising from the duality theorem can be solved uniquely for one variable as a function of the other [46] (see also [12], [36] and [14]). In light of this, one might expect that, for $m \geq 3$, if $c$ is $(i, j)$-twisted for all $i \neq j$, then the Kantorovich solution $\mu$ induces a Monge solution. This is not true, as our examples in chapter 3 demonstrate; see Propositions 3.2.1.3, 3.2.1.4 and 3.2.1.5. In the multi-marginal problem, duality yields $m$ first order conditions; our strategy in this paper is to show that if we fix the first variable, these equations can be uniquely solved for the other $m-1$ variables. In the problems considered by Gangbo and Święch [37] and Heinich [39], these equations turn out to have a particularly simple form and can be solved explicitly. For more general cost functions, this becomes a much more subtle issue. Our proof will combine a second order, differential condition with tools from convex analysis and will require that the tensor $T$, defined below, is negative definite.

**Definition 4.1.4.** *Suppose $c$ is $(1,m)$-non-degenerate. Let $\vec{y} = (y_1, y_2, ..., y_m) \in M_1 \times M_2 \times ... \times M_m$. For each $i := 2, 3, ..., m-1$ choose a point $\vec{y}(i) = (y_1(i), y_2(i), ..., y_m(i)) \in \overline{M_1} \times \overline{M_2} \times ... \times \overline{M_m}$ such that $y_i(i) = y(i)$. Define the following bi-linear maps on $T_{y_2}M_2 \times T_{y_3}M_3 \times ... \times T_{y_{m-1}}M_{m-1}$:*

$$S_{\vec{y}} = -\sum_{j=2}^{m-1}\sum_{\substack{i=2\\ i\neq j}}^{m-1} D^2_{x_ix_j}c(\vec{y}) + \sum_{i,j=2}^{m-1}(D^2_{x_ix_m}c(D^2_{x_1x_m}c)^{-1}D^2_{x_1x_j}c)(\vec{y})$$

$$H_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)} = \sum_{i=2}^{m-1}(Hess_{x_i}c(\vec{y}(i)) - Hess_{x_i}c(\vec{y}))$$

$$T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)} = S_{\vec{y}} + H_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)}$$

Note that $D^2_{x_ix_j}c(x_1, x_2, ..., x_m), Hess_{x_i}c(x_1, x_2, ..., x_m)$ and the composition

$$\big(D_{x_ix_m}c(D^2_{x_1x_m}c)^{-1}D^2_{x_1x_j}c\big)(x_1, x_2, ..., x_m)$$

are actually bi-linear maps on the spaces $T_{x_i}M_i \times T_{x_j}M_j$, $T_{x_i}M_i \times T_{x_i}M_i$ and $T_{x_i}M_i \times T_{x_j}M_j$, respectively, but we can extend them to maps on the product space $(T_{x_2}M_2 \times T_{x_3}M_3 \times ... \times T_{x_{m-1}}M_{m-1})^2$ by considering only the appropriate components of the tangent vectors.

Though $T$ looks complicated, it appears naturally in our argument. The condition $T < 0$ is in one sense analogous to the twist and non-degeneracy conditions that are so important in the two marginal problem. Like the non-degeneracy condition, negativity of $S$ is an inherently local property on $M_1 \times M_2 \times ... \times M_m$; under this condition, one can show that our system of equations is locally uniquely solvable. To show that the solution is actually globally unique requires something more; in the two marginal case, this is the twist condition, which can be seen as a global extension of non-degeneracy. In our setting, requiring that the sum $T = S + H < 0$ turns out to be enough to ensure that the locally unique solution is in fact globally unique.

## 4.2 Monge solutions

We are now in a position to precisely state our main theorem:

**Theorem 4.2.1.** *Suppose that:*

1. *$c$ is $(1, m)$-non-degenerate.*

2. *$c$ is $(1, m)$-twisted.*

3. *For all choices of $\vec{y} = (y_1, y_2, ..., y_m) \in M_1 \times M_2 \times ... \times M_m$ and of $\vec{y}(i) = (y_1(i), y_2(i), ..., y_m(i)) \in \overline{M_1} \times \overline{M_2} \times ... \times \overline{M_m}$ such that $y_i(i) = y_i$ for $i = 2, ..., m-1$, we have*

$$T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)} < 0. \tag{4.3}$$

4. *The first marginal $\mu_1$ does not charge sets of Hausdorff dimension less than or equal to $n-1$.*

*Then any solution $\mu$ to the Kantorovich problem is concentrated on the graph of a function; that is, there exist functions $G_i : M_1 \to M_i$ such that*

$$graph(\vec{G}) = \{(x_1, G_2(x_1), G_3(x_1), ..., G_m(x_1))\}$$

*satisfies $\mu(graph(\vec{G})) = 1$*

*Proof.* Let $u_i$ be a $c$-conjugate solution to the dual problem. Now, $u_1$ is semi-concave and hence differentiable off a set of Hausdorff dimension $n-1$; as $\mu_1$ vanishes on every set of Hausdorff dimension less than or equal to $n-1$, by Theorem 4.1.1 it suffices to show that for every $x_1 \in M_1$ where $u_1$ is differentiable, there is at most one $(x_2, x_3, ..., x_m) \in M_2 \times M_3 \times ... \times M_m$ such that $\sum_{i=1}^m u_i(x_i) = c(x_1, x_2, x_3, ..., x_m)$. Note that this equality implies that $D_{x_i}c(x_1, x_2, ..., x_m) \in \overline{\partial} u_i(x_i)$ for all $i = 1, 2..., m$; in particular, as $u_1$ is differentiable at $x_1$, $Du_1(x_1) = D_{x_1}c(x_1, x_2, ..., x_m)$. Our strategy will be to show that these inclusions can hold for at most one $(x_2, x_3, ..., x_m)$.

Fix a point $x_1$ where $u_1$ is differentiable. Twistedness implies that the equation $Du_1(x_1) = D_{x_1}c(x_1, x_2, ..., x_m)$ defines $x_m$ as a function $x_m = F_{x_1}(x_2, ..., x_{m-1})$ of the variables $x_2, x_3, ..., x_{m-1}$; non-degeneracy and the implicit function theorem then imply that $F_{x_1}$ is continuously differentiable with respect to $x_2, x_3, ..., x_{m-1}$ and

$$D_{x_i}F_{x_1}(x_2, ..., x_{m-1}) = -\big((D^2_{x_1x_m}c)^{-1}D^2_{x_1x_i}c)(x_1, x_2, ..., F_{x_1}(x_2, ..., x_{m-1})\big)$$

for $i = 2, ..., m-1$. We will show that there exists at most one point $(x_2, x_3, ..., x_{m-1}) \in M_2 \times M_3 \times ... \times M_{m-1}$ such that

$$D_{x_i}c(x_1, x_2, ..., F_{x_1}(x_2, ..x_{m-1})) \in \overline{\partial} u_i(x_i)$$

for all $i = 2, ..., m-1$.

The proof is by contradiction; suppose there are two such points, $(x_2, x_3, ..., x_{m-1})$ and $(\overline{x_2}, \overline{x_3}, ..., \overline{x_{m-1}})$. For $i = 2, ..., m-1$, we can choose Riemannian geodesics $\gamma_i(t)$ in $M_i$ such that $\gamma_i(0) = x_i$ and $\gamma_i(1) = \overline{x_i}$. Take a measurable selection of covectors

$V_i(t) \in \partial u_i(\gamma_i(t))$. We will show that $f(1) < f(0)$, where

$$f(t) := \sum_{i=2}^{m-1} [V_i(t) - D_{x_i} c(x_1, \vec{\gamma}(t)] \langle \frac{d\gamma_i}{dt} \rangle$$

and we have used $(x_1, \vec{\gamma}(t))$ as a shorthand for

$$(x_1, \gamma_2(t), ..., \gamma_{m-1}(t), F_{x_1}(\gamma_2(t), ..., \gamma_{m-1}(t)))$$

and $a < b >$ to denote denote the duality pairing between a 1-form $a$ and a vector $b$. This will clearly imply the desired result.

For each $t$ and each $i = 2, ..., m-1$, by $c$-conjugacy of $u_i$ and the compactness of $\overline{M_j}$, we have

$$u_i(\gamma_i(t)) = \min_{\substack{x_j \in M_j \\ j \neq i}} \Big( c(x_1, x_2, ..., x_m) - \sum_{j \neq i} u_j(x_j) \Big)$$

For $j \neq i$, choose points $y_j(i;t) \in M_j$ where the minimum above is attained. Set $y_i(i;t) = \gamma_i(t)$ and denote $\vec{y}(i;t) = (y_1(i;t), y_2(i;t), ..., y_m(i:t)) \in \overline{M_1} \times \overline{M_2} \times ... \times \overline{M_m}$. We then have

$$\sum_{j=1}^{m} u_j(y_j(i;t)) = c(y_1(i;t), y_2(i;t), ..., y_m(i;t))$$

Note that $V_i(t)\langle \frac{d\gamma_i}{dt} \rangle$ supports the semi-concave function $T \in [0,1] \mapsto u_i(\gamma_i(t))$. But $u_i(\gamma_i(t))$ is twice differentiable almost everywhere and hence we have $V_i(t)\langle \frac{d\gamma_i}{dt} \rangle = \frac{d(u_i(\gamma_i(t)))}{dt}$ for almost all $t$ and, by semi-concavity, $V_i(1)\langle \frac{d\gamma_i}{dt} \rangle - V_i(0)\langle \frac{d\gamma_i}{dt} \rangle \leq \int_0^1 \frac{d^2(u_i(\gamma_i(t)))}{dt^2} dt$. Now, for any $t, s \in [0,1]$

$$u_i(\gamma_i(t)) \leq c(y_1(i;s), y_2(i;s), ..., y_{i-1}(i;s), \gamma_i(t), y_{i+1}(i;s)..., y_m(i;s)) - \sum_{j \neq i} u_j(y_j(i;s))$$

and we have equality when $t = s$, as $\gamma_i(s) = y_i(i;s)$. Hence, whenever $\frac{d^2(u_i(\gamma_i(t)))}{dt^2}$ exists, we have

$$\begin{aligned} \frac{d^2(u_i(\gamma_i(t)))}{dt^2}\bigg|_{t=s} &\leq \frac{d^2(c(y_1(i;s), y_2(i;s), ..., y_{i-1}(i;s), \gamma_i(t), y_{i+1}(i;s)..., y_m(i;s)))}{dt^2}\bigg|_{t=s} \\ &= Hess_{x_i} c(y_1(i;s), y_2(i;s), ..., y_m(i;s)) \langle \frac{d\gamma_i}{ds}, \frac{d\gamma_i}{ds} \rangle \end{aligned}$$

We conclude that

$$V_i(1)\langle\frac{d\gamma_i}{dt}\rangle - V_i(0)\langle\frac{d\gamma_i}{dt}\rangle \leq \int_0^1 Hess_{x_i}c(y_1(i;t), y_2(i;t), ..., y_m(i;t))\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_i}{dt}\rangle dt \tag{4.4}$$

Turning now to the other term in $f(1) - f(0)$, we have

$$\begin{aligned}
& D_{x_i}c(x_1, \vec{\gamma(1)})\langle\frac{d\gamma_i}{dt}\rangle - D_{x_i}c(x_1, \vec{\gamma(0)})\langle\frac{d\gamma_i}{dt}\rangle \\
= & \int_0^1 \frac{d}{dt}\big(D_{x_i}c(x_1, \vec{\gamma}(t))\langle\frac{d\gamma_i}{dt}\rangle\big)dt \\
= & \int_0^1 \Bigg(\sum_{\substack{j=2 \\ j\neq i}}^{m-1}\Big(D^2_{x_ix_j}c(x_1, \vec{\gamma}(t))\Big)\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_j}{dt}\rangle \ + Hess_{x_i}c(x_1, \vec{\gamma}(t))\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_i}{dt}\rangle \\
& + \sum_{j=2}^{m-1}\Big(D^2_{x_ix_m}c(x_1, \vec{\gamma}(t))D_{x_j}F_{x_1}(\vec{\gamma}(t))\Big)\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_j}{dt}\rangle\Bigg)dt \\
= & \int_0^1 \Bigg(\sum_{\substack{j=2 \\ j\neq i}}^{m-1}\Big(D^2_{x_ix_j}c(x_1, \vec{\gamma}(t))\Big)\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_j}{dt}\rangle + Hess_{x_i}c(x_1, \vec{\gamma}(t))\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_i}{dt}\rangle \\
& - \sum_{j=2}^{m-1}\Big((D^2_{x_ix_m}c(D^2_{x_1x_m}c)^{-1}D^2_{x_1x_j}c)(x_1, \vec{\gamma}(t))\Big)\langle\frac{d\gamma_i}{dt}, \frac{d\gamma_j}{dt}\rangle\Bigg)dt
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\begin{aligned}
f(1) - f(0) \ & \leq \textstyle\int_0^1 T_{\left(x_1, \vec{\gamma}(t)\right), \vec{y}(2;t), \vec{y}(3;t), ..., \vec{y}(m-1;t)}\langle\frac{d\vec{\gamma}}{dt}, \frac{d\vec{\gamma}}{dt}\rangle dt \\
& < 0
\end{aligned}$$

□

**Corollary 4.2.2.** *Under the same conditions as Theorem 1, the Monge problem* $\boldsymbol{M}$ *admits a unique solution and the solution to the Kantorovich problem* $\boldsymbol{K}$ *is unique.*

*Proof.* We first show that the $G_i$ defined in Theorem 4.2.1 push $\mu_1$ to $\mu_i$ for all $i = 2, 3, ..m$. Pick a Borel set $B \in M_i$. We have

$$\begin{aligned}
\mu_i(B) &= \mu(M_1 \times M_2 \times ... \times M_{i-1} \times B \times M_{i+1} \times ... \times M_m) \\
&= \mu\Big((M_1 \times M_2 \times ... \times M_{i-1} \times B \times M_{i+1} \times ... \times M_m) \cap \text{graph}(\vec{G})\Big) \\
&= \mu\Big(\{(x_1, G_2(x_1), ..., G_m(x_1)|G_i(x_1) \in B\}\Big) \\
&= \mu\Big((G_i^{-1}(B) \times M_2 \times ... \times M_m) \cap \text{graph}(\vec{G})\Big) \\
&= \mu(G_i^{-1}(B) \times M_2 \times ... \times M_m) \\
&= \mu_1(G_i^{-1}(B))
\end{aligned}$$

This implies that $(G_2, G_3, ..., G_m)$ solves **M**. To prove uniqueness of $\mu$, note that any other optimizer $\overline{\mu}$ must also be concentrated on $\text{graph}(\vec{G})$, which in turn implies $\overline{\mu} = (\text{Id}, G_2, ..., G_m)_{\#}\mu_1 = \mu$. Uniqueness of $(G_2, G_3, ..G_m)$ now follows immediately; if $(\overline{G_2}, \overline{G_3}, ..., \overline{G_m})$ is another solution to **M** then $(\text{Id}, \overline{G_2}, \overline{G_3}, ..., \overline{G_m})_{\#}\mu_1$ is another solution to **K**, which must then be concentrated on $\text{graph}(\vec{G})$. This means that $G_i = \overline{G}_i$, $\mu_1$ almost everywhere. □

## 4.3 Examples

In this section, we discuss several types of cost functions to which Theorem 4.2.1 applies. In these examples, the complicated tensor $T$ simplifies considerably.

**Example 4.3.1.** *(Perturbations of concave functions of the sum) Gangbo and Święch [37] and Heinich [39] treated cost functions defined on $(\mathbb{R}^n)^m$ by $c(x_1, x_2, ..., x_m) = h(\sum_{k=1}^m x_k)$ where $h : \mathbb{R}^n \to \mathbb{R}$ is strictly concave. Here, we make the slightly stronger assumption that $h$ is $C^2$ with $D^2h < 0$. Assuming each $\mu_i$ is compactly supported, we can take each $M_i$ to be a bounded, convex domain in $\mathbb{R}^n$. Now, $D_{x_i}c(x_1, x_2, ..., x_m) = Dh(\sum_{k=1}^m x_k)$ and $D^2_{x_ix_j}c(x_1, x_2, ..., x_m)) = D^2h(\sum_{k=1}^m x_k)$, where we have made the obvious identification between tangent spaces at different points. $c$ is then clearly $(1, m)$-twisted and $(1, m)$-non-degenerate. Furthermore, the bi-linear map $S_{\vec{y}}$ on $(\mathbb{R}^n)^{m-2}$ is block diagonal, and each of* *its diagonal blocks is*

$$D^2h(\sum_{k=1}^{m} y_k).$$

*Similarly, as* $Hess_{x_i}c(\vec{y}(i)) = D^2h(\sum_{k=1}^{m} y_k(i))$ *and* $Hess_{x_i}c(\vec{y}) = D^2h(\sum_{k=1}^{m} y_k)$, $H_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)}$ *is block diagonal and its* $i$*th diagonal block is*

$$D^2h(\sum_{k=1}^{m} y_k(i)) - D^2h(\sum_{k=1}^{m} y_k).$$

*Therefore,* $T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)}$ *is block diagonal and its* $i$*th diagonal block is*

$$D^2h(\sum_{k=1}^{m} y_k(i)).$$

*This is clearly negative definite. Furthermore,* $C^2$ *perturbations of this cost function will also satisfy* $T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)} < 0$*; this shows that the results of Gangbo and Święch and Heinich are robust with respect to perturbations of the cost function.*

**Example 4.3.2.** *(Bi-linear costs) We now turn to bi-linear costs; suppose* $c : (\mathbb{R}^n)^m \to \mathbb{R}$ *is given by* $c(x_1, x_2, ..., x_m) = \sum_{i \neq j}(x_i)^T A_{ij} x_j$ *for* $n$ *by* $n$ *matrices* $A_{ij}$*. If* $A_{1m}$ *is non-singular,* $c$ *is* $(1, m)$*-twisted and* $(1, m)$*-non-degenerate. Now, the Hessian terms in* $T$ *vanish and so the condition* $T < 0$ *becomes a condition on the* $A_{ij}$*. For example, when* $m = 3$*, we have* $T = A_{21}(A_{31})^{-1}A_{32}$*;* $T < 0$ *is the same condition that ensures the solution to* ***K*** *is contained in an* $n$ *dimensional submanifold in the preceding chapter; see Theorem 3.1.3 and Lemma 3.3.2.*

*Note that after changing coordinates in* $x_2$ *and* $x_3$*, we can assume any bi-linear three-marginal cost is of the form*

$$c(x_1, x_2, x_3) = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2^T A x_3$$

*In these coordinates, the threefold product* $A_{21}(A_{31})^{-1}A_{32} = A^T$*. Applying the linear change of coordinates*

$$x_1 \mapsto U_1 x_1$$

$$x_2 \mapsto U_2 x_2$$

$$x_3 \mapsto U_3 x_3$$

*yields*

$$c(x_1, x_2, x_3) = x_1^T U_1^T U_2 x_2 + x_1^T U_1^T U_3 x_3 + x_2^T U_2^T A U_3 x_3$$

*If $A$ is negative definite and symmetric, then we can choose $U_3 = U_2$ such that $U_2^T A U_3 = -I$ and $U_1 = -(U_2^T)^{-1}$ to obtain*

$$c(x_1, x_2, x_3) = -x_1^T x_2 - x_1^T x_3 - x_2^T x_3$$

*which is equivalent* [2] *to the cost of Gangbo and Święch. As the symmetry of $D^2_{x_2x_1}c(D^2_{x_3x_1}c)^{-1}D^2_{x_3x_2}c$ is independent of our choice of coordinates, we conclude that $c$ is equivalent to Gangbo and Święch's cost if and only if $A_{21}(A_{31})^{-1}A_{32}$ is symmetric and negative definite. Thus, when $m = 3$ our result restricted to bi-linear costs generalizes Gangbo and Święch's theorem from costs for which $A_{21}(A_{31})^{-1}A_{32}$ is symmetric and negative definite to ones for which it is only negative definite.*

**Example 4.3.3.** *There is another class of three marginal problems which Theorem 4.2.1 applies to: on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, set*

$$c(x_1, x_2, x_3) = g(x_1, x_3) + \frac{|x_1 - x_2|^2}{2} + \frac{|x_3 - x_2|^2}{2}.$$

*If $g(x_1, x_3) = \frac{|x_1 - x_3|^2}{2}$, this is equivalent to the cost of Gangbo and Święch. More generally, if $g$ is $(1,3)$-twisted and non-degenerate, then $c$ is as well. Moreover, if we make the usual identification between tangent spaces at different points in $\mathbb{R}^n$, we have*

$$T_{\vec{y},\vec{y}(2)} = \left(D^2_{x_1x_3}g(y_1, y_3)\right)^{-1}.$$

*Hence, if $D^2_{x_1x_3}g(y_1, y_3) < 0$, we have $T_{\vec{y},\vec{y}(2)} < 0$. This will be the case if, for example, $g(x_1, x_3) = h(x_1 - x_3)$ for $h$ uniformly convex or $g(x_1, x_3) = h(x_1 + x_3)$ for $h$ uniformly concave.*

[2] We say cost functions $c$ and $\bar{c}$ are equivalent if $\bar{c}(x_1, x_2, ..., x_m) = c(x_1, x_2, ..., x_m) + \sum_{i=1}^m g_i(x_i)$. As the effect of the $g_i$'s is to shift the functionals $C(G_2, G_3, ..., G_m)$ and $C(\mu)$ by the constant $\sum_{i=1}^m \int_{M_i} g_i(x_i) d\mu_i$, studying $c$ is essentially equivalent to studying $\bar{c}$.

**Example 4.3.4.** *(Hedonic Pricing) As was outlined in chapter 3, Chiappori, McCann and Nesheim [22] and Carlier and Ekeland [20] showed that finding equilibrium in a certain hedonic pricing model is equivalent to solving a multi-marginal optimal transportation problem with a cost function of the form* $c(x_1, x_2, ..., x_m) = \inf_{z \in Z} \sum_{i=1}^m f_i(x_i, z)$. *Let us assume:*

1. *$Z$ is a $C^2$ smooth $n$-dimensional manifold.*

2. *For all $i$, $f_i$ is $C^2$ and non-degenerate.*

3. *For each $(x_1, x_2, ..., x_m)$ the infinum is attained by a unique $z(x_1, x_2, ..., x_m) \in Z$ and*

4. *$\sum_{i=1}^m D^2_{zz} f_i(x_i, z(x_1, x_2, ..., x_m))$ is non-singular.*

*In chapter 3, we showed that these conditions implied that $c$ is $C^2$ and $(i, j)$-non-degenerate for all $i \neq j$; we then showed that the support of any optimizer is contained in an $n$-dimensional Lipschitz submanifold of the product $M_1 \times M_2 \times ...M_m$. Here we examine conditions on the $f_i$ that ensure the hypotheses of Theorem 4.2.1 are satisfied. If, for fixed $i \neq j$, we assume in addition that:*

5. *$f_i$ is $x_i, z$ twisted (that is, $z \mapsto D_{x_i} f_i(x_i, z)$ is injective) and*

6. *$f_j$ is $z, x_j$ twisted.*

*then $c$ is $(i, j)$-twisted. Indeed, note that $c(x_1, x_2, ..., x_m) \leq \sum_{i=1}^m f_i(x_i, z))$ with equality when $z = z(x_1, x_2, ..., x_m)$; therefore,*

$$D_{x_i} c(x_1, x_2, ..., x_m) = D_{x_i} f(x_i, z(x_1, x_2, ..., x_m)) \tag{4.6}$$

*Therefore, for fixed $x_k$ for all $k \neq j$, the map $x_j \mapsto D_{x_i} c(x_1, x_2, ..., x_m)$ is the composition of the maps $x_j \mapsto z(x_1, x_2, ..., x_m)$ and $z \mapsto D_{x_i} f(x_i, z)$. The later map is injective by assumption. Now, note that*

$$\sum_{k=1}^m D_z f_i(x_i, z(x_1, x_2, ..., x_m)) = 0;$$

*hence,*

$$D_z f_j(x_j, z(x_1, x_2, ..., x_m)) = -\sum_{k \neq j} D_z f_k(x_k, z(x_1, x_2, ..., x_m)).$$

*Twistedness of $f_j$ now immediately implies injectivity of the first map.*

*We now investigate the form of the tensor $T$.*

*As $A(x_1, x_2, ..., x_m) := \sum_{i=1}^m D^2_{zz} f_i(x_i, z(x_1, x_2, ..., x_m))$ is non-singular by assumption, the implicit function theorem implies that $z(x_1, x_2, ..., x_m)$ is differentiable and*

$$D_{x_i} z(x_1, x_2, ..., x_m) = -\big(A(x_1, x_2, ..., x_m)\big)^{-1} D^2_{zx_i} f_i(x_i, z(x_1, x_2, ..., x_m))$$

*Furthermore, note that as $A$ is positive semi-definite by the minimality of $z \mapsto \sum_{i=1}^m f_i(x_i, z))$ at $z(x_1, x_2, ..., x_m)$, the non-singular assumption implies that it is in fact positive definite.*

*Differentiating (4.6) with respect to $x_i$ for $i = 2, 3, ..m-1$ yields:*

$$Hess_{x_i} c = -(D^2_{x_i z} f_i) D_{x_i} z + Hess_{x_i} f_i = -(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_i} f_i) + Hess_{x_i} f_i.$$

*where we have suppressed the arguments $x_1, x_2, ..x_m$ and $z(x_1, x_2, ..., x_m)$. A similar calculation yields, for all $i \neq j$,*

$$D^2_{x_i x_j} c = (D^2_{x_i z} f_i) D_{x_j} z = -(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_j} f_i)$$

*Thus, for all $i \neq j$, a straightforward calculation yields*

$$D^2_{x_i x_m} c (D^2_{x_1 x_m} c)^{-1} D^2_{x_1 x_j} c = -(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_j} f_i) = D^2_{x_i x_j} c,$$

*Hence, $S_{\vec{y}}$ is block diagonal. Furthermore, another simple calculation implies that its ith diagonal block is*

$$\Big[D^2_{x_i x_m} c (D^2_{x_1 x_m} c)^{-1} D^2_{x_1 x_i} c\Big]\Big(\vec{y}\Big) = -\Big[(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_i} f_i)\Big]\Big(\vec{y}, z(\vec{y})\Big).$$

*In addition, $H_{\vec{y}, \vec{y}(2), \vec{y}(3), ..., \vec{y}(m-1)}$ is block diagonal and its ith block is*

$$-\Big[(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_i} f_i)\Big]\Big(\vec{y}(i), z\big(\vec{y}(i)\big)\Big) + Hess_{x_i} f_i \Big(y_i, z\big(\vec{y}(i)\big)\Big)$$
$$+ \Big[(D^2_{x_i z} f_i) A^{-1} (D^2_{z x_i} f_i)\Big]\Big(\vec{y}, z(\vec{y})\Big) - Hess_{x_i} f_i \Big(y_i, z(\vec{y})\Big)$$

*Hence,* $T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)}$ *is block diagonal and its* $i$*th block is*

$$-\Big[(D_{x_i z} f_i) A^{-1} (D^2_{z x_i} f_i)\Big]\Big(\vec{y}(i), z\big(\vec{y}(i)\big)\Big) + Hess_{x_i} f_i\Big(y_i, z\big(\vec{y}(i)\big)\Big) - Hess_{x_i} f_i\Big(y_i, z(\vec{y})\Big) \tag{4.7}$$

*Therefore,* $T_{\vec{y},\vec{y}(2),\vec{y}(3),...,\vec{y}(m-1)}$ *is negative definite if and only if each of its diagonal blocks is. Now,* $A$ *is symmetric and positive definite; therefore* $A^{-1}$ *is as well. The first term in the* $i$*th block of (4.7) is therefore negative definite; the entire block will be negative definite if this term dominates the difference of the Hessian terms. This is the case if, for example,* $M_i = \mathbb{R}^n$ *and* $f_i$ *takes the form* $f_i(x_i, z) = x_i \alpha_i(z) + \beta_i(x_i) + \lambda_i(z)$ *for all* $i = 2, 3, ..., m-1$*, in which case* $Hess_{x_i} f_i\Big(y_i, z\big(\vec{y}(i)\big)\Big) = Hess_{x_i} f_i\Big(y_i, z(\vec{y})\Big)$.

# Chapter 5

# Regularity of optimal maps when $m = 2$ and $n_1 \neq n_2$.

In this chapter, we study how the regularity theory for two marginals developed by Ma, Trudinger and Wang [52] and Loeper [49] extends to the case when the dimensions are uneven, $n_1 > n_2$.

Explicity, we use a counter example of Ma, Trudinger and Wang to show that unless $c$ takes the form in equation (1.1), there are smooth densities $\mu_1$ and $\mu_2$, bounded above and below, for which the optimal map is discontinuous.

In the first section, we will introduce preliminary concepts from the regularity theory of optimal transportation, suitably adapted for general values of $n_1 \geq n_2$. In the second section, we prove that $c$-convexity (a necessary condition for regularity) implies the existence of a quotient map $Q$ as in equation (1.1). We then show that the properties on $Z$ which are necessary for the optimal map to be continuous follow from analogous properties on $M_1$.

For cost functions that are not of the special form (1.1), there are smooth marginals for which the optimal map is discontinuous. However, as the condition equation (1.1) is so restrictive, it is natural to ask about regularity for costs which are not of this form; any result in this direction will require stronger conditions on the marginals than smoothness. In the final section of this chapter, we address this problem when $n_1 = 2$ and $n_2 = 1$.

As in chapter 2, we will denote variables in $M_1$ and $M_2$ by $x$ and $y$, respectively.

## 5.1 Conditions and definitions

Here we develop several definitions and conditions which we will require in the following sections; many of them are similar to the definitions found in the preceding chapters. We begin with some basic notation. In what follows, we will assume that $M_1$ and $M_2$ may be smoothly embedded in larger manifolds, in which their closures, $\overline{M_1}$ and $\overline{M_2}$, are compact. If $c$ is differentiable, we will denote by $D_x c(x, y)$ its differential with respect to $x$. If $c$ is twice differentiable, $D^2_{xy} c(x, y)$ will denote the map from the tangent space of $M_2$ at $y$, $T_y M_2$, to the cotangent space of $M_1$ at $x$, $T^*_x M_1$, defined in local coordinates by

$$\frac{\partial}{\partial y^i} \mapsto \frac{\partial^2 c(x, y)}{\partial y^i \partial x^j} dx^j$$

where summation on $j$ is implicit, in accordance with the Einstein summation convention. $D_y c(x, y)$ and $D^2_{yx} c(x, y)$ are defined analogously.

A function $u : M_1 \to \mathbb{R}$ is called $c$-concave if $u(x) = \inf_{y \in M_2} c(x, y) - u^c(y)$, where $u^c(y) := \inf_{x \in M_1} c(x, y) - u(x)$.

Next, we introduce the concept of $c$-convexity, which first appeared in Ma, Trudinger and Wang.

**Definition 5.1.1.** *We say domain $M_2$ looks c-convex from $x \in M_1$ if $D_x c(x, M_2) = \{D^2_x c(x, y) | y \in M_2\}$ is a convex subset of $T_x M_1$. We say $M_2$ is c-convex with respect to $M_1$ if it looks c-convex from every $x \in M_1$.*

Our next definition is novel, as it is completely irrelevant when $n_1 = n_2$. It will, however, play a vital role in the present setting.

**Definition 5.1.2.** *We say domain $M_2$ looks c-linear from $x \in M_1$ if $D_x c(x, M_2)$ is contained in a shifted $n_2$-dimensional, linear subspace of $T_x M_1$. We say $M_2$ is c-linear with respect to $M_1$ if it looks c-linear from every $x \in M_1$.*

When $n_1 = n_2$, $c$-linearity is automatically satisfied. When $n_1 > n_2$, this is no longer true, although $c$-convexity clearly implies $c$-linearity.

We will also have reason to consider the level set of $\overline{x} \mapsto D_y c(\overline{x}, y)$ passing through $x$, $L_x(y) := \{\overline{x} \in M_1 : D_y c(\overline{x}, y) = D_y c(x, y)\}$.

Let us now state the first three regularity conditions introduced by Ma, Trudinger and Wang:

**(A0)**: The function $c \in C^4(\overline{M_1} \times \overline{M_2})$.

**(A1)**: (Twist) For all $x \in M_1$, the map $y \mapsto D_x c(x, y)$ is injective on $\overline{M_2}$.

**(A2)**: (Non-degeneracy) For all $x \in M_1$ and $y \in M_2$, the map $D^2_{xy} c(x, y) : T_y M_2 \to T^*_x M_1$ is injective.

**Remark 5.1.3.** *When $n_1 = n_2$, a bi-twist hypothesis is required to prove regularity of the optimal map; in addition to **(A1)**, one must assume $x \mapsto D_y c(x, y)$ is injective on $M_1$ for all $y \in M_2$. Clearly, such a condition cannot hold if $n_1 > n_2$; in fact, the non-degeneracy condition and the implicit function theorem imply that the level sets $L_x(y)$ of this mapping are smooth $n_1 - n_2$ dimensional hypersurfaces. Later, we will assume that the these level sets are connected. When $n_1 = n_2$, non-degeneracy implies that each $L_x(y)$ consists of finitely many isolated points, in which case connectedness implies that it is in fact a singleton, or, equivalently, that $x \mapsto D_y c(x, y)$ is injective.*

The statements of **(A3w)** and **(A3s)**, the most important regularity conditions, require a little more machinery. For a twisted cost, the mapping $y \mapsto D_x c(x, y)$ is invertible on its range. We define the $c$-exponential map at $x$, denoted by $c\text{-}exp_x(\cdot)$, to be its inverse; that is, $D_x c(x, c\text{-}exp_x(p)) = p$ for all $p \in D_x c(x, M_2)$.

**Definition 5.1.4.** *Let $x \in M_1$ and $y \in M_2$. Choose tangent vectors $\boldsymbol{u} \in T_x M_1$ and*

*$\boldsymbol{v} \in T_yM_2$. Set $\boldsymbol{p} = D_xc(x,y) \in T^*_xM_1$ and $\boldsymbol{q} = (D^2_{xy}c(x,y)) \cdot \boldsymbol{v} \in T^*_xM_1$; note that if $M_2$ looks c-linear at $x$, $\boldsymbol{p} + t\boldsymbol{q} \in D_xc(x,M_2)$ for small $t$. For any smooth curve $\beta(s)$ in $M_1$ with $\beta(0) = x$ and $\frac{d\beta}{ds}(0) = \boldsymbol{u}$, we define the Ma, Trudinger Wang curvature at $x$ and $y$ in the directions $\boldsymbol{u}$ and $\boldsymbol{v}$ by:*

$$MTW_{xy}\langle \boldsymbol{u}, \boldsymbol{v}\rangle := -\frac{3}{2}\frac{\partial^4 c}{\partial s^2 \partial t^2} c(\beta(s), c\text{-}exp_x(\boldsymbol{p} + t\boldsymbol{q}))$$

We are now ready to state the final conditions of Ma, Trudinger and Wang. Because they are designed to deal with the general case $n_1 \geq n_2$, our formulations look somewhat different from those found in [52]; when $n_1 = n_2$, they reduce to the standard conditions.

**(A3w)**: For all $x \in M_1$, $y \in M_2$, $\mathbf{u} \in T_xM_1$ and $\mathbf{v} \in T_yM_2$ such that $\mathbf{u} \cdot D^2_{xy}c(x,y) \cdot \mathbf{v} = 0$, $MTW_{xy}\langle \mathbf{u}, \mathbf{v}\rangle \geq 0$.

**(A3s)**: For all $x \in M_1$, $y \in M_2$, $\mathbf{u} \in T_xM_1$ and $\mathbf{v} \in T_yM_2$ such that $\mathbf{u} \cdot (D^2_{xy}c(x,y)) \cdot \mathbf{v} = 0$, $\mathbf{u} \cdot (D^2_{xy}c(x,y)) \neq 0$ and $\mathbf{v} \neq 0$ we have $MTW_{xy}\langle \mathbf{u}, \mathbf{v}\rangle > 0$.

If $n_1 = n_2$, non-degeneracy implies that the condition $\mathbf{u} \cdot (D^2_{xy}c(x,y)) \neq 0$ is equivalent to $\mathbf{u} \neq 0$.

## 5.2 Regularity of optimal maps

The following theorem asserts the existence of an optimal map. It is due to Levin [46] in the case where $M_1$ is a bounded domain in $\mathbb{R}^{n_1}$ and $\mu_1$ is absolutely continuous with respect to Lebesgue measure. The following version can be proved in the same way; see also Brenier [12], Gangbo [35], Gangbo and McCann [36] and Caffarelli [14].

**Theorem 5.2.1.** *Suppose $c$ is twisted and $\mu_1(A) = 0$ for all Borel sets $A \subseteq M_1$ of Hausdorff dimension less than or equal to $n_1 - 1$. Then the Monge problem admits a unique solution $F$ of the form $F(x) = c\text{-}exp(x, Du(x))$ for some c-concave function $u$.*

The following example confirms the necessity of $c$-convexity to regularity. It is due to Ma, Trudinger and Wang [52] in the case where $n_1 = n_2$; their proof applies to the $n_1 \geq n_2$ case as well.

**Theorem 5.2.2.** *Suppose there exists some $x \in M_1$ such that $M_2$ does not look c-convex from $x$. Then there exist smooth measures $\mu_1$ and $\mu_2$ for which the optimal map is discontinuous.*

As $c$-convexity implies $c$-linearity, this example verifies that we cannot hope to develop a regularity theory in the absence of $c$-linearity. The following lemma demonstrates that, under the $c$-linearity hypothesis, the level sets $L_x(y)$ are the same for each $y$, yielding a canonical foliation of the space $M_1$.

**Lemma 5.2.3.** *(i) $M_2$ looks c-linear from $x \in M_1$ if and only if $T_x(L_x(y))$ is independent of $y$; that is $T_x(L_x(y_0)) = T_x(L_x(y_1))$ for all $y_0, y_1 \in M_2$.*
*(ii) If the level sets $L_x(y)$ are all connected, then $M_2$ is c-linear with respect to $M_1$ if and only if $L_x(y)$ is independent of $y$ for all $x$*

*Proof.* We first prove (i). The tangent space to $L_x(y)$ at $x$ is the null space of the map $D^2_{yx}c(x,y) : T_xM_1 \mapsto T^*_yM_2$, which, in turn, is the orthogonal complement of the range of $D^2_{xy}c(x,y) : T_yM_2 \mapsto T^*_xM_1$. Therefore, $T_x(L_x(y))$ is independent of $y$ if and only if the range of $D^2_{xy}c(x,y)$ is independent of $y$. But $D^2_{xy}c(x,y)$ is the differential of the map $y \mapsto D_xc(x,y)$ (making the obvious identification between $T^*_xM_1$ and its tangent space at a point) and so its range is independent of $y$ if and only if the image of this map is linear.

To see (ii), note that (i) implies $M_2$ is $c$-linear with respect to $M_1$ if and only if $T_x(L_x(y_0)) = T_x(L_x(y_1))$ for all $x \in M_1$ and all $y_0, y_1 \in M_2$. But $T_x(L_x(y_0)) = T_x(L_x(y_1))$ for all $x$ is equivalent to $L_x(y_0) = L_x(y_1)$ for all $x$; this immediately yields (ii). □

For the remainder of this section, we will assume that $L_x(y)$ is connected and independent of $y$ for all $x$ and we will denote it simply by $L_x$. In this case, we will demonstrate now that points in the same level set are indistinguishable from an optimal transportation perspective. The $L_x$'s define a canonical foliation of $M_1$ and our problem will be reduced to an optimal transportation problem between $M_2$ and the space of leaves of this foliation. More precisely, we define an equivalence relation on $M_1$ by $x \sim \overline{x}$ if $\overline{x} \in L_x$. We then define the quotient space $Z = M_1/\sim$ and the quotient map $Q : M_1 \to Z$. Note that, for any fixed $y_0 \in M_2$, the map $x \mapsto D_y c(x, y_0) \in T_{y_0} M_2$ has the same level sets as $Q$ (namely the $L_x$'s) and is smooth by assumption. Furthermore, the non-degeneracy condition implies that this map is open and hence a quotient map. We can therefore identify $Z \approx D_y c(M_1, y_0)$ with a subse! t of the cotangent space $T^*_{y_0} M_2$. In particular, $Z$ has a smooth structure, and, if $c$ satisfies **(A0)**, $Q$ is $C^3$.

Our strategy now will be to show that if $F : M_1 \to M_2$ is the optimal map, then $F$ factors through $Q$; $F = T \circ Q$. As $Q$ is smooth, this will imply that treating the smoothness of $F$ reduces to studying the smoothness of $T$. To this end, we will show that $T$ itself solves an optimal transportation problem with marginals $\alpha = Q_{\#}\mu_1$ on $Z$ and $\mu_2$ on $M_2$ relative to the cost function $b(z, y)$ defined uniquely by:

$$D_y b(z, y) = D_y c(x, y), \text{ for } x \in Q^{-1}(z)$$

$$b(z, y_0) = 0$$

As $Z$ and $M_2$ share the same dimension, the regularity theory of Ma, Trudinger and Wang will apply in this context.

We first obtain a useful formula for the cost function $b$.

**Proposition 5.2.4.** *For any $z \in Z$, $y \in M_2$ and $x \in Q^{-1}(z)$, we have $b(z, y) = c(x, y) - c(x, y_0)$.*

*Proof.* For $y = y_0$ the result follows immediately from the definition of $h$. As $D_y b(z, y) = D_y c(x, y)$ for all $y$, the formula holds everywhere. □

Note that this implies $c(x, y) = b(Q(x), y) + c(x, y_0)$, which is equivalent to $b(Q(x), y)$ for optimal transportation purposes.

**Lemma 5.2.5.** *For any $x_0, x_1 \in L_x$, $\overline{y} \in M_2$ and $c$-concave $u$ we have $u(x_0) = c(x_0, \overline{y}) - u^c(\overline{y})$ if and only if $u(x_1) = c(x_1, \overline{y}) - u^c(\overline{y})$.*

*Proof.* First note that as $D_yc(x_0, y) - D_yc(x_1, y) = 0$ for all $y \in M_2$, the difference $c(x_0, y) - c(x_1, y)$ is independent of $y$. Now, suppose $u(x_0) = c(x_0, \overline{y}) - u^c(\overline{y})$. Then

$$\begin{aligned}
u(x_1) &= \inf_{y \in M_2} c(x_1, y) - u^c(y) \\
&= \inf_{y \in M_2} \big(c(x_1, y) - c(x_0, y) + c(x_0, y) - u^c(y)\big) \\
&= c(x_1, \overline{y}) - c(x_0, \overline{y}) + \inf_{y \in M_2} \big(c(x_0, y) - u^c(y)\big) \\
&= c(x_1, \overline{y}) - c(x_0, \overline{y}) + u(x_0) \\
&= c(x_1, \overline{y}) - u^c(\overline{y})
\end{aligned}$$

The proof of the converse is identical. □

**Proposition 5.2.6.** *Suppose $c$ is twisted and $\mu_1$ doesn't charge sets of Hausdorff dimension $n_1 - 1$. Let $F : M_1 \to M_2$ be the optimal map. Then there exists a map $T : Z \to M_2$ such that $F = T \circ Q$, $\mu_1$ almost everywhere. Moreover, $T$ solves the optimal transportation problem on $Z \times M_2$ with cost function $b$ and marginals $\alpha$ and $\mu_2$.*

*Proof.* It is well known that there exists a $c$-concave functions $u(x)$ such that, for $\mu_1$ almost every $x$, there is a unique $y \in M_2$ such that $u(x) = c(x, y) - u^c(y)$; in this case, $F(x) = y$.

For $\alpha$ almost every $z \in Z$, Lemma 5.2.5 now implies that there is a unique $y \in M_2$ such that $u(x) = c(x, y) - u^c(y)$ for all $x \in Q^{-1}(z)$; define $T(z)$ to be this $y$. In then follows immediately that $F = T \circ Q$, $\mu_1$ almost everywhere, and that $T$ pushes $\alpha$ to $\mu_2$.

Now, suppose $G : Z \to M_2$ is another map pushing $\alpha$ to $\mu_2$. Then $G \circ Q$ pushes $\mu_1$ to $\mu_2$ and because of the optimality of $F = Q \circ T$ we have

$$\int_{M_1} c(x, T \circ Q(x)) d\mu_1 \leq \int_{M_1} c(x, G \circ Q(x)) d\mu_1. \tag{5.1}$$

Now, using Proposition 5.2.4 we have

$$\begin{aligned}\int_{M_1} c(x, T \circ Q(x)) d\mu_1 &= \int_{M_1} b(Q(x), T \circ Q(x)) + c(x, y_0) d\mu_1 \\ &= \int_Z b(z, T(z)) d\alpha + \int_{M_1} c(x, y_0) d\mu_1\end{aligned}$$

Similarly,

$$\int_{M_1} c(x, G \circ Q(x)) d\mu_1 = \int_Z b(z, G(z)) d\alpha + \int_{M_1} c(x, y_0) d\mu_1$$

and so (5.1) becomes

$$\int_Z b(z, T(z)) d\alpha \leq \int_Z c(z, G(z)) d\alpha$$

Hence, $T$ is optimal. □

Having established that the optimal map $F$ from $M_1$ to $M_2$ factors through $Z$ via the quotient $Q$ and the optimal map $T$ from $Z$ to $M_2$, we will now study how the regularity conditions **(A1)-(A3s)** for $c$ translate to $b$.

Proposition 5.2.4 also allows us to understand the derivatives of $b$ with respect to $z$. Pick a point $z_0 \in Z$ and select $x_0 \in Q^{-1}(z_0)$. Now, let $S$ be an $n_2$-dimensional surface passing though $x_0$ which intersects $L_{x_0}$ transversely. As the null space of the map $D^2_{yx}c(x, y_0) : T_xM_1 \to T^*_yM_2$ is precisely $T_xL_x$ for any $y$, it is invertible when restricted to $T_xS$; by the inverse function theorem, the map $D_yc(\cdot, y_0)$ restricts to a local diffeomorphism on $S$. For all $z$ near $z_0$, there is a unique $x \in S \cap Q^{-1}(z)$ and we have $b(z, y) = c(x, y) - c(x, y_0)$; we can now identify $D_zb(z, y) \approx D_xc|_{S \times M_2}(x, y) - D_xc|_{S \times M_2}(x, y_0)$ and $D^2_{zy}b(z, y) \approx D^2_{xy}c|_{S \times M_2}(x, y)$. We use this observation to prove the following result.

**Theorem 5.2.7.** *(i) If $c$ is twisted, $b$ is bi-twisted.*

*(ii) If $c$ is non-degenerate, $b$ is non-degenerate.*

*(iii)If $M_2$ is $c$-convex, it is also $b$-convex.*

*Proof.* The injectivity of $z \mapsto D_yb(z, y)$ follows immediately from the the definition of $b$. Injectivity of $y \mapsto D_zb(z, y)$ and non-degeneracy follow from the preceding identification.

Note that transversality implies $T_x^* M_1 = T_x^* L_x \oplus T_x^* S$. Our local identification between $Z$ and $S$ identifies the projection of the range $D_x c(x, M_2)$ onto $T_x^* S$ with $D_z b(z, M_2)$. As the projection of a convex set is convex, the $b$-convexity of $M_2$ now follows from its $c$-convexity. □

**Theorem 5.2.8.** *The following are equivalent:*

1. *$b$ satisfies **(A3w)**.*

2. *$c$ satisfies **(A3w)**.*

3. *$c$ satisfies **(A3w)** when restricted to any smooth surface $S \subseteq M_1$ of dimension $n_2$ which is transverse to each $L_x$ that it intersects.*

*Proof.* The equivalence of (1) and (3) follow immediately from our identification. Clearly, (2) implies (3); to see that (3) implies (2) it suffices to show $MTW_{xy}\langle \mathbf{u}, \mathbf{v}\rangle = 0$ when $\mathbf{u} \in T_x L_x$, as $MTW_{xy}$ is linear in $\mathbf{u}$. Choosing a curve $\beta(s) \in L_x$ such that $\beta(0) = x$ and $\frac{d\beta}{ds}(0) = \mathbf{u}$ and $\mathbf{p}, \mathbf{q}$ as in the definition, we have

$$\frac{d\beta}{ds}(s) \in T_{\beta(s)} L_{\beta(s)} = \mathrm{null}\big(D^2_{xy} c(\beta(s), c\text{-}exp_x(\mathbf{p} + t\mathbf{q}))\big).$$

for all $s$ and $t$, yielding

$$\frac{d^2}{dsdt} c(\beta(s), c\text{-}exp_x(\mathbf{p} + t\mathbf{q})) = \frac{d\beta}{ds} \cdot D^2_{xy} c\big(\beta(s), c\text{-}exp_x(\mathbf{p} + t\mathbf{q})\big) \cdot \frac{d(c\text{-}exp(\mathbf{p} + t\mathbf{q}))}{dt} = 0$$

Hence, $MTW_{xy}\langle \mathbf{u}, \mathbf{v}\rangle = 0$ □

**Theorem 5.2.9.** *The following are equivalent:*

1. *$b$ satisfies **(A3s)**.*

2. *$c$ satisfies **(A3s)**.*

3. *$c$ satisfies **(A3s)** when restricted to any smooth surface $S \subseteq M_1$ of dimension $m$ which is transverse to each $L_x$ that it intersects.*

*Proof.* The equivalence follows immediately from the identification, after observing that the $v \cdot (D^2_{xy}c(x, y)) \neq 0$ condition in the definition of **(A3s)** excludes the non-transverse directions. □

Various regularity results for $T$ (and therefore $F$) now follow from the regularity results of Ma, Trudinger and Wang [52], Loeper [49] and Liu [48]. Note, however, that these results all require certain regularity hypotheses on the marginals; to apply them in the present context, we must check these conditions on $\alpha$, rather than $\mu_1$. A brief discussion on whether the relevant regularity conditions on $\mu_1$ translate to $\alpha$ therefore seems in order.

First, suppose $M_1$ is a bounded domain in $\mathbb{R}^n$ and $\mu_1 = f(x)dx$ is absolutely continuous with respect to $m$-dimensional Lebesgue measure. Then $\alpha$ is absolutely continuous with respect to $n$-dimensional Lebesgue measure with density $h(z)$ given by the coarea formula:

$$h(z) := \int_{Q^{-1}(z)} \frac{f(x)}{JQ(x)} dH^{m-n}(x)$$

where $JQ$ is the Jacobian of the map $Q$, restricted to the orthogonal complement of $T_xL_x$.

**Lemma 5.2.10.** *Suppose $f \in L^p(M_1)$ (with respect to Lebesgue measure on $M_1$) for some $p \in [1, \infty]$. Then $h \in L^p(Z)$.*

*Proof.* We have $h^p(z) = (\int_{Q^{-1}(z)} \frac{f(x)}{JQ(x)} dH^{m-n}(x))^p$. Normalizing and applying Jensen's inequality yields:

$$\begin{aligned} \frac{h^p(z)}{C^p(z)} &\leq \int_{Q^{-1}(z)} \frac{f^p(x)}{(JQ(x))^p C(z)} dH^{m-n}(x) \\ &\leq \int_{Q^{-1}(z)} \frac{f^p(x)}{JQ(x)C(z)K^{p-1}} dH^{m-n}(x) \end{aligned}$$

where $C(z)$ is the $(m-n)$-dimensional Hausdorff measure of $Q^{-1}(z)$ and $K > 0$ is a global lower bound on $JQ(x)$. Letting $C$ be a global upper bound on $C(z)$ and integrating over $z$ implies:

$$\begin{aligned}\int h^p(z)dz &\leq \int\int_{Q^{-1}(z)} \frac{f^p(x)C^{p-1}(z)}{JQ(x)^p K^{p-1}} dH^{n_1-n_2}(x)dz \\ &\leq \frac{C^{p-1}}{K^{p-1}} \int\int_{Q^{-1}(z)} \frac{f^p(x)}{JQ(x)^p} dH^{n_1-n_2}(x)dz \\ &= \frac{C^{p-1}}{K^{p-1}} \int f^p(x)dx < \infty\end{aligned}$$

where we have again used the coarea formula in the last step. □

Let us note, however, that an analogous result does not hold for the weaker condition introduced by Loeper [49], which requires that for all $x \in M_1$ and $\epsilon > 0$

$$\mu_1(B_\epsilon(x)) \leq K\epsilon^{n(1-\frac{1}{p})}$$

for some $p > n_2$ and $K > 0$. Indeed, if $n_1 - n_2 \geq n_2$, we can take $\mu_1$ to be $(n_1 - n_2)$-dimensional Hausdorff measure on a single level set $L_x$. Then $\mu_1$ will satisfy the above condition for any $p$, but $\alpha$ will consist of a single Dirac mass.

The preceding lemma allows use to immediately translate the regularity results of Loeper [49] and Liu [48] to the present setting.

**Corollary 5.2.11.** *Suppose that $M_2$ is c-convex with connected level sets $L_x(y)$ for all $x \in M_1$ and $y \in M_2$, and that **(A0)**, **(A1)**, **(A2)** and **(A3s)** hold. Suppose that $f \in L^p(M_1)$ for some $p > \frac{n_2+1}{2}$. Then the optimal map is Holder continuous with Holder exponent $\frac{\beta(n_2+1)}{2n_2^2+\beta(n_2-1)}$, where $\beta = 1 - \frac{n_2+1}{2p}$.*

The higher regularity results of Ma, Trudinger and Wang require $C^2$ smoothness of the density $h$. As the following example demonstrates, however, smoothness of $f$ does not even imply continuity of $h$.

**Example 5.2.12.** *Let*

$$M_1 = \{x = (x^1, x^2) : -1 < x^1 < 1, -1 < x^2 < \phi(x^1)\} \subseteq \mathbb{R}^2$$

*where* $\phi : (-1, 1) \to (-1, 1)$ *is a* $C^\infty$ *function such that* $\phi(x^1) = 0$ *for all* $-1 < x^1 < 0$, $\phi(1) = 1$ *and* $\phi$ *is strictly increasing on* $(0, 1)$. *Let* $M_2 = (0, 1) \subseteq \mathbb{R}$ *and* $c(x, y) = x^2 y$. *Then* $M_2$ *is c-convex and c satisfies* ***(A0)-(A3s)***. *The level sets* $L_x$ *are simply the curves* $\{x : x^2 = c\}$ *for constant values of* $c \in (-1, 1)$ *and* $Z = (-1, 1)$. *Set* $f(x) = k$, *where* $k$ *is a constant chosen so that* $\mu_1$ *has total mass 1. The density* $h$ *is then easy to compute; it is simply the length of the line segment* $Q^{-1}(z)$. *For* $z < 0$, $h(z) = 2k$; *however, for* $z > 0$, $h(z) = k(1 - \phi^{-1}(z)) < k$. [1]

On the other hand, we should note that is possible for $\alpha$ to be smooth even when $\mu_1$ is singular. This will be the case if, for example, $\mu_1$ is $n_2$-dimensional Hausdorff measure concentrated on some smooth $n_2$-dimensional surface $S$ which intersects the $L_x$'s transversely.

Finally, we exploit Loeper's counterexample, which shows that, when $n_1 = n_2$ and **(A3w)** fails, there are smooth densities for which the optimal map is not continuous.

**Corollary 5.2.13.** *Suppose that* $M_2$ *is c-convex and that the level sets* $L_x(y)$ *are connected for all* $x \in M_1$ *and* $y \in M_2$. *Assume* ***(A0)***, ***(A1)***, *and* ***(A2)*** *hold but* ***(A3w)*** *fails. Then there are smooth marginals* $\mu_1$ *on* $M_1$ *and* $\mu_2$ *on* $M_2$ *such that the optimal map is discontinuous.*

*Proof.* Using Proposition 5.2.4, it is easy to check that $u : M_1 \to \mathbb{R}$ is $c$-concave if and only it $u(x) = v(Q(x)) + c(x, y_0)$ for some $b$-concave $v : Z \to \mathbb{R}$. By [49], we know that if **(A3w)** fails, then the set of $C^1$, $b$-concave functions is not dense in the set of all $b$-concave functions in the $L^\infty(Z)$ topology. From this it follows easily that the set of $C^1$, $c$-concave functions is not dense in the set of all $c$-concave functions in the $L^\infty(M_1)$ topology. The argument in [49] now implies the desired result. □

[1] It should be noted that the while the boundary of $M_1$ is not smooth here, this is not the reason for the discontinuity in $h$; the corners of the boundary can be mollified and the density will still be discontinuous at 0.

## 5.3 Regularity for non c-convex targets

The counterexamples of Ma, Trudinger and Wang, combined with the results in the previous section imply that we cannot hope that the optimizer is continuous for arbitrary smooth data if the level sets $L_x(y)$ are not independent of $y$. It is then natural to ask for which marginals *can* we expect the optimal map to smooth? In this section, we study this question in the special case when $n_1 = 2$ and $n_2 = 1$. We identify conditions on the interaction between the marginals and the cost that allow us to find an explicit formula for the optimal map and prove that it is continuous.

We will assume $M_2 = (a, b) \subset \mathbb{R}$ is an open interval and that $M_1$ is a bounded domain in $\mathbb{R}^2$. We will also assume that $c \in C^2(\overline{M_1} \times \overline{M_2})$ satisfies **(A2)**, which in this setting simply means that the gradient $\nabla_x(\frac{\partial c}{\partial y})$ never vanishes. Therefore, the level sets $L_x(y)$ will all be $C^1$ curves. We define the following set:

$$P = \Big\{\tilde{x} \in \overline{M_1} : \forall\ y_0 < y_1 \in M_2,\ x \in L_{\tilde{x}}(y_0),\ \text{we have } \frac{\partial c(\tilde{x}, y_1)}{\partial y} \leq \frac{\partial c(x, y_1)}{\partial y}\Big\}$$

When the level sets $L_x(y)$ are independent of $y$, $P$ is the entire domain $M_1$. If not, $P$ consists of points $\tilde{x}$ for which the level sets $L_{\tilde{x}}(y)$ evolve with $y$ in a monotonic way. $L_{\tilde{x}}(y_1)$ divides the region $M_1$ into two subregions: $\{x : \frac{\partial c(\tilde{x},y_1)}{\partial y} > \frac{\partial c(x,y_1)}{\partial y}\}$ and $\{x : \frac{\partial c(\tilde{x},y_1)}{\partial y} \leq \frac{\partial c(x,y_1)}{\partial y}\}$. $\tilde{x} \in P$ ensures that for $y_0 < y_1$, the set $L_{\tilde{x}}(y_0)$ will lie entirely in the latter region. For interior points, the curves $L_{\tilde{x}}(y_0)$ and $L_{\tilde{x}}(y_1)$ will generically intersect transversely and so $L_{\tilde{x}}(y_0)$ will interect both of these regions; therefore, $P$ will typically consist only of boundary points. At each boundary point $\tilde{x}$, we can heuristically view the level curves $L_{\tilde{x}}(y)$ as rotating about the point $\tilde{x}$; $P$ consists of those points which rotate in a particular fixed direction.

In what follows, $\mu$ will be a solution to the Kantorovich problem. Recall that the support of $\mu$, or $spt(\mu)$, is the smallest closed subset of $M_1 \times M_2$ of full mass.

**Lemma 5.3.1.** *Suppose $\tilde{x} \in P, x \in M_1, y_0, y_1 \in M_2$ and $(\tilde{x}, y_1), (x, y_0) \in spt(\mu)$. Then $\frac{\partial c(x,y_1)}{\partial y} \leq \frac{\partial c(\tilde{x},y_1)}{\partial y}$ if $y_0 < y_1$ and $\frac{\partial c(x,y_1)}{\partial y} \geq \frac{\partial c(\tilde{x},y_1)}{\partial y}$ if $y_0 > y_1$.*

*Proof.* The support of $\mu$ is $c$-monotone (see [68] for a proof); this means that $c(\tilde{x}, y_1) + c(x, y_0) \leq c(\tilde{x}, y_0) + c(x, y_1)$. If $y_0 < y_1$, this implies

$$\int_{y_0}^{y_1} \frac{\partial c(\tilde{x}, y)}{\partial y} dy \leq \int_{y_0}^{y_1} \frac{\partial c(x, y)}{\partial y} dy. \tag{5.2}$$

Assume $\frac{\partial c(\tilde{x},y_1)}{\partial y} > \frac{\partial c(x,y_1)}{\partial y}$. We claim that this implies $\frac{\partial c(\tilde{x},y)}{\partial y} > \frac{\partial c(x,y)}{\partial y}$ for all $y \in [y_0, y_1]$, which contradicts (5.2). To see this, suppose that there is some $y \in [y_0, y_1]$ such that $\frac{\partial c(\tilde{x},y)}{\partial y} \leq \frac{\partial c(x,y)}{\partial y}$; the Intermediate Value Theorem then implies the existence of a $\overline{y} \in [y, y_1)$ such that $\frac{\partial c(\tilde{x},\overline{y})}{\partial y} = \frac{\partial c(x,\overline{y})}{\partial y}$, or $x \in L_{\tilde{x}}(\overline{y})$. This, together with our assumption $\frac{\partial c(\tilde{x},y_1)}{\partial y} > \frac{\partial c(x,y_1)}{\partial y}$, violates the condition $\tilde{x} \in P$.

A similar argument shows $\frac{\partial c(\tilde{x},y_1)}{\partial y} \geq \frac{\partial c(x,y_1)}{\partial y}$ if $y_0 > y_1$. □

**Definition 5.3.2.** *We say $y$ splits the mass at $x$ if*

$$\mu_1\Big(\{\overline{x} : \frac{\partial c(x, y)}{\partial y} < \frac{\partial c(\overline{x}, y)}{\partial y}\}\Big) = \mu_2([0, y))$$

*If $\mu_1$ and $\mu_2$ are absolutely continuous with respect to Lebesgue measure, this is equivalent to*

$$\mu_1\Big(\{\overline{x} : \frac{\partial c(x, y)}{\partial y} > \frac{\partial c(\overline{x}, y)}{\partial y}\}\Big) = \mu_2([y, 1])$$

Lemma 5.3.1 immediately implies the following.

**Lemma 5.3.3.** *Suppose $\mu_1$ and $\mu_2$ are absolutely continuous with respect to Lebesgue measure. Then if $\tilde{x} \in P, y \in M_2$ and $(\tilde{x}, y) \in spt(\mu)$, $y$ splits the mass at $\tilde{x}$.*

**Lemma 5.3.4.** *Suppose $\mu$ and $\mu_2$ are absolutely continuous with respect to Lebesgue. Then, for each $x \in M_1$ there is a $y \in M_2$ that splits the mass at $x$.*

*Proof.* The function $y \mapsto f_x(y) := \mu_1\Big(\{\overline{x} : \frac{\partial c(x,y)}{\partial y} < \frac{\partial c(\overline{x},y)}{\partial y}\}\Big) - \mu_2\big([0, y)\big)$ is continuous. Observe that $f_x(0) \geq 0$ and $f_x(1) \leq 0$; the result now follows from the Intermediate Value Theorem. □

Similarly, it is straightforward to prove the following lemma.

**Lemma 5.3.1.** *Suppose $\mu_1$ and $\mu_2$ are absolutely continuous with respect to Lebesgue. Then, for each $y \in M_2$ there is an $x \in M_1$ such that $y$ splits the mass at $\overline{x}$ if and only if $\overline{M_1} \in L_x(y)$.*

**Definition 5.3.2.** *Let $\tilde{x} \in P$. We say $\tilde{x}$ satisfies the mass comparison property (MCP) if for all $y_0 < y_1 \in M_2$ we have*

$$\mu_1\Big(\bigcup_{y\in[y_0,y_1]} L_{\tilde{x}}(y)\Big) < \mu_2\big([y_0, y_1]\big)$$

In the case when the level sets $L_x(y)$ are independent of $y$, the MCP is satisfied for all $x \in P = \overline{M_1}$ as long as $\mu_1$ assigns zero mass to every $L_x(y)$ and $\mu_2$ assigns non-zero mass to every open interval. Alternatively, in view of the previous section, we know that in this case the cost has the form $c(Q(x), y)$, where $Q : M_1 \to Z$ and $Z = [z_0, z_1] \subseteq \mathbb{R}$ is an interval; the MCP boils down to the assumption that $\alpha$ assigns zero mass to all singletons and $\mu_2$ assigns non-zero mass to every open interval.

**Lemma 5.3.3.** *Suppose $\mu_1$ and $\mu_2$ are absolutely continuous with respect to Lebesgue measure and that $\tilde{x} \in P$ satisfies the MCP. Then there is a unique $y \in M_2$ that splits the mass at $\tilde{x}$.*

*Proof.* Existence follows from Lemma 5.3.4; we must only show uniqueness. Suppose $y_0 < y_1 \in M_2$ both split the mass at $\tilde{x}$. For any $x$ such that $\frac{\partial c(x,y_0)}{\partial y} > \frac{\partial c(\tilde{x},y_0)}{\partial y}$ and $\frac{\partial c(x,y_1)}{\partial y} < \frac{\partial c(\tilde{x},y_1)}{\partial y}$ the Intermediate Value Theorem yields a $y \in [y_0, y_1]$ such that $x \in L_{\tilde{x}}(y)$; hence,

$$\Big\{x : \frac{\partial c(x, y_0)}{\partial y} > \frac{\partial c(\tilde{x}, y_0)}{\partial y}\Big\} \cap \Big\{x : \frac{\partial c(x, y_1)}{\partial y} < \frac{\partial c(\tilde{x}, y_1)}{\partial y}\Big\} \subseteq \bigcup_{y\in[y_0,y_1]} L_{\tilde{x}}(y)$$

Therefore

$$\mu_1\Big(\{x : \frac{\partial c(x,y_0)}{\partial y} > \frac{\partial c(\tilde{x},y_0)}{\partial y}\} \ \bigcap \ \{x : \frac{\partial c(x,y_1)}{\partial y} < \frac{\partial c(\tilde{x},y_1)}{\partial y}\}\Big) \\ \leq \ \mu_1\big( \bigcup_{y\in[y_0,y_1]} L_{\tilde{x}}(y)\big) \tag{5.3}$$

Now, absolute continuity of $\mu_1$ and $\mu_2$ together with the assumption that $y_0$ and $y_1$ split the mass at $\tilde{x}$ yield

$$\mu_1\Big(\{x : \frac{\partial c(x,y_0)}{\partial y} > \frac{\partial c(\tilde{x},y_0)}{\partial y}\} \ \bigcap \ \{x : \frac{\partial c(\tilde{x},y_1)}{\partial y} < \frac{\partial c(\tilde{x},y_1)}{\partial y}\}\Big) \\ = \ \mu_2\big([y_0,y_1]\big) \tag{5.4}$$

Combining (5.3) and (5.4) and the MCP now yields a contradiction. □

We are now ready to prove the main result of this section.

**Theorem 5.3.5.** *Suppose $\mu_1$ and $\mu_2$ are absolutely continuous with respect to Lebesgue. Suppose that for all $x, y \in \overline{M_1} \times \overline{M_2}$ such that $y$ splits the mass at $x$ there exists an $\tilde{x} \in P \cap L_x(y)$ satisfying the MCP. Then for each $x \in \overline{M_1}$ there is a unique $y \in \overline{M_2}$ that splits the mass at $x$. Moreover, $(x, y) \in spt(\mu)$ and $(x, \overline{y}) \notin spt(\mu)$ for all other $\overline{y} \in \overline{M_2}$. Therefore, the optimal map is well defined everywhere.*

*Proof.* For each $x \in M_1$, by Lemma 5.3.4 we can choose $y \in M_2$ that splits the mass at $x$; the hypothesis then implies the existence of $\tilde{x} \in P \cap L_x(y)$ satisfying the MCP. Lemmas 5.3.3 and 5.3.3 imply that $(\tilde{x}, y) \in spt(\mu)$.

We now show that

$$(x, y') \notin spt(\mu) \text{ for all } y' \neq y. \tag{5.5}$$

The proof is by contradiction; to this end, assume $(x, y') \in spt(\mu)$ for some $y' \neq y$. Suppose $y' > y$; choose $\overline{y} \in (y, y')$. By Lemma 5.3.1, we can choose $\overline{x}$ such that $\overline{y}$ splits the mass at $\overline{x}$. Now use the hypothesis of the theorem again to find $\tilde{\overline{x}} \in P \cap L_{\overline{x}}(\overline{y})$ satisfying the MCP and note that $(\tilde{\overline{x}}, \overline{y}) \in spt(\mu)$. By Lemma 5.3.3, $\tilde{x} \notin L_{\tilde{\overline{x}}}(\overline{y})$, and so Lemma 5.3.1 implies $\frac{\partial c(\tilde{x},\overline{y})}{\partial y} < \frac{\partial c(\tilde{\overline{x}},\overline{y})}{\partial y}$.

Therefore,

$$\begin{aligned}\frac{\partial c(x,\overline{y})}{\partial y} &\leq \frac{\partial c(\tilde{x},\overline{y})}{\partial y} \\ &< \frac{\partial c(\tilde{\tilde{x}},\overline{y})}{\partial y}\end{aligned}$$

But now $(x,y'), (\tilde{\tilde{x}},\overline{y}) \in spt(\mu)$ and $y' > \overline{y}$ contradicts Lemma 5.3.1. An analogous argument implies that we cannot have $(x,y') \in spt(\mu)$ for $y' < y$, completing the proof of (5.5).

Now, note that we must have $(x,\overline{y}) \in spt(\mu)$ for *some* $\overline{y} \in M_2$ and so the preceding argument implies $(x,y) \in spt(\mu)$.

Finally, we must show that there is no other $y' \in M_2$ which splits the mass at $x$; this follows immediately, as if there were such a $y'$, an argument analogous to the preceding one would imply that $(x,y') \in spt(\mu)$, contradicting (5.5).

□

Note that we can use Theorem 5.3.5 to derive a formula for the optimal map:

$$F(x) := \sup_y \Big\{ y : \mu_1\Big(\{\overline{x} : \frac{\partial c(x,y)}{\partial y} < \frac{\partial c(\overline{x},y)}{\partial y}\}\Big) > \mu_2([0,y))\Big\}$$

**Corollary 5.3.6.** *Under the assumptions of the preceding theorem, the optimal map is continuous on* $\overline{M_1}$.

*Proof.* Choose $x_k \to x \in \overline{M_1}$ and set $y_k = F(x_k)$; we need to show $y_k \to F(x)$. Set $\overline{y} = \limsup_{k\to\infty} y_k \in \overline{M_2}$; by passing to a subsequence we can assume $y_k \to \overline{y}$. As $spt(\mu)$ is closed by definition, we must have $(x,\overline{y}) \in spt(\mu)$ and so Theorem 5.3.5 implies $\overline{y} = F(x)$. A similar argument implies $\liminf_{k\to\infty} y_k = F(x)$, completing the proof. □

The following example illustrates the implications of the preceding Corollary.

**Example 5.3.7.** *Let* $M_1$ *be the quarter disk:*

$$M_1 = \big\{(x^1,x^2) : x^1 > 0, x^2 > 0, (x^1)^2 + (x^2)^2 < 1\big\}$$

*Let* $M_2 = (0, \frac{\pi}{2})$ *and take* $\mu_1$ *and* $\mu_2$ *to be uniform measures on* $M_1$ *and* $M_2$*, respectively, scaled so that both have total mass 1. Let* $c(x,y) = -x^1 \cos(y) - x^2 \sin(y)$*; this is equivalent to the Euclidean distance between* $x$ *and the point on the unit circle parametrized by the polar angle* $y$*. We claim that the optimal map takes the form* $F(x) = \arctan(\frac{x^2}{x^1})$*; that is, each point* $x$ *is mapped to the point* $\frac{x}{|x|}$ *on the unit circle. Indeed, note that*

$$c(x,y) \geq -\sqrt{(x^1)^2 + (x^2)^2} \tag{5.6}$$

*with equality if and only if* $y = F(x)$*, and that uniform measure on the graph* $(x, F(x))$ *projects to* $\mu_1$ *and* $\mu_1$*, implying the desired result. Now observe that* $F$ *is discontinuous at* $(0,0)$*; in fact,* $((0,0), y)$ *satisfies (5.6) for all* $y \in M_2$ *so the optimal measure pairs the origin with every point. Note that the conditions of Theorem 5.3.5 fail in this case, as every* $y \in M_2$ *splits the mass at* $(0,0) \in \overline{M_1}$*.*

*Now suppose instead that* $\mu_2$ *is uniform measure on* $[0, \frac{\pi}{4}]$*, rescaled to have total mass* 1*. It is not hard to check that* $(0, x^2)$ *is in* $P$ *and satisfies the MCP for all* $x^2$*. Now, for all* $(x,y) \in M_2$ *such that* $y$ *splits the mass at* $x$*, it is straightforward to verify that we have some* $(0, x^2) \in L_x(y)$*; hence, Corollary 5.6 implies continuity of the optimizer.*

# Chapter 6

# An application to the principal-agent problem

In this chapter we apply the techniques from chapter 5 to the principal-agent problem of mathematical economics outlined in the introduction. After formulating the problem in the first, we show that $b$-convexity of the space of products is necessary for the set of $b$-convex functions to be convex in section 6.2. In sections 6.3 and 6.4 we study the cases $n_1 > n_2$ and $n_1 < n_2$, respectively. When $n_1 > n_2$, we show that if the space of products is $b$-convex, then the problem can be reduced to an equal dimensional problem; in this case, the extra dimensions in the space of types do not encode independent economic information. When $n_1 < n_2$, we show that if the space of types is $b$-convex, the problem can again be reduced to an equal dimensional problem; we establish that it is always optimal for the principal to only offer goods from a certain, $n_1$-dimensional submanifold of $Y$.

## 6.1 Assumptions and mathematical formulation

We will assume that the space of types $X \subseteq \mathbb{R}^{n_1}$ and the space of goods $Y \subseteq \mathbb{R}^{n_2}$ are open and bounded.

Before formulating the problem mathematically, we recall the conditions on $b$ imposed by Figalli, Kim and McCann [31], which are also reminiscent of the conditions **(A0)**-**(A3s)** introduced by Ma, Trudinger and Wang [52] and reformulated in the last chapter. Our formulations will appear slightly different than those in [31], as they must apply to the more general case $n_1 \neq n_2$; when $n_1 = n_2$ they coincide exactly with the conditions in [31].

**(B0)**: The function $b \in C^4(\overline{X} \times \overline{Y})$.

**(B1)**: (bi-twist) For all $x_0 \in \overline{X}$ and $y_0 \in \overline{Y}$, the level sets of the maps $y \mapsto D_x c(x_0, y)$ and $x \mapsto D_y c(x, y_0)$ are connected and the matrix of mixed, second order, partial derivatives, $D^2_{xy} c(x_0, y_0)$ has full rank.

**(B2)**: For all $x_0 \in \overline{X}$ and $y_0 \in \overline{Y}$, the images $D_x b(x_0, \overline{Y})$ and $D_y b(\overline{X}, y_0)$ are convex. If $D_x b(x_0, \overline{Y})$ is convex for all $x_0$, we say that $\overline{Y}$ is $b$-convex, while if $D_y b(\overline{X}, y_0)$ is convex for all $y_0$ we say that $\overline{X}$ is $b$-convex.

**(B3)**: For all $x_0 \in X$ and $y_0 \in Y$, we have

$$\frac{\partial^4}{\partial t^2 \partial s^2} b(x(s), y(t))\Big|_{(s,t)=(0,0)} \geq 0,$$

whenever the curves $s \in [-1, 1] \mapsto D_y b(x(s), y_0)$ and $t \in [-1, 1] \mapsto D_x b(x_0, y(t))$ form affinely parameterized line segments.

**(B3u)**: **(B3)** holds and, whenever $\dot{x}(0) \cdot D^2_{xy} b(x_0, y_0) \neq 0$ and $D^2_{xy} b(x_0, y_0) \cdot \dot{y}(0) \neq 0$, the inequality is strict.

As was emphasized by Figalli, Kim and McCann, these conditions are invariant under reparameterizations of $X$ and $Y$. This means that they are in some sense economically natural; they do not depend on the coordinates used to parametrize the problem [31].

Let us take a moment to explain the meaning of condition **(B1)**. Assume momentarily that $n_1 \geq n_2$. Then the full rank condition implies that $y \mapsto D_x c(x_0, y)$ is locally injective and so connectedness of its level sets implies its global injectivity. Hence, we recover the generalized Spence-Mirrlees, or generalized single crossing, condition found in, for example, Basov [10] (more precisely, we obtain the strengthened version in [31]). On the other hand, if $n_1 < n_2$, the generalized Spence-Mirrlees condition cannot hold; however, as we will establish, in certain cases **(B1)** is a suitable replacement.

Much of our attention here will be devoted to **(B2)**. For a bilinear $b$, this condition coincides with the usual notion of convexity of the sets $\overline{X}$ and $\overline{Y}$; for more general $b$, it implies convexity of $\overline{X}$ and $\overline{Y}$ after an appropriate change of coordinates [31]. We will see in the next section that the convexity of $D_x b(x_0, \overline{Y})$ is a necessary condition for the monopolist's problem to be a convex program; in section 5, we will show that when $n_1 < n_2$ the convexity of $D_y b(\overline{X}, y_0)$ reduces the problem to a more tractable problem in equal dimensions.

The relevance of **(B3)** and **(B3u)** to economic problems was established in [31]. They are, respectively, strengthenings of the conditions **(A3w)** and **(A3s)**, which are well known in optimal transportation due to their intimate connection with the regularity of optimal maps [52] [49].

We are now ready to review the mathematical formulation of the principal-agent problem. Suppose that the monopolist sets a price schedule $v(y)$; $v(y)$ is the price she charges for good $y$. Buyer $x$ chooses to buy the good that maximizes $b(x, y) - v(y)$. We therefore define the utility for buyer $x$ to be

$$v^b(x) = \sup_{y \in Y} b(x, y) - v(y)$$

Functions of this type are called $b$-convex functions; we will denote by $U_b$ the set of all such functions.

We assume the existence of a $y_\phi \in \overline{Y}$ that the monopolist *must* offer at cost; that is, for any price schedule $v$

$$v(y_\phi) = c(y_\phi) \tag{6.1}$$

If both sides in equation (6.1) are equal to zero, we can interpret $y_\phi$ as the null good, and the this condition represents the consumers' option not to purchase any product (and the monopolist's obligation not to charge them should they exercise this option). Note that the restriction $v(y_\phi) = c(y_\phi)$ immediately implies $v^b(x) \geq u_\phi(x) := b(x, y_\phi) - c(y_\phi)$.

Let $y_{v^b}(x) \in \operatorname{argmax}_{y \in \overline{Y}}(b(x, y) - v(y))$. Assuming that a buyer of type $x$ chooses to buy good $y_{v^b}(x)$ [1], the monopolist's profits from this buyer is then $v(y_{v^b}(x)) - c(y_{v^b}(x)) = b(x, y_{v^b}(x)) - v^b(x) - c(y_{v^b}(x))$ and her total profits are:

$$P(v^b) := \int_x b(x, y_{v^b}(x) - v^b(x) - c(y_{v^b}(x))d\mu(x)$$

The monopolist's goal, of course, is to maximize her profits. That is, to maximize $P(v^b)$ over the set $U_{b,\phi}$ of $b$-convex functions which are everywhere greater than $u_\phi$ (and, if the generalized Spence-Mirrlees condition fails to hold, over all functions $y_{v^b}(x) \in \operatorname{argmax}_{y \in \overline{Y}}(b(x, y) - v(y)))$.

The main result of [31] is that when $n_1 = n_2$, under hypotheses **(B0)**-**(B2)** convexity of the of the set $U_{b,\phi}$ is equivalent to **(B3)**.

## 6.2 b-convexity of the space of products

This section establishes the following result, which is novel even when $n_1 = n_2$.

**Proposition 6.2.1.** *If $\overline{Y}$ is not $b$-convex at some point $x \in X$, the set $U_{b,\phi}$ is not convex.*

*Proof.* Suppose $Y$ is not $b$-convex at $x \in X$. Then there exist $y_0, y_1 \in \overline{Y}$ and a $t \in (0, 1)$ such that $(1 - t) \cdot D_x b(x, y_0) + t \cdot D_x b(x, y_1) \notin D_x b(x, Y)$.

Now, choose $b$-convex functions $v_0^b, v_1^b \geq u_\phi$ such that $v_i^b$ is differentiable at $x$ and $Dv_i^b(x) = D_x b(x, y_i)$, for $i = 0, 1$. Define $v_t^b = (1 - t) \cdot v_0^b + t \cdot v_1^b$; we will show that $v_t^b$ is

[1] The generalized Spence-Mirrlees condition implies that for almost all $x$, there is exactly one $y$ maximizing $b(x, y) - v(y)$, and so under this condition, the function $y_{v^b}$ is uniquely determined from $v^b$ almost everywhere.

not $b$-convex. Now,

$$\begin{aligned} Dv_t^b(x) &= (1-t)\cdot Dv_0^b(x) + t\cdot Dv_1^b(x) \\ &= (1-t)\cdot D_x b(x,y_0) + t\cdot D_x b(x,y_1) \notin D_x b(x,\overline{Y}) \end{aligned} \tag{6.2}$$

Now, assume $v_t^b$ is $b$-convex; then

$$v_t^b(x) = \sup_{y\in Y} b(x,y) - v_t(y) \tag{6.3}$$

for some price schedule $v_t$. Without loss of generality, we may assume $v_t$ is $b$-convex: $v_t(x) = \sup_{x\in X} b(x,y) - v_t^b(x)$, which implies that $v_t(x)$ is continuous [36]. By compactness of $\overline{Y}$ and continuity of $y \mapsto b(x,y) - v_t(y)$, the supremum in 6.3 is attained by some $y_t \in \overline{Y}$, $v_t^b(x) = b(x,y_t) - v^b(y_t)$. Now, for all $\overline{x} \in X$, we have $v_t^b(\overline{x}) \geq b(\overline{x},y_t) - u^b(y_t)$ and so the function $\overline{x} \mapsto v_t^b(\overline{x}) - b(\overline{x},y_t)$ is minimized at $\overline{x} = x$. It now follows that $Dv_t^b(x) = D_x b(x,y_t) \in D_x b(x,Y)$, contradicting (6.2). We conclude that $v_t^b$ cannot be $b$-convex. As $v_t^b$ is a convex combination of $b$-convex functions, this yields the desired result. □

**Remark 6.2.2.** *This result can be seen as a slight strengthening of the result of Figalli, Kim and McCann [31]; assuming $n_1 = n_2$, **(B0)**, **(B1)** and the b-convexity of $X$, the main result of [31] combines with Proposition 6.2.1 to imply that the convexity of $U_{b,\phi}$ is equivalent to the b-convexity of $Y$ and **(B3)**. We will see in the next section that this extends nominally to the case $n_1 > n_2$, although it should be stressed that in that case $\overline{Y}$ cannot be b-convex unless all the economic information encoded to $X$ can actually be encoded in an $n_2$-dimensional space.*

The following elementary example shows that when $b$-convexity of $\overline{Y}$ fails, the principal's optimal strategy may not be unique.

**Example 6.2.3.** *Let $X = [0,1]$ be the unit interval and $Y = \{0,1\}$ be a set of two points, including the null good 0. Take $b(x,y) = xy + y$ to be bilinear and $c(y) = y^2$. Let the* *density of consumer types be* $f(x) = 60x^2 - 80x + 29$. *To make a profit, the price* $v$ *the principal sets for her good must be between* 1 *and* 2; *a straightforward calculation shows that her profits are* $(v - \frac{3}{2})^2 - 20(v - \frac{3}{2})^4 + 1$ *which is maximized at* $v = \frac{3}{2} \pm \frac{1}{2\sqrt{10}}$.

*The profit functional, written in terms of the utility functions* $v^b(x) = \sup_{y \in Y} xy + y - v(y)$ *is*

$$\int_0^1 x\frac{dv^b}{dx} + \frac{dv^b}{dx} - v^b - (\frac{dv^b}{dx})^2 dx$$

*which is strictly concave. However, the only allowable utility functions are of the form* $v^b(x) = \max\{0, x - v + 1\}$ *for some constant* $v \in [1, 2]$. *The convex interpolant of two functions of this form fails to have the same form; that is, the set of allowable utilities is not convex, precisely because the space* $Y$ *is not convex (recall that convexity and b-convexity are equivalent for bilinear preferences). Hence, uniqueness fails. If the principal had access to a convex set of goods (for example, the whole space* $[0, 1]$*) she could construct a more sophisticated pricing strategy which would earn her a higher profit than either of the maxima exhibited in this example.*

## 6.3 $n_1 > n_2$

In this section we focus on the case where $n_1 > n_2$. We will show that the $b$-convexity of the space of products implies that $X$ can be reduced to an $n$-dimensional space without losing any economic information. The analysis in this section strongly parallels the work in the last chapter.

First we recall the definition of $b$-linearity

**Definition 6.3.1.** *We say the domain* $Y$ *looks b-linear from* $x \in X$ *if* $D_x b(x, Y)$ *is contained in a shifted* $n_2$*-dimensional, linear subspace of* $T_x X$. *We say* $Y$ *is b-linear with respect to* $X$ *if it looks b-linear from every* $x \in X$.

As in chapter 5, $L_x(y)$ will denote the level set of $\overline{x} \mapsto D_y b(\overline{x}, y)$ passing through $x$, $L_x(y) := \{\overline{x} \in X : D_y b(\overline{x}, y) = D_y b(x, y)\}$.

We also recall the following lemma, expressing the relationship between $b$-linearity and the sets $L_x(y)$. The proof can be found in chapter 5 (Lemma 5.2.3).

**Lemma 6.3.2.** *(i) $Y$ looks $b$-linear from $x \in X$ if and only if $T_x(L_x(y))$ is independent of $y$; that is $T_x(L_x(y_0)) = T_x(L_x(y_1))$ for all $y_0, y_1 \in Y$.*
*(ii) If the level sets $L_x(y)$ are all connected, then $Y$ is $b$-linear with respect to $X$ if and only if $L_x(y)$ is independent of $y$ for all $x$*

For the rest of this section, we will assume that the sets $L_x(y)$ are in fact independent of $y$ (as otherwise Proposition 6.2.1 implies that the principal's program cannot be convex); we will henceforth denote them simply by $L_x$. We will show next that, no matter what pricing schedule the principal chooses, consumers in the same $L_x$ will always choose the same good and so, at least for the purposes of this problem, different points in the same $L_x$ do not really represent different types.

We can now reformulate the monopolist's problem as a problem between two $n$-dimensional spaces. To do this, we define an *effective* space of types, by essentially identifying all consumer types in a single $L_x$ as a single *effective* type.

Fix some $y_0 \in Y$ and define the space of effective types $Z := D_y b(X, y_0) \subseteq \mathbb{R}^n$ and the map $Q : X \to Z$ via $Q(x) := D_y b(x, y_0)$. We define an effective preference function: $h : Z \times Y \to \mathbb{R}$ via

$$h(z, y) = b(x, y) - b(x, y_0),$$

where $x \in Q^{-1}(z)$. We must check that $h$ is well defined, that is

$$b(x, y) - b(x, y_0) = b(\overline{x}, y) - b(\overline{x}, y_0),$$

or equivalently

$$B(x, \overline{x}, y, y_0) := b(x, y) - b(x, y_0) - b(\overline{x}, y) + b(\overline{x}, y_0) = 0,$$

for all $\overline{x} \in L_x$ and $y \in Y$. This is easily verified; the identity clearly holds at $y = y_0$ and as $D_y B(x, \overline{x}, y, y_0)$ vanishes, it must hold for all $y$.

Given a price schedule $v(y)$, the corresponding effective utility is,

$$\begin{aligned} v^h(z) &= \sup_{y \in Y} h(z,y) - v(y) \\ &= \sup_{y \in Y} b(x,y) - b(x,y_0) - v(y) \\ &= -b(x,y_0) + \sup_{y \in Y} b(x,y) - v(y) \\ &= -b(x,y_0) + v^b(x) \end{aligned}$$

for any $x \in Q^{-1}(z)$. An effective consumer of type $z$ chooses the product at which this supremum is attained; we define this product to be $y_{v^h}(z)$. It is clear from the preceding calculation that, for every $x \in Q^{-1}(z)$ we have $y_{v^b}(x) = y_{v^h}(z)$. Define $\nu = Q_{\#}\mu$ to be the distribution of effective consumer types. Hence, if we define the monopolist's effective profits to be

$$P_{eff}(v^h) = \int_Z (h(z, y_{v^h}(z)) - v^h(z) - c(y_{v^h}(z)))d\nu(z)$$

we have

$$\begin{aligned} P(v^b) &= \int_X (b(x, y_{v^b}(x)) - v^b(x) - c(y_{v^b}(x)))d\mu(x) \\ &= \int_X (b(x, y_{v^h}(Q(x))) - v^h(Q(x)) - b(x,y_0) - c(y_{v^h}(Q(x))))d\mu(x) \\ &= \int_X (h(Q(x), y_{v^h}(Q(x))) - v^h(Q(x)) - c(y_{v^h}(Q(x))))d\mu(x) \\ &= \int_Z (h(z, y_{v^h}(z)) - v^h(z) - c(y_{v^h}(z)))d\nu(x) \\ &= P_{eff}(v^h) \end{aligned}$$

Therefore, maximizing the monopolist's effective profits is equivalent to maximizing her profits.

According to Figalli, Kim and McCann [31], this new, equal dimensional problem is a maximization over a convex set, provided that the conditions **(B0)**-**(B3)** hold for $h$,

$Z$ and $Y$; meanwhile, to ensure convexity of the functional $P_{eff}$ and uniqueness of the optimizer, one needs **(B3u)** and the $h$-convexity of $c$. It is therefore desirable to be able to test these properties using only the information present in the original problem; that is, using $b$ and $X$ rather than $h$ and $Z$.

The proof the following theorem is identical to the proof of Theorem 5.2.7.

**Theorem 6.3.3.** *(i) If $b$ satisfies* ***(B1)*** *on $X \times Y$, then $h$ satisfies* ***(B1)*** *on $Z \times Y$.*
*(ii) If $b$ satisfies* ***(B2)*** *on $X \times Y$, then $h$ satisfies* ***(B2)*** *on $Z \times Y$.*
*(iii) If $b$ satisfies* ***(B3)*** *on $X \times Y$, then $h$ satisfies* ***(B3)*** *on $Z \times Y$.*
*(iv) If $b$ satisfies* ***(B3u)*** *on $X \times Y$, then $h$ satisfies* ***(B3u)*** *on $Z \times Y$.*

Finally, we verify that the $b$-convexity of $c$ implies its $h$-convexity.

**Proposition 6.3.4.** *If $c$ is $b$-convex it is $h$-convex.*

*Proof.* If $c$ is $b$-convex we have:

$$\begin{aligned} c(y) &= \sup_{x \in X} b(x,y) - c^b(x) \\ &= \sup_{x \in X} b(x,y) - b(x,y_0) + b(x,y_0) - c^b(x) \\ &= \sup_{x \in X} h(Q(x),y) - c^h(Q(x)) \\ &= \sup_{z \in Z} h(z,y) - c^h(z) \end{aligned}$$

□

Under these conditions, economic phenomena such as uniqueness of the optimal pricing strategy, bunching and the desirability of exclusion follow from the results in [31]. In particular, let us say a few words about bunching, or the phenomenon that sees different types choose the same good. Of course, in this setting one naturally expects bunching because, as was noted by Basov [9], the $n_1 > n_2$ condition precludes the full separation of types. When $X$ is $b$-convex, the bunching that occurs as a result of the difference in dimensions corresponds to identifying all types in a single level set $L_x$. The results of this section imply that these are not genuinely different types; that is, that they can be treated as a single type $z$ without any loss of pertinent information. However, genuine bunching occurs when types in different level sets opt for the same good. This occurs under **(B3)**, according to the results in [31].

**Remark 6.3.5.** *In light of the previous section, these results mean that the monopolist's problem cannot be reduced to a maximization over a convex space when $n_1 > n_2$ (at least as long as the extra dimensions encode real, economic information); this means that this class of problems is especially daunting. However, in certain special cases these problems can be treated without relying on convexity. Basov, for example, treats the case where $Y$ is a convex graph embedded in $\mathbb{R}^{n_2}$ and $b(x, y) = x \cdot y$ [9]. He then uses the techniques of Rochet and Chone [64] to solve the monopolists problem in the epigraph (a convex, $n_1$-dimensional set) and shows that it is actually optimal to sell each consumer a product in the original graph. The case $(n_1, n_2) = (2, 1)$ with a general preference function is treated by Deneckere and Severinov [27], again in the absence of a b-convex space of products.*

## 6.4 $n_2 > n_1$

When $n_2 > n_1$, the generalized Spence Mirrlees condition cannot hold; that is, $y \mapsto D_x b(x, y)$ cannot be injective. Therefore, when faced with a pricing schedule, a consumer's utility will typically be maximized by a continuum of products. The principal has the ability to offer only the good which will maximize her profits from that consumer; however, in doing so, she may exclude products that maximize her profits from another consumer. One way around this difficulty is to assume a tie-breaking rule as in Buttazzo and Carlier [13]; that is, assume that the principal can persuade each consumer to select the product that maximizes her profits (among those which maximize that consumer's utility function). This is in fact inherent in Carlier's formulation of the problem and proof of existence [18]. [2]

As we show in this section, this difficulty can be avoided by assuming $b$-convexity of $X$. Much like in the last section, this condition will allow us to reduce to a problem where the dimensions of the two spaces are the same. Intuitively, given a price schedule $v(y)$, a consumer $x$ will see the space of goods disintegrate into sub-manifolds. If the price schedule is $b$-convex, then the consumer's preference $b(x, y) - v(y)$ for good $y$ will be maximized at every point in (at least) one of these submanifolds. The $b$-convexity of $X$ will ensure that this disintegration will be the same for each $x \in X$. The principal can then choose to offer only the good in each of these submanifolds which will maximize her profits from consumers whose preferences are maximized one that sub-manifold; the resulting space will be $m$ dimensional. A special case of this structure was exploited by Basov to prove a similar result for bilinear preference functions [9].

The motivation behind the the $b$-convexity of $X$ is not as clear the motivation behind as the $b$-convexity of $Y$, which we saw in section 6.2 was necessary for the convexity of $U_b$. It is, however, a fairly standard hypothesis in the fairly limited literature on multi-dimensional screening. It is present in the work of Rochet and Chone [64] and Basov [9] on bilinear preference functions (where it reduces to ordinary convexity) as well as that of Figalli, Kim and McCann [31]. In the latter work, it is noted that $b$-convexity of $X$ implies ordinary convexity after a change of coordinates, which is essential in their proof of the genericity of exclusion modeled on the work of Armstrong [7].

Using the method from the previous section, we note that if $X$ is $b$-convex, then the level sets $L_y(x) := \{\overline{y} \in X : D_x b(\overline{y}, x) = D_x b(x, y)\}$ are independent of $x$ and so we will denote them simply by $L_y$. Letting $Z$ be the image of $y \mapsto D_x b(x_0, y) := Q(y)$, for any fixed $x_0$, the new preference function defined by $h(x, z) = b(x, y) - b(x_0, y)$, where $y$ is

[2] In [18], no extended Spence Mirrlees condition is assumed and so the function $y_{v^b}(x) \in \operatorname{argmax} b(x, y) - v^b(x)$ need not be uniquely determined by the $b$-convex $v^b(x)$. If $v^b$ and $y_{v^b}$ maximize $P$, then $y_{v^b}$ must satisfy a tie-breaking rule; that is, $y_{v^b}(x)$ must be chosen among elements in $\operatorname{argmax} b(x, y) - v^b(x)$ so as to maximize the profits $v(y_{v^b}(x)) - c(y_{v^b}(x))$.

such that $Q(y) = z$, is well defined. We then define a new, effective cost function by

$$g(z) = \inf_{\{y:Q(y)=z\}} c(y) - b(x_0, y).$$

**Proposition 6.4.1.** *Given a price schedule $v(y)$ and corresponding utility $v^b(x)$, let $y \in Y$ be such that $v^b(x) + v(y) = b(x, y)$; equality then holds as well for $\tilde{y} \in L(y)$. The maximal profit the monopolist can make by selling a consumer of type $x$ a good $\tilde{y} \in L_y$ is $h(x, Q(y)) - v^b(x) + g(Q(y))$.*

The interpretation of this result is that, in order to maximize her profits, the monopolist should only offer those goods which maximize $y \mapsto b(x_0, y) - c(y)$ over over the set $Q^{-1}(z)$ for some $z$. Any utility function $v^b$ can be implemented by offering only these goods, and by doing this for a given $v^b$, the monopolist forces each consumer to buy the good which offers her the highest possible profit.

*Proof.* The fact that $v^b(x) + v(\tilde{y}) = b(x, \tilde{y})$ if and only if $\tilde{y} \in L(y)$ follows exactly as in the last section.

We must now show that the maximum of $v(\tilde{y}) - c(\tilde{y})$ over the set $L_y = Q^{-1}(Q(y))$ is equal to $h(x, z(y)) - u(x) + g(z(y))$. We have

$$\begin{aligned}
\sup_{\tilde{y}\in L_y} v(y) - c(y) &= \sup_{\tilde{y}\in L_y} b(x, y) - u(x) - c(y) \\
&= \sup_{\tilde{y}\in L_y} h(x, z) + b(x_0, y) - u(x) - c(y) \\
&= \sup_{\tilde{y}\in L_y} h(x, z) + b(x_0, y) - u(x) - c(y) \\
&= h(x, z) - u(x) + \sup_{\tilde{y}\in L_y} b(x_0, y) - c(y) \\
&= h(x, z) - u(x) - \inf_{\tilde{y}\in L_y} c(y) - b(x_0, y)
\end{aligned}$$

□

The uniqueness argument in [31] relies on the $b$-convexity of $c$; we verify below that this convexity carries over when we reduce to an equal dimensional problem.

**Proposition 6.4.2.** *If $c$ is $b$-convex, $g$ is $h$ convex.*

*Proof.* We have

$$\begin{aligned} g(z) &= \inf_{\{y: D_x b(x_0,y)=z\}} c(y) - b(x_0,y) \\ &= \inf_{\{y: D_x b(x_0,y)=z\}} \sup_{x\in X} b(x,y) - c^b(x) - b(x_0,y) \\ &= \inf_{\{y: D_x b(x_0,y)=z\}} \sup_{x\in X} h(x,z) - c^b(x) \\ &= \sup_{x\in X} h(x,z) - c^b(x) \end{aligned}$$

□

It now follows that, under these conditions, rather than solving the principal-agent problem on $X \times Y$ with preference function $b$ and cost $c$ we can solve it on $X \times Z$ with preference function $h$ and cost $g$. Computing the conditions **(B0)**-**(B3u)** on $Z$ is equivalent to computing them on $Y$, or alternatively on any smooth $n_2$ dimensional surface which intersects the $L_y$ transversely; the proof of this is nearly identical to the proof of the analogous results proven in the last section.

## 6.5 Conclusions

We have shown that, nominally, the result of Figalli, Kim and McCann holds for *any* $n_1$ and $n_2$: assuming **(B0)**-**(B2)**, $U_{b,\phi}$ is convex if and only if **(B3)** holds. However, we should bear in mind that **(B2)** is a very strong condition when $n_1 \neq n_2$; it effectively reduces the problem to a new screening problem where both spaces have dimension $\min(n_1, n_2)$. We have also shown that the $b$-convexity of $Y$ is necessary for the convexity of $U_{b,\phi}$ and so in problems where $n_1 > n_2$ and this reduction is not possible, $U_{b,\phi}$ cannot be convex.

Economic consequences can then be deduced as in [31] under condition **(B3)**.

# Appendix A

# Differential topology notation

In this appendix, we explain in detail the notational conventions used in Chapter 3. We begin by reviewing some basic notation from differential topology.

Given a manifold $M^n$ and a point $x \in M$, recall that the tangent space of $M$ at $x$, denoted by $T_xM$ consists of all derivations (or tangent vectors) at $x$. That is, all linear maps $v : C^\infty(M) \to \mathbb{R}$ satisfying the product rule: for all $f, g \in C^\infty(M)$, we have

$$v(fg) = v(f)g(x) + v(g)f(x)$$

Fix local coordinates $(x^1, x^2, ..., x^n)$ on $M$. We denote by $\frac{\partial}{\partial x^\alpha}$ the derivation that sends the function $f$ to $\frac{\partial f}{\partial x^\alpha}(x)$. The set $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\}$ forms a basis for $T_xM$. We will use the Einstein summation convention; given a tangent vector $v = \sum_{\alpha=1}^n v^\alpha \frac{\partial}{\partial x^\alpha}$, we write $v = v^\alpha \frac{\partial}{\partial x^\alpha}$; the summation on the repeated index $\alpha$ is implicit.

A covector at $x$ is a linear functional on $T_xM$; that is, a linear mapping $F : T_xM \to \mathbb{R}$. The cotangent space $T_x^*M$ of $M$ at $x$ is the vector space of all covectors at $x$. Given local coordinates $(x^1, x^2, ..., x^n)$ on $M$, we will denote by $dx^\alpha$ the unique covector that maps $\frac{\partial}{\partial x^\alpha}$ to 1 and $\frac{\partial}{\partial x^\beta}$ to 0 for all $\beta \neq \alpha$. The set $\{dx^1, dx^2, ..., dx^n\}$ forms a basis for $T_x^*M$. The tensor product $F \otimes G$ of covectors $F$ and $G$ is a bilinear map on $T_xM \times T_xM$; it maps the ordered pair of vectors $(v, w)$ to the product $F(v)G(w)$.

In this thesis, we often work with the product of several manifolds. Given such a product $M_1^{n_1} \times M_2^{n_2} \times ... \times M_m^{n_m}$, we will use Latin indices to indicate which manifold we are in and Greek indices (indexed themselves by the appropriate Latin index), as before, to indicate local coordinates within each manifold. That is, $x_i$ will denote a point in $M_i$, for $i = 1, 2, ..., m$ and we will use the index $\alpha_i$ to denote local coordinates within $M_i$; a vector $v_i$ in $T_{x_i}M_i$ will be represented in local coordinates as $v_i = v_i^{\alpha_i} \frac{\partial}{\partial x_i^{\alpha_i}}$, where the summation on the repeated index $\alpha_i$ is implicit. Generally, summation on repeated Greek indices will be implicit, as these indices represent local coordinates in a particular manifold, whereas summation on Latin indices, indicating which manifold we are working in, will not be implicit.

The tangent space $T_x(M_1 \times M_2 \times ... \times M_m)$ at a point $x = (x_1, x_2, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$ is naturally isomorphic to the product $T_{x_1}M_1 \times T_{x_2}M_2 \times ... \times T_{x_m}M_m$ and the cotangent space $T_x^*(M_1 \times M_2 \times ... \times M_m)$ at $x$ is naturally isomorphic to $T_{x_1}^*M_1 \times T_{x_2}^*M_2 \times ... \times T_{x_m}^*M_m$. We will represent vectors $v \in T_x(M_1 \times M_2 \times ... \times M_m)$ using direct sum notation; that is, $v = \bigoplus_{i=0}^m v_i$, where $v_i \in T_{x_i}M_i$. We will extend covectors $F_i$ on $M_i$ to the product $M_1 \times M_2 \times ... \times M_m$ in the obvious way; that is, $F_i(v_1, v_2, ..., v_m) = F_i(v_i)$. In particular, note that $dx_i^{\alpha_i} \otimes dx_j^{\alpha_j}$ represents a bilinear map on

$$T_x(M_1 \times M_2 \times ... \times M_m) \times T_x(M_1 \times M_2 \times ... \times M_m)$$

which maps $(v, w) = (\bigoplus_{k=0}^m v_k, \bigoplus_{k=0}^m w_k)$ to $dx_i^{\alpha_i}(v_i) dx_j^{\alpha_j}(v_j) = v_i^{\alpha_i} w_j^{\alpha_j}$.

We will often deal with a $C^2$ function $c : M_1 \times M_2 \times ... \times M_m \to \mathbb{R}$. We are especially interested in bilinear maps of the form $\frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}} (dx_j^{\alpha_j} \otimes dx_k^{\alpha_k} + dx_k^{\alpha_k} \otimes dx_j^{\alpha_j})$, which takes $(v, w) = (\bigoplus_{k=0}^m v_k, \bigoplus_{k=0}^m w_k)$ to $\frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}} (v_j^{\alpha_j} w_k^{\alpha_k} + v_k^{\alpha_k} w_j^{\alpha_j})$. In particular, the bilinear map $g_p$ in (3.1) takes $(v, w)$ to

$$g_p(v, w) = \sum_{j \in p_+, k \in p_-} \frac{\partial^2 c}{\partial x_j^{\alpha_j} \partial x_k^{\alpha_k}} (v_j^{\alpha_j} w_k^{\alpha_k} + v_k^{\alpha_k} w_j^{\alpha_j})$$

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