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REGULARITY FOR TRANSPORT AND NONLINEAR DIFFUSION PROBLEMS

Doctoral thesis by

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Contents

Introduction

This doctoral thesis contains the research activity I carried out during my PhD program in 'Matematica e Statistica' at the University of Pavia. I touched two fields of Mathematical Analysis, which are Optimal Transportation and Partial Differential Equations.

Thanks to the opportunity to spend part of the PhD program abroad, I visited the University of Toronto, where Prof. Robert McCann introduced me to Optimal Transportation.

Generally speaking, Optimal Transportation is the study of how to minimize the cost of moving a certain mass from a location to another one. This kind of problems is of interest in a number of applications that span from Physics to Economics. For example, in Physics, our problem can be to move an object from a position to another one. In this case the cost can be identified with the energy spent to move the object of mass m in a gravitational potential in the usual 3d-space (4d-space if we consider time as an additional variable). In Economics, mass and space may not have their physical meanings, and we are not limited to work in a 3d-space. On the contrary, the cost may indeed refer to the 'price' of an option in the stock market.

Mathematically speaking, Optimal Transportation consists in solving the so called *Monge-Kantorovich problem*: given two probability measures μ and ν supported on $\mathbf{R}^N$, minimize the following integral

$$\int_{\mathbf{R}^N \times \mathbf{R}^N} c(x,y) d\gamma(x,y)$$

when γ varies between the probability measures on $\mathbf{R}^N \times \mathbf{R}^N$ with μ and ν as marginals, and where $c : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is a measurable function called *cost function*. In the Monge-Kantorovich problem the mass has been normalized to 1, and the probability measures μ and ν represent how the mass is distributed at the initial and final location, respectively. The value $c(x,y)$ is the cost of transporting the mass from $x \in \mathbf{R}^N$ to $y \in \mathbf{R}^N$.

In my studies I concentrated on Gangbo and McCann's work of 1999, [27]. This work finds applications in shape recognition algorithms. The authors took inspiration from Fry's thesis [25], where he elaborated an algorithm to identify unknown leaves from New England, comparing them to a catalog of standard leaves. Fry's innovative idea was to distribute unit mass uniformly along each leaf boundary, and then calculate the total cost of transporting the mass from the boundary of the sample leaf to the specified distribution on the catalog leaf. Thus, once the appropriate cost had been chosen, each comparison involved computing the solution to a Monge-Kantorovich transportation problem. In [27], Gangbo and McCann examined this kind of problems computing the distance between leaf boundaries by mean of the Wasserstein distance. Hence, they analyzed the following problem

$$\mathrm{d}^2(\mu,\nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbf{R}^{N+1} \times \mathbf{R}^{N+1}} |x-y|^2 d\gamma(x,y), \tag{1}$$

where μ, ν are two Borel probability measures on hypersurfaces of $\mathbf{R}^{N+1}$, and $\Gamma(\mu,\nu)$ denotes the set of all Borel measures on $\mathbf{R}^{N+1} \times \mathbf{R}^{N+1}$ having μ and ν as marginals.

It is well known that a minimizer γ for (1) exists. It is also known that, when μ is absolutely continuous with respect to the Lebesgue measure, the optimizer γ is unique and it lies on the graph of the gradient of a convex function ψ, also called Kantorovich's potential (see [55] for a complete collection of these results with proper references). But, as Fry's numerical evidence suggested ([25], Fig. 3.5), since both the measures concentrate on hypersurfaces, the optimal measure γ might fail to be unique or to concentrate on the graph of any map.

In trying to understand the theory that lies beyond Fry's numerical results, Gangbo and McCann presented some natural examples; in some of them the optimal measure γ fails to be unique (see Example 2.1 of [27]), in others it is unique but its support fails to concentrate on the graph of a single map (see Examples 3.12 and 3.13 of [27]). Nevertheless they proved that, as long as one of the two hypersurfaces is a boundary of a strictly convex set -say $\partial\Omega$, where $\Omega \subset \mathbf{R}^{N+1}$-, and the measure supported on it -say μ- is absolutely continuous with respect to its surface measure $\mathcal{H}^d\lfloor_{\partial\Omega}$, then the optimal measure $\gamma \in \Gamma(\mu,\nu)$ is the unique. Moreover they proved that the images of $\mu-$a.e. x are collinear: they lie on a line parallel to the first hypersurface's normal at x. This follows from the tangential differentiability of the Kantorovich potential ψ: in those points which disintegrate into multiple images it is the normal differentiability which fails.

Now, when we consider two measures μ,ν supported on the boundaries of two convex sets, respectively Ω and Λ, each point $x \in \partial\Omega$ can have at most two images, since each line intersects a strictly convex boundary twice at most. Gangbo and McCann [27] denoted these two images $t^+(x)$ and $t^-(x) \in \mathrm{spt}\nu$, and pointed out that they correspond to the two limits of $\nabla\psi(x_k)$ obtained as $x_k \to x$ from outside or inside Ω. Moreover they showed the outer trace $t^+ : \mathrm{spt}\,\mu \to \mathrm{spt}\,\nu$ gives a global homeomorphism between the hypersurfaces, while the inner trace t^- to be continuous and continuously invertible on the closure of the set $S_2 := \{x \in \mathrm{spt}\,\mu \mid t^+(x) \neq t^-(x)\}$. Together, the graphs of these two maps cover the support of the optimal measure γ.

Besides its application, the theory of transportation between hypersurfaces revealed itself to be very interesting even from a purely mathematical point of view. Recently, some authors achieved innovative regularity results related to this kind of problems ([42],[41],[34],[35],[40]).

The second part of my thesis deals with regularity results for weak solutions to some Partial Differential Equations. This part has been carried out at the University of Pavia under the supervision of Prof. Ugo Gianazza. It is based on recent works of DiBenedetto, Gianazza, and Vespri ([18],[17],[19],[21],[20]), which deal with some classes of parabolic differential equations. Two well known parabolic equations are the p-Laplace equation

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = 0, \qquad p > 1, \tag{2}$$

and the Porous Medium equation

$$u_t - m\mathrm{div}(|u|^{m-1}Du) = 0, \qquad m > 0.$$

Their modulus of ellipticity is

$$|Du|^{p-2}, \qquad |u|^{m-1},$$

respectively.

DiBenedetto, Gianazza, and Vespri introduced a novel set of analytical tools and techniques that allow to deduce, by means of purely measure theoretical arguments, regularity properties, such as Harnack inequalities and Hölder continuity, for weak solutions to p-Laplacian and Porous Medium type equations. The main achievement of DiBenedetto, Gianazza, and Vespri is the proof that degeneracy and/or singularity of an equation limits the degree of regularity of its solutions.

A differential equation is degenerate or singular if the modulus of ellipticity of its principal part tends to zero or to infinity at points of its domain of definition. Such a behavior may be intrinsic when the vanishing or blowing up of the modulus of ellipticity occurs through the solution or its gradient. For example, for $p > 2$, the p-Laplace equation is *degenerate* on the set $[|Du| = 0]$, while, for $p \in (1, 2)$, the p-Laplace equation is *singular* on the set $[Du = 0]$.

In this thesis I will extend some of the results of [17]-[21] to a class of parabolic, doubly nonlinear, partial differential equations whose prototype is

$$u_t - \operatorname{div}(u^{m-1}|Du|^{p-2}Du) = 0. \tag{3}$$

In particular, in a 4d-space, with variables (x_1, x_2, x_3, t), when $m > 1$ and $p > 2$, such equation describes the dynamics of a non-Newtonian polytropic fluid in a porous medium. It can be seen as a combination of the p-Laplacian equation and the Porous Medium equation. When $m + p > 3$, it is degenerate on the set $[|u| = 0] \cap [|Du| = 0]$; when $m + p < 3$ it is singular on the set $[|u| = 0] \cap [|Du| = 0]$.

I will show that, in the degenerate case ($m + p > 3$), an intrinsic Harnack inequality holds for the weak solutions to (3), and that such inequality implies Hölder continuity, while in the singular case a critical threshold for regularity will emerge ($m + p + \frac{p}{N} = 3$, where N is the dimension of the space). In the singular supercritical range

$$3 - \frac{p}{N} < p + m < 3,$$

a Harnack inequality holds in the same intrinsic form of the degenerate case; in addition, another family of Harnack inequalities will be proved. These will be simultaneously *forward* in time, *backward* in time, and *elliptic*. In the sub-critical range

$$2 < p + m < 3 - \frac{p}{N}$$

no Harnack estimate in any of the forms mentioned above seems to hold. In [54], Vespri claims that the solutions to (3), combined with proper initial data, become extinct after a finite time. Following the insightful statements of [53], I will consider alternative forms of Harnack-type inequalities. All the previous results for (3), will actually be proved for the entire class of parabolic, doubly nonlinear, partial differential equations it represents.

Chapter 1 is devoted to the extension of Gangbo and McCann's results [27] on the regularity of the optimal multi-valued mappings t^+ and t^-. In particular I will show that t^+ and t^- are locally Hölder continuous on those subsets of their domains where $n_\Omega(x) \cdot n_\Lambda(t^+(x)) \neq 0$ and $n_\Omega(x) \cdot n_\Lambda(t^-(x)) \neq 0$. Chapter 2 contains some preliminary results to the following Chapters. More specifically, I will deal with integral estimates and some DeGiorgi-type lemmas. In Chapter 3 and 4 I will prove intrinsic Harnack inequalities and Hölder continuity for weak solutions to a class of doubly nonlinear parabolic equations in the degenerate and singular case, respectively. To make the thesis more readable, I postponed some technical results and some theorems already known in the literature to the appendices A and B.

Chapter 1

Hölder continuity for optimal multivalued mappings

1.1 Introduction

Let X, Y be two measure spaces, μ, ν two probability measures defined on X and Y, respectively, and c a measurable map from $X \times Y$ to $[0, +\infty]$. Let us denote with Γ the set of all the probability measures on $X \times Y$ that have marginals μ and ν. More explicitly, $\gamma \in \Gamma(\mu, \nu)$ if and only if γ is a nonnegative measure satisfying $\gamma(A \times Y) = \mu(A)$, $\gamma(X \times B) = \nu(B)$, for all measurable subsets A of X and B of Y. The minimization problem

$$\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x, y) d\gamma(x, y) \tag{1.1}$$

is known as *Kantorovich's optimal transportation problem*; c is called the *cost function*, and every probability measures in $\Gamma(\mu, \nu)$ is called a *transference plan*. Kantorovich's problem is meant to investigate how a certain mass μ distributed on a domain X is transported to another location (described by ν and Y) at a minimal cost (see [55] for an exhaustive description).

When $X = Y = \mathbf{R}^N$, and the cost function is the Euclidean squared distance, the minimizers of (1.1) are characterized by the existence of a convex function $\psi : \mathbf{R}^N \to \mathbf{R} \cup \{+\infty\}$, whose subdifferential $\partial\psi \subset \mathbf{R}^N \times \mathbf{R}^N$ contains the support of every optimal transference plan $\gamma \in \Gamma(\mu, \nu)$(see Brenier [4] for references). This convex function is called *Brenier's potential*. When μ is absolutely continuous with respect to the Hausdorff measure of dimension N, ψ is differentiable on a set of full μ–measure, the optimizer γ is unique, and it full mass lies on the graph $\{(x, \nabla\psi(x)) \mid x \in \mathrm{dom}\nabla\psi\}$ of the gradient of ψ. This means μ–a.e. point x must be mapped to the unique destination $y = \nabla\psi$ for transportation to be efficient. Therefore the optimal transference plan is the pushed–forward of μ by $\mathrm{Id} \times \nabla\psi$,

$$\gamma = (\mathrm{Id} \times \nabla\psi)_\sharp \mu.$$

The measurable map $T = \nabla\psi$ is called *optimal map*.

Several authors treated the regularity of optimal maps when the cost function is the Euclidean squared distance; among them Caffarelli [6] [7] [8] [9] [10] [11], Delanoë [13], and Urbas [52]. In particular, Caffarelli showed that if the domain Y is convex, $d\mu = f dVol, d\nu = g dVol$, where $dVol$

denotes the Lebesgue measure, and the densities $f, 1/g$ are bounded, then the optimal map is Hölder continuous.

In some applications of Optimal Transportation to Physics or Economics, also other cost functions are of interest. For example, the problem of the reflector antenna (see [56] by Wang, and [44] by Oliker and Waltman) has been shown to be equivalent to optimal transportation of measures on the Euclidean unit sphere with respect to the cost function $-\log|x-y|$ (see [57] and [28]). Inspired by these works on the reflector antenna, Ma, Trudinger, and Wang found a condition on the cost function, which implies the regularity of the optimal map (see [42]). It is a structural condition depending upon derivatives up to the order four of the cost function. Following their notation, we name it (A3). It will be stated in Section 1.5.

Loeper [41], Kim, and McCann [34] [35] clarified the role of (A3) when an optimal map exists and is unique. More precisely, when the cost function is sufficiently smooth and (A3) holds, under suitable convexity hypotheses on the domains, and the absolute continuity of the Lebesgue measure with respect to ν, Loeper was able to prove the Hölder continuity of the optimal map (see Section 1.5 for a precise statement of the hypothesis). On the other hand, Kim and McCann [35] found a covariant expression of (A3), named (A3s) in their paper, and extended Loeper's results to transportation problems set on a pair of smooth manifolds.

Our work makes use of Loeper, Kim, and McCann's argument to improve the regularity results obtained by Gangbo and McCann [27] for a transportation problem between boundaries of convex sets. Optimal transportation between boundaries of convex sets does not generally lead to a single-valued optimal map, but rather to multi-valued mappings. This means that an optimizer $\gamma \in \Gamma(\mu, \nu)$ takes the form

$$\gamma = \sum_{i=1}^{m} \gamma_i, \qquad \gamma_i = (\mathrm{Id} \times t_i)_\sharp \mu_i,$$

where t_i are measurable maps from X to Y, and $\mu = \sum_{i=1}^{m} \mu_i$. This is the case of the Kantorovich problem analyzed by Gangbo and McCann, who found a bivalent mapping. The novelty of our result is the quantification of the continuity in this setting of multi-valued mappings.

Let Ω and Λ be two bounded, strongly convex (in the sense of Section 1.2), open sets in $\mathbf{R}^{N+1}$, with Borel probability measures μ on $\partial\Omega$ and ν on $\partial\Lambda$. We consider the Monge-Kantorovich problem

$$\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbf{R}^{N+1} \times \mathbf{R}^{N+1}} |x-y|^2 d\gamma(x,y). \tag{1.2}$$

When μ is absolutely continuous with respect to the Hausdorff measure of dimension N, ($\mathcal{H}^N$), and Ω is strictly convex, the optimal transference plan is unique, but its support fail to concentrate on the graph of a single map (see Theorem 2.6 of [27]). Gangbo and McCann [27] showed that the unique optimizer $\gamma \in \Gamma(\mu, \nu)$ is supported by two maps, named t^+ and t^-, i.e.

$$\gamma = \gamma_1 + \gamma_2, \quad \gamma_1 = (\mathrm{id} \times t^+)_\sharp \mu_1, \quad \gamma_2 = (\mathrm{id} \times t^-)_\sharp \mu_2,$$

where $\mu = \mu_1 + \mu_2$. This means that the mass at a point $x \in \partial\Omega$ does not always have a unique destination on $\partial\Lambda$, but can be split into two different destinations, $t^+(x)$ and $t^-(x)$, which correspond to the two limits $\nabla\psi(x_k)$ obtained as $x_k \to x$ from outside or inside Ω, respectively. Indeed, while Brenier's potential ψ is tangentially differentiable at $\mathcal{H}^N$–a.e. boundary point $x \in \partial\Omega$, the normal differentiability might fail. This implies that the subdifferential $\partial\psi$ consists of a segments with endpoints $t^+(x), t^-(x)$ on $\partial\Lambda$ (see Lemma 1.6 of [27]).

Gangbo and McCann proved that t^+ is a homeomorphism between $\partial\Omega$ and $\partial\Lambda$. Moreover, they conjectured Hölder regularity for t^+ on $\partial\Omega \setminus S_0$, where

$$S_0 := \{x \in \partial\Omega \mid n_\Omega(x) \cdot n_\Lambda(t^+(x)) = 0\}$$

represents a part of the "boundary" between the region where the mass splits and the region where it does not. More precisely, if S_2 denotes the region where the mass splits (bivalent region), then S_0 contains those limit points of S_2 at which the split images degenerate to a single image. In the present work, we will prove a slight modification of their conjecture, i.e. that t^+ is locally Hölder continuous on S_2 and on $S_1 = \partial\Omega \setminus (S_0 \cup S_2)$.

The peculiarity of (1.2) is the "hybrid" setting given by the choice of the Euclidean squared distance cost for a transportation problem set on embedded hypersurfaces. One of the difficulties we encountered has been to combine the convexity notion deriving from the Euclidean cost with the dimension and the pseudo-Riemannian structure of the manifolds where the measures are supported. Since the Hausdorff dimension of sptμ and sptν is N rather than $N+1$, we are not able to adapt Caffarelli's regularity theory to our problem; (see however [23]). Nevertheless Gangbo and McCann's conjecture about Hölder continuity is reinforced by Example 2.4 of Kim-McCann [34]: the authors showed that the Euclidean squared distance cost, in the settings of (1.2), satisfies (A3) on

$$\mathcal{N} := \{(x, y) \in \partial\Omega \times \partial\Lambda \mid n_\Omega(x) \cdot n_\Lambda(y) > 0\}.$$

Despite this comforting result, the regularity of t^+ is not immediate. Loeper's results needs to be adapted to our "hybrid" setting. Moreover, the target measure with respect to t^+, ν_1, which is the portion of mass "transferred" by t^+, does not inherit the hypothesis on ν of having a positive lower bound on its density with respect to the Lebesgue surface measure. This means there are regions in $\partial\Omega$ where the Lebesgue surface measure is not absolutely continuous with respect to ν_1, so one necessary hypothesis of Loeper's argument is not satisfied. We will treat these regions separately with a different argument.

This chapter is organized as follows. In Section 1.2 we report the main results of Gangbo and McCann's paper [27]; we also discuss the most important statement of our work and the strategy we are going to adopt to prove it. We will restrict our argument to the case of spherical domains, $\partial\Omega = \partial\Lambda = \mathbf{S}^N$, though we believe that our regularity result can be extended to more general uniformly convex domains. In Section 1.3 we introduce and clarify some notation. In Section 1.4 we comment on some questions related to our problem. In Section 1.5 we adapt Loeper's theory to our transportation problem, restricting his argument to the regions of $\partial\Omega$ where the necessary hypothesis on the measures holds. The regularity result on the remaining regions is derived in Section 1.6 . Section 1.7 gives an explicit dependence of the Hölder constant appearing in Section 1.5 on the distance from $\mathcal{N}$. Finally, in Section 1.8, we prove the bi-Lipschitz estimates that t^+ satisfies when $N = 1$.

The results of the present chapter will appear on [46].

1.2 Preliminaries, strategy, and results

We recall the following definitions from [27]. For a smooth convex domain Ω, *strong* convexity asserts the existence of a positive lower bound for all principal curvatures of $\partial\Omega$.

Definition 1.2.1 *A pair of Borel measures μ on $\partial\Omega$, ν on $\partial\Lambda$ is said to be* suitable *if* *(i) there exists $\epsilon > 0$ such that $\mu < \frac{1}{\epsilon}\mathcal{H}^N\lfloor_{\partial\Omega}$ and $\nu > \epsilon\mathcal{H}^N\lfloor_{\partial\Lambda}$, and*

(ii) Ω is strongly convex.

If the above hypotheses are satisfied also when the roles of $\mu \leftrightarrow \nu$ and $\Omega \leftrightarrow \Lambda$ are interchanged, we say that the pair (μ, ν) is symmetrically suitable.

Under these assumptions on the measures, Gangbo and McCann were able to prove the following optimality results.

Theorem 1.2.2 *Fix bounded, strictly convex domains $\Omega, \Lambda \in \mathbf{R}^{N+1}$ with suitable measures μ on $\partial\Omega$ and ν on $\partial\Lambda$. Then the infimum of (1.2) is uniquely attained.*

Let $N_\Omega(x)$ denote the set of all outward unit normals to $\partial\Omega$ at x. When $N_\Omega(x)$ contains only one element, we denote that unit vector by $n_\Omega(x)$.

Proposition 1.2.3 *Fix bounded, strictly convex domains $\Omega, \Lambda \in \mathbf{R}^{N+1}$ with suitable measures μ on $\partial\Omega$ and ν on $\partial\Lambda$. Let ψ be the Brenier convex potential. For each $x \in \partial\Omega$ exactly one of the following statements holds:*

(o) $\partial\psi(x) = \{y_1\}$ with $n \cdot q_1 = 0$ for some pair $n \in N_\Omega(x), q_1 \in N_\Lambda(y_1)$;

(i) $\partial\psi(x) = \{y_1\}$ with $n \cdot q_1 > 0$ for all pairs $n \in N_\Omega(x), q_1 \in N_\Lambda(y_1)$;

(ii) $\partial\psi(x) = [y_1, y_2]$, in which case $\partial\Omega$ is differentiable at x and $n_\Omega(x) \cdot q_1 > 0, n_\Omega(x) \cdot q_2 < 0$ for all $q_i \in N_\Lambda(y_i)$, $i = 1, 2$.

Definition 1.2.4 *Given $\Omega, \Lambda, (\mu, \nu)$, and ψ as in Proposition 1.2.3, we decompose $\partial\Omega = S_0 \cup S_1 \cup S_2$ into three disjoint sets such that (o) holds for $x \in S_0$, (i) holds for $x \in S_1$, (ii) holds for $x \in S_2$. Moreover we use the extreme images $y_1, y_2 \in \partial\psi$ of the proposition to define an outer map $t^+ : \partial\Omega \to \partial\Lambda$, and an inner map $t^- : S_2 \to \partial\Lambda$ by $t^+(x) = y_1$, and $t^-(x) = y_2$. It is convenient to extend the definition of t^- to $\partial\Omega$ by setting $t^-(x) = t^+(x)$ for $x \in S_0 \cup S_1$.*

Theorem 1.2.5 *Fix bounded, strictly convex domains $\Omega, \Lambda \in \mathbf{R}^{N+1}$ with symmetrically suitable measures μ on $\partial\Omega$ and ν on $\partial\Lambda$. Then the minimizer γ can be expressed by*

$$\gamma = \gamma_1 + \gamma_2, \quad \gamma_1 = (\mathrm{id} \times t^+)_\sharp \mu_1, \quad \gamma_2 = (\mathrm{id} \times t^-)_\sharp \mu_2,$$

where $\mu_1 := (t^+)^{-1}_\sharp \nu_1$, $\mu_2 := \mu - \mu_1$, and $\nu_1 := \nu\lfloor_{T_2^c}$, with $T_2^c := \partial\Lambda \setminus t^-(S_2)$. Whenever $x \in S_2$, $t^+(x) - t^-(x) \neq 0$ is an outward normal for $\partial\Omega$ at x. Moreover $t^+ : \partial\Omega \to \partial\Lambda$ and $t^-\lfloor_{\bar{S}_2} : \bar{S}_2 \to \bar{T}_2$ are homeomorphisms.

The partition $\partial\Omega = S_0 \cup S_1 \cup S_2$ will play an important role in our work, so it is essential to understand the meaning of these sets. The mass lying on $S_0 \cup S_1$ is transferred without splitting to a target set on $\partial\Lambda$ by t^+, while the mass lying on S_2 splits into two destinations, which are described by t^+ and t^-. For this reason we will call S_0 the *degenerate set*, S_1 the *non-degenerate univalent set*, and S_2 the *bivalent set*. When the measures (μ, ν) are symmetrically suitable, an analogous decomposition of $\partial\Lambda = T_0 \cup T_1 \cup T_2$ can be introduced (see Definition 3.6 of [27]). In particular T_2 is the bivalent set for the Kantorovich transportation problem (1.2), where (Ω, ν) and (Λ, ν) are exchanged, with (Λ, ν) playing the role of the source.

Our aim is to prove that the map $t^+ : \partial\Omega \longrightarrow \partial\Lambda$ is Hölder continuous on S_1 and S_2. We will show that t^+ satisfies bi-Lipschitz estimates when $N = 1$, via an argument relying on the results of Ahmad [1], which cannot be extended to higher dimensions. Here we are developing a different strategy which works for all $N > 1$, when $\partial\Omega, \partial\Lambda = \mathbf{S}^N$. We will proceed in two steps. First we will show that t^+ is Hölder continuous on the preimage $(t^+)^{-1}(T_1) \subset \mathbf{S}^N$ of the set T_1 where

$$\nu_1 > \epsilon \mathcal{H}^N \lfloor_{\partial\Lambda},$$

where ϵ is the constant from Definition 1.2.1. This lower bound on ν_1 allows us to adapt the argument used by Kim and McCann in [35]. On $(t^+)^{-1}(T_2)$, where the lower bound fails, the regularity of t^+ will be derived from the Hölder continuity of t^+ on S_2. In the end we will be able to obtain the following result.

Theorem 1.2.6 (Hölder continuity of multi-valued maps outside the degenerate set) *If (μ, ν) are symmetrically suitable measures on $(\mathbf{S}^N, \mathbf{S}^N)$, $N > 1$, then*

$$t^+ \in C_{loc}^{\frac{1}{4N-1}}(S_1) \qquad \textit{and} \qquad t^+ \in C_{loc}^{\frac{1}{4N-1}}(S_2).$$

1.3 Notation

The notation we are going to use is similar to that of [27] and [34], in particular we refer to Example 2.4 of [34], with $\partial\Omega = \partial\Lambda = \mathbf{S}^N$, $c : \mathbf{S}^N \times \mathbf{S}^N \to \mathbf{R}$, $c(x, y) = |x - y|^2$, $\mathcal{N} := \{(x, y) \in \mathbf{S}^N \times \mathbf{S}^N \mid n_{\mathbf{S}^N}(x) \cdot n_{\mathbf{S}^N}(y) > 0\}$, and $\hat{\mathcal{N}}(x) := \{y \in \mathbf{S}^N \mid (x, y) \in \mathcal{N}\}$. We will always use the variable x for points on the source domain $\partial\Omega = \mathbf{S}^N$, and the variable y for points on the target domain $\partial\Lambda = \mathbf{S}^N$.

Let us recall the usual system of local coordinates for the points of $\mathbf{S}^N$

$$\varphi_i : \mathbf{S}^N \cap \{x \in \mathbf{S}^N | x_i > 0\} \to \mathbf{R}^N, \quad \varphi_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Following this example, given $x \in \mathbf{S}^N$ and $y \in \hat{\mathcal{N}}(x)$ we can consider a system π_x of local coordinates projecting on the hyperplane perpendicular to x. In this way both x and y can be represented in local coordinates by means of the same map π_x

$$x \xrightarrow{\pi_x} X, \qquad y \xrightarrow{\pi_x} Y,$$

where the capital letters stand for the image of the projection. To simplify the notation, given a function $F : \mathbf{R}^{N+1} \to \mathbf{R}$ and a projection π_{x_0}, whenever $x \in \hat{\mathcal{N}}(x_0)$ we will write $F(X)$ to denote $F(\pi_x^{-1}(X)) = F(x)$. We will therefore write $\psi(X)$, $c(X, Y)$ instead of $\psi(\pi_{x_0}^{-1}(X))$, $c((\pi_{x_0}^{-1}(X), (\pi_{x_0}^{-1}(Y))$. For example, given $x \in \mathbf{S}^N$ and $y \in \mathcal{N}(x)$, by mean of π_x we can write

$$c(X, Y) = |X - Y|^2 + (\sqrt{1 - |X|^2} - \sqrt{1 - |Y|^2})^2.$$

In local coordinates, we use the notation $Dc = (\frac{\partial c}{\partial X_1}, \ldots, \frac{\partial c}{\partial X_n}$ and $\bar{D}c = (\frac{\partial c}{\partial Y_1}, \ldots, \frac{\partial c}{\partial Y_n})$ to denote the partial derivatives. The cross partial derivatives $\bar{D}Dc$ at $(x, y) \in N$ define an unambiguous linear map from vectors at y to covectors at x.

Hereafter $d\mathcal{H}^N$ denotes the Hausdorff measure of dimension $N, \mathcal{U}_\rho(B)$ represents the ρ-neighbourhood of a set B, and $[Y_0, Y_1]$ indicate the Euclidean segment whose extreme points are Y_0 and Y_1.

In Section 1.6 we will use the expression 'angle between two vectors z_1 and $z_2 \in \mathbf{R}^{N+1}$'. The term angle refers to the arccos $\frac{z_1 \cdot z_2}{|z_1||z_2|}$.

1.4 Some related questions

1.4.1 Relation between the convex potential ψ and the mappings t^+, t^-

Let ψ be the Brenier potential associated to (1.2). It is well known that the subdifferential $\partial\psi$ includes the support $\mathrm{spt}\gamma \subset \mathbf{R}^{N+1} \times \mathbf{R}^{N+1}$ of all minimizers $\gamma \in \Gamma(\mu, \nu)$ for (1.2)(see [4][5] for references). Under the hypothesis of Theorem 1.2.5, there exists a unique optimizer $\gamma \in \Gamma(\mu, \nu)$ for (1.2), and there exist two continuous maps $t^{\pm} : \partial\Omega \to \partial\Lambda$, such that

$$\{(x, t^+(x))\}_{x\in \mathrm{spt}\mu} \quad \subset \mathrm{spt}\gamma \subset \quad \{(x, t^+(x))\}_{x\in\partial\Omega} \cup \{(x, t^-(x)))\}_{x\in S_2} \\ (= \partial\psi \cap (\partial\Omega \times \partial\Lambda)).$$

So, what is the relation between the optimal mappings t^+, t^-, and the convex potential ψ? Can we derive any regularity for ψ from Theorem 1.2.6? Gangbo and McCann answered to the first question in Lemma 1.6 of [27]. Indeed the maps t^+ and t^- correspond to the outer and inner trace of $\nabla\psi$, respectively. So we can write the subdifferential of ψ in terms of the optimal mappings: $\partial\psi(x) = [t^+(x), t^-(x)]$ at any boundary point $x \in \partial\Omega$. Moreover, in Corollary 4.4 of [27], Gangbo and McCann proved that, when Ω is bounded and strongly convex, Λ is bounded and strictly convex, and (μ, ν) are suitable measures on $\partial\Omega, \partial\Lambda$, then ψ is tangentially differentiable along $\partial\Omega$. This answers the second question. From Theorem 1.2.6 it follows immediately that

$$\psi \in C^{1,\frac{1}{4N-1}}_{loc} \text{ on } S_1 \subset \mathbf{S}^N,$$

i.e. on the non-degenerate univalent set, where $\partial\psi(x) = \{\nabla\psi(x)\} = \{t^+(x)\}$. Notice that the conclusion of Theorem 1.2.6 does not imply $\psi \in C^{1,\frac{1}{4N-1}}_{loc}$ on S_2, since ψ is not differentiable in the normal direction to the sphere on S_2. Nevertheless, choosing the coordinates of Lemma A.1 of [27], $\frac{\partial\psi}{\partial x_1}$ exists for $i = 2, 3, \ldots, n+1$, and

$$\frac{\partial\psi}{\partial x_i}(x) = t^+(x)_i = t^-(x)_i, \text{ for } i = 2, 3, \ldots, n+1, \text{ and } x \in \mathbf{S}^N.$$

We conclude that the restriction of ψ to $\mathbf{S}^N$ has a derivative which is Hölder continuous locally on S_1 and S_2.

1.4.2 The regularity of t^+ on S_0

We do not presently have any regularity result for t^+ on the degenerate set S_0, except continuity from [27], On the contrary, we will see in the statements of Theorem 1.5.3 and Theorem 1.6.1 that, on $\mathbf{S}^N \setminus S_0$, close to S_0 the Hölder constant of t^+ provided by our proof may become very big. Moreover, as noticed in Example 2.4 of [34], the nondegeneracy hypothesis (A2) fails on S_0. Therefore, we cannot apply Loeper's argument on S_0. On the other hand we believe the set S_0 to be small. In dimension $N = 1$, with Ω and Λ bounded strictly convex planar domains, Ahmad [1] proved that S_0 consists of at most two points.

1.4.3 Extending the results to more general domains

Theorem 1.2.6 can be extended to the problem of transporting a measure on a given Euclidean sphere to a measure on any other Euclidean sphere, possibly with a different centre and radius.

Indeed, identities (9) and (10) of [27] indicate how to reduce this more general problem to the case treated in this work.

Thanks to the results in Example 2.4 of [34], Theorem 1.5.3 can be extended to the transportation problem where the measures (μ, ν) are supported on $(\partial\Omega, \partial\Lambda)$, with $\Omega, \Lambda \subset \mathbf{R}^{N+1}$ bounded convex domains with C^2-smooth boundaries. We believe that the same extension is possible for Theorem 1.6.1, but we cannot presently provide any proof. Our argument has a critical point in Lemma 1.6.9, which exploit the peculiar geometric properties of $\mathbf{S}^N$, and cannot be easily extended to more general convex domains.

1.4.4 Nearly constant measures on $\mathbf{S}^N$

J. Kitagawa and M. Warren [36] proved that when the measures μ, ν are nearly constant on $\mathbf{S}^N$ (in C^1 topology), then the optimizer $\gamma \in \Gamma(\mu, \nu)$ is supported on the graph of a single map.

1.4.5 Sharp Hölder exponent

The Hölder exponent in Theorem 1.2.6 is not sharp. It is the same exponent provided by Loeper's argument [41], i.e. $1/(4N-1)$, where N is the dimension of the sphere where μ and ν are supported. Recently, Liu [40] improved Loeper's Hölder exponent to the sharp exponent $1/(2N-1)$.

1.5 t^+ is Hölder continuous on $(t^+)^{-1}(T_1) \subset \mathbf{S}^N$

In this section we are going to adapt Kim-McCann's version of Loeper's argument (Appendices B,C and D of [35]) to our mapping t^+, which satisfies $(t^+)_\sharp \mu_1 = \nu_1$. Thus, let us recall the regularity conditions (A0),(A1), (A2), and (A3s) from [34] [42] on a cost function $c : \mathbf{S}^N \times \mathbf{S}^N \to \mathbf{R}$

(A0)(Smoothness) $c \in C^4(\mathcal{N})$*, where* $\mathcal{N}$ *has been define in Section 1.3.*
(A1)(Twist condition) $c \in C^1(\mathcal{N})$ *and for all* $x \in \mathbf{S}^N$ *the map* $y \to -Dc(x, y)$ *from* $\hat{\mathcal{N}}(x) \subset \partial\Lambda$ *to* $T^*_x(\mathbf{S}^N)$ *is injective.*
(A2)(Non-degeneracy) $c \in C^2(\mathcal{N})$ *and for all* $(x, y) \in \mathcal{N}$ *the linear map* $\bar{D}Dc : T_y\mathbf{S}^N \to T^*_x\mathbf{S}^N$ *is bijective.*
(A3s)(Strictly regular costs) $c \in C^4(\mathcal{N})$ *satisfies (A2) and for every* $(x, y) \in \mathcal{N}$

$$sec_{(x,y)}(p \oplus 0) \wedge (0 \oplus \bar{p}) \geq 0 \text{ for all null vectors } p \oplus \bar{p} \in T_{(x,y)}\mathcal{N}, \tag{1.3}$$

and equality in (1.3) implies $p = 0$ *or* $\bar{p} = 0$.

The notation "sec" refers to the sectional curvature of a two–plane. We define it by means of the Riemann curvature tensor $R_{i'j'k'l'}$ induced by the symmetric bilinear form

$$h = \frac{1}{2} \begin{pmatrix} 0 & -\bar{D}Dc \\ -D\bar{D}c & 0 \end{pmatrix} \tag{1.4}$$

on $\mathcal{N}$. If $c \in C^4(\mathcal{N})$, the *sectional curvature* of a two–plane $P \wedge Q$ at $(x, y) \in \mathcal{N}$ is given by

$$\sec_{(x,y)} P \wedge Q = \sum_{i'=1}^{2N} \sum_{j'=1}^{2N} \sum_{k'=1}^{2N} \sum_{l'=1}^{2N} R_{i'j'k'l'} P^{i'} Q^{j'} P^{k'} Q^{l'}.$$

We recall also some notions of convexity from Definition 2.5 of [34]. Though we are assuming $\partial\Omega = \partial\Lambda = \mathbf{S}^N$, the following definition holds for more general convex domains.

Definition 1.5.1 *A subset $W \subseteq N \subseteq \partial\Omega \times \partial\Lambda$ is* geodesically convex *if each pair of points in W is linked by a curve satisfying the geodesic equation on $(\mathcal{N}, h)$. We say that $B \subset \partial\Lambda$* appears convex *from $x \in \partial\Omega$ if $\{x\} \times B$ is geodesically convex and $B \subset \hat{\mathcal{N}}(x)$. We say $W \subseteq \partial\Omega \times \partial\Lambda$ is* vertically convex *if $\hat{W}(x) := \{y \in \partial\Lambda \mid (x, y) \in W\}$ appears convex from x for each $x \in \partial\Omega$. We say that $A \subset \partial\Omega$* appears convex *from $y \in \partial\Lambda$ if $A \times \{y\}$ is geodesically convex and $A \subset \mathcal{N}(y)$. We say $W \subseteq \partial\Omega \times \partial\Lambda$ is* horizontally convex *if $W(y) := \{x \in \partial\Omega \mid (x, y) \in W\}$ appears convex from y for each $y \in \partial\Lambda$. If W is both vertically and horizontally convex, we say it is* bi–convex*.*

The regularity result that we are going to exploit is Theorem D.1 of [35]. We now state in a reductive form, referring to our particular settings, to avoid the introduction of new unnecessary notations.

Theorem 1.5.2 (Simplified version of Theorem D.1 of [35]) *Assume $c \in C^4(\mathcal{M})$ satisfies (A1),(A2), and (A3s) on the closure of $\mathcal{M}$, where $\mathcal{M} \subset \mathbf{S}^N \times \mathbf{S}^N$ is a bounded domain bi–convex with respect to* (1.4)*. Fix $m > 0$, and let $\rho, \bar{\rho}$ be probability measures on $\mathbf{S}^N$ with Lebesgue densities $d\bar{\rho}/d\mathrm{vol} \geq m$ throughout $\mathbf{S}^N$ and $d\rho/d\mathrm{vol} \in L^\infty(\mathbf{S}^N)$. Then there exists a map $F \in C^{1/\max\{5,4N-1\}}_{loc}(\mathbf{S}^N, \mathbf{S}^N)$ between ρ and $\bar{\rho}$ which is optimal with respect to the transportation cost c.*

Assuming (μ, ν) to be suitable measures on $(\mathbf{S}^N, \mathbf{S}^N)$, in order to apply Kim–McCann's argument we need ν_1 to satisfy

$$\text{there exists } \epsilon_1 \text{ such that } \nu_1 > \epsilon_1 \mathcal{H}^N\lfloor_{\partial\Lambda}. \tag{1.5}$$

From the definition of ν_1 in Theorem 1.2.2 we see that ν_1 satisfies (1.5) only outside the bivalent set $T_2 \in \partial\Lambda = \mathbf{S}^N$, i.e. outside the set where the image of t^+ is bivalent. This is the reason why we can state a regularity result only on a portion of the source domain, $(t^+)^{-1}(T_1) \subset \mathbf{S}^N$. Hereafter we will assume $N > 1$.

Theorem 1.5.3 *Suppose (μ, ν) are symmetrically suitable measures on $(\mathbf{S}^N, \mathbf{S}^N)$ (in particular, from Definition 1.2.1, there exists $\epsilon > 0$ such that $\nu > \epsilon\mathcal{H}^N\lfloor_{\mathbf{S}^N}$). Then t^+ is locally Hölder continuous on $(t^+)^{-1}(T_1)$, with Hölder exponent at least $\frac{1}{4N-1}$. Our control on the local Hölder constant depends on ϵ, n, and tends to infinity when one approaches the boundary of $\mathcal{N}$.*

Remark 1.5.4 *Computations that show the explicit dependence of the Hölder constant on the distance of the boundary of $\mathcal{N}$ can be found in Section 1.7.*

Lemma 1.5.5 *The set*

$$\mathcal{N} = \{(x, y) \in \mathbf{S}^N \times \mathbf{S}^N \mid n_{\mathbf{S}^N}(x) \cdot n_{\mathbf{S}^N}(y) > 0\}$$

is bi-convex in the sense of Definition 2.5 of [34].

Proof: Fix $x_0 \in \mathbf{S}^N$. $\hat{\mathcal{N}}(x_0)$ appears convex from x_0 if and only if $Dc(x_0, \hat{\mathcal{N}}(x_0))$ is convex in $T^*_{x_0}(\mathbf{S}^N)$. Suppose $Dc(x_0, y_0), Dc(x_0, y_1) \in Dc(x, \hat{\mathcal{N}}(x))$, where $y_0, y_1 \in \hat{\mathcal{N}}(x)$. We are going to show that for every $\theta \in (0, 1)$

$$\theta Dc(x_0, y_1) + (1 - \theta) Dc(x_0, y_0) \in Dc(x_0, \hat{\mathcal{N}}(x)). \tag{1.6}$$

Let's consider a system of local coordinates. Given $x_0 \in \mathbf{S}^N$ we project x_0 and $y \in \hat{\mathcal{N}}(x_0)$ to the hyperplane perpendicular to $\hat{n}_\Omega(x_0)$ and containing the origin (notice that this choice of local coordinates is well defined since $\hat{n}_\Omega(x_0) \cdot \hat{n}_\Lambda(y_k) > 0$, when $y_k \in \hat{\mathcal{N}}(x_0)$, $k = 0, 1$)

$$x_0 \xrightarrow{\pi_{x_0}} 0, \quad y \xrightarrow{\pi_{x_0}} Y \tag{1.7}$$

so that, in local coordinates,

$$\begin{gathered} x_0 = (0,1), \quad y = (Y, \sqrt{1-|Y|^2}) \\ c(X,Y) = |X-Y|^2 + (\sqrt{1-|X|^2} - \sqrt{1-|Y|^2})^2. \end{gathered}$$

We easily get

$$\frac{\partial c}{\partial X_i}(0,Y) = -2Y_i.$$

If $v \in T_x(\partial\Omega)$ and v_i are its coordinate with respect to the basis $\frac{\partial}{\partial X_i}$, I can write

$$Dc(v)(x_0,y) = v(c)(x_0,y) = \sum_{i=1}^N v_i \frac{\partial c}{\partial X_i}(0,Y).$$

Hence we can compute

$$\begin{aligned} \theta Dc(v)(x_0,y_1) + (1-\theta)Dc(v)(x_0,y_0) &= \theta v(c)(x_0,y_1) + (1-\theta)v(c)(x_0,y_0) \\ &= \textstyle\sum_{i=1}^N \left[\theta v_i \frac{\partial c}{\partial X_i}(0,Y_1) + (1-\theta) v_i \frac{\partial c}{\partial X_i}(0,Y_0)\right] \\ &= \textstyle\sum_{i=1}^N 2v_i \left[\theta\,(-Y_{1,i}) + (1-\theta)\,(-Y_{0,i})\right] \\ &= \textstyle\sum_{i=1}^N -2v_i(\theta Y_{1,i} + (1-\theta)Y_{0,i}). \end{aligned} \tag{1.8}$$

Therefore, for all $\theta \in (0,1)$

$$\theta Dc(x_0,y_1) + (1-\theta)Dc(x_0,y_0) = Dc(x_0, \pi_{x_0}^{-1}(\theta Y_1 + (1-\theta)Y_0) \in Dc(x_0, \hat{\mathcal{N}}(x_0)).$$

Since x_0 is an arbitrary point of $\mathbf{S}^N$, we conclude that $\mathcal{N}$ is vertically convex. By a similar argument, it is easy to show that $\mathcal{N}$ is also horizontally convex. We conclude that $\mathcal{N}$ is bi-convex. □

Proof of Theorem 1.5.3: Fix $(x,y) = (x, t^+(x)) \in \mathcal{N}$, with $t^+(x) \in T_1$. Since T_1 is open, and t^+ is continuous, we can choose R and then r small enough that $B_r(y) \subset t^+(B_R(x)) \subset T_1$; as asserted by Trudinger and Wang in [51], since $\mathcal{N}$ is bi-convex, taking R and r even smaller, $P = B_R(x) \times B_r(y) \subset \mathcal{N}$ is bi-convex (alternatively, we could show directly that P is bi-convex, by means of the same argument used for $\mathcal{N}$ in Lemma 1.5.5. We replace ν_1 with its restriction ν_1' to $B_r(y)$ and we denote $\mu_1' = s^+_\#(\nu_1')$. Taking R and then r even smaller than before gives us local coordinates over both domains simultaneously (for example through the chart π_x). Let $X = \pi_x(x), Y = \pi_x(y)$, and $P' = \pi_x(B_R(x)) \times \pi_x(B_r(y))$. Since P is bi-convex and the notion of bi-convexity is coordinate invariant (as manifest from Definition 2.5 of [34]), P' is bi-convex with respect to the cost

$$c(X,Y) = |X-Y|^2 + (\sqrt{1-|X|^2} - \sqrt{1-|Y|^2})^2. \tag{1.9}$$

Kim and McCann showed that the cost in the original coordinates satisfies condition (A2) and (A3s) (see Example 2.4 of [34]), and that the quantities in these conditions have an intrinsic meaning independent of coordinates, since they are geometric quantities (i.e. pseudo-Riemannian curvatures in the case of (A3s) and non-degeneracy of the metric in the case (A2)). This implies that also the cost (1.9) satisfies (A2) and (A3s). Only the constant C_0' of (A3s) will depend on the coordinates. Since we know that the equation $D_X c(X,Y) = D\psi(X)$ has at most two solutions, $Y^+ = t^+(X)$ and $Y^- = t^-(X)$ and only Y^+ lies in P', the cost satisfies (A1) on P'.

At this point we can apply Theorem 1.5.2 to the cost (1.9) on P', with probability measures μ_1^x and ν_I^x, on $\pi_x(B_R(x))$ and $\pi_x(B_r(y))$ respectively, defined by

$$\mu_1^x := (\pi_x)_\sharp \mu_1', \qquad \nu_1^x := (\pi_x)_\sharp \nu_1'.$$

The source μ_1^x is supported (and bounded above) in $\pi_x(B_R(x))$ and target ν_1^x supported (and bounded below) in $\pi_x(B_r(y))$, We deduce the existence of a locally Hölder continuous optimal map pushing μ_1' forward to ν_1'. By the uniqueness of optimal transport, this map must coincide μ_1'-a.e. with t^+. Since both maps are continuous they agree on the (closed) support of μ_1'. Since sptμ_1' contains a small ball around x, this shows t^+ is locally Hölder at x. □

1.6 t^+ is locally Hölder continuous where its image is bivalent

The previous section established local Hölder continuity for the outer map $t^+ = (s^+)^{-1}$ on the source domain $s^+(T_1) \subset \mathbf{S}^N$, but not on $s^+(T_0 \cup T_2) = S_0 \cup s^+(T_2)$. Our strategy for extending this estimate to $s^+(T_2)$ is described at the end of this paragraph. First note, however, that Gangbo and McCann's *Sole Supplier Lemma*, 2.5 of [27], implies the outer image of the bivalent source is contained in the univalent target $t^+(S_2) \subset T_1$, and similarly $s^+(T_2) \subset S_1$. Since $s^+ : \mathbf{S}^N \longrightarrow \mathbf{S}^N$ is a homeomorphism, from $S_1 \cup S_2 = s^+(T_1) \cup s^+(T_2)$, it follows that the bivalent source $S_2 \subset s^+(T_1)$ belongs to the domain where Hölder continuity of t^+ has already been shown. On this bivalent set S_2, the inner map t^- is related to the outer map $t^+(x) = t^-(x) + \lambda(x)x$ by the geometry of the target. In Proposition 1.6.2, this relation will be used to deduce (i) Hölder continuity of t^- from that of t^+. This quantifies injectivity (ii) of the inverse map $s^- = (t^-)^{-1}$ (through a bi-Hölder estimate in Proposition 1.6.6), whose relation to the outer map $s^+(y) = s^-(y) + \omega(y)y$ is then used in Proposition 1.6.10 to quantify injectivity (iii) of $s^+ = (t^+)^{-1}$ on the bivalent target $T_2 = t^-(S_2)$. This yields the desired local Hölder continuity of t^+ on the source set $s^+(T_2)$ mentioned at the outset.

Let us recall the geometric characterization of t^+ and t^- from Proposition 1.2.3 and Definition 1.2.4. Remembering that, on $\mathbf{S}^N$, $n_{\mathbf{S}^N}(x) = x$, we have

- If $x \in S_0$ then $x \cdot t^+(x) = 0$.
- If $x \in S_1$ then $x \cdot t^+(x) > 0$.
- If $x \in S_2$ then $x \cdot t^+(x) > 0$ and $x \cdot t^-(x) < 0$.

We are going to introduce a geometric approach, based on the previous characterization, which allows us to prove the following theorem. Hereafter we will assume $n > 1$.

Theorem 1.6.1 *If (μ, ν) are symmetrically suitable measures on $(\mathbf{S}^N, \mathbf{S}^N)$, then t^+ is locally Hölder continuous on $(t^+)^{-1}(T_2)$.*

From Lemma 1.6 of [27] we know that t^+ and t^- are related by

$$\forall x \in S_2 \subset \mathbf{S}^N \qquad t^+(x) - t^-(x) = \lambda(x)x,$$

where λ is a continuous positive function on S_2. Given x_0, x_1 in S_2 we then have

$$|t^-(x_1) - t^-(x_0)| \le |t^+(x_1) - t^+(x_0) - \lambda(x_1)x_1 + \lambda(x_0)x_0|. \tag{1.10}$$

We would like to exploit the regularity of t^+ on $S_2 \subset (t^+)^{-1}(T_1)$, proved in the previous section, to prove that also t^- is Hölder continuous on S_2. For this purpose we also need to estimate the term $\lambda(x_1)x_1 + \lambda(x_0)x_0$. This will be done applying the Mean Value Theorem to a suitable function and utilizing the geometric properties of the target.

Proposition 1.6.2 (Hölder continuity of t^-) *If $t^+ \in C^\alpha_{loc}(S_2)$ then $t^- \in C^\alpha_{loc}(S_2)$. Let $U \subset S_2$ and $0 < k_U := \min\{-x \cdot t^+(x) \mid x \in U\}$. If C_U^+ bounds the Hölder constant for t^+ on U, then*

$$C_U^- := \left(1 + \frac{1}{k_U}\right)(C_U^+ + 2)$$

is the Hölder constant for t^- on U.

Proof: The function $h(y) := d(y, \mathbf{S}^N) = 1 - |y|$ is differentiable on $\Lambda = B_1(0)$ except at $y = 0$. Notice that $h(t^-(x)) = h(t^+(x) - \lambda(x)x) = h(t^+(x)) = 0$ whenever $x \in S_2$. Consider a neighbourhood $U \subset S_2$ and the corresponding k_U, C_U^+ from the statement of Proposition 1.6.2. Let $x_0, x_1 \in U$, $|x_1 - x_0| < 2$ (we need ∇h to be well defined on the line segment between $t^-(x_0)$ and $t^-(x_1)$, i.e. $0 \notin [t^-(x_0, t^-(x_1)])$. Applying the Mean Value Theorem, we get

$$\begin{aligned} 0 &= h(t^+(x_1) - \lambda(x_1)x_1) - h(t^+(x_0) - \lambda(x_0)x_0) \\ &= \nabla h(u) \cdot (t^+(x_1) - t^+(x_0) - \lambda(x_1)x_1 + \lambda(x_0)x_0), \end{aligned}$$

for some u on the line segment between $t^-(x_0)$ and $t^-(x_1)$. It follows

$$(\lambda(x_1)x_1 - \lambda(x_0)x_0) \cdot \nabla h(u) = (t^+(x_1) - t^+(x_0)) \cdot \nabla h(u). \tag{1.11}$$

We can rewrite (1.11) as

$$\begin{aligned} (t^+(x_1) - t^+(x_0)) \cdot \nabla h(u) + \lambda(x_0)(x_0 - x_1) \cdot \nabla h(u) \\ = (\lambda(x_1) - \lambda(x_0))x_1 \cdot \nabla h(u); \end{aligned}$$

then, using $|\nabla h(u)| = 1$,

$$\begin{aligned} &|\lambda(x_1) - \lambda(x_0)||x_1 \cdot \nabla h(u)| \\ &\le |(t^+(x_1) - t^+(x_0))| + \lambda(x_0)|x_0 - x_1|. \end{aligned} \tag{1.12}$$

We now state a claim, whose demonstration is postponed to the end of this proof.

Lemma 1.6.3 *Under the hypotheses of Proposition 1.6.2, fix $\epsilon \in (0,1)$, such that $\epsilon^2 < \frac{k_U}{2}$. Since t^- is uniformly continuous on $\bar{S}_2$, there exists δ_ϵ, depending on the data through ψ, such that*

$$|x_1 - x_0| < \delta_\epsilon \Rightarrow |t^-(x_1) - t^-(x_0)| < \epsilon.$$

Then, taking x_0, x_1 such that $|x_1 - x_0| < \delta_\epsilon$, we have

$$x_i \cdot \nabla h(u) > \frac{k_U}{2} > 0 \qquad \text{for } i = 1, 2.$$

Recalling that $\lambda(x) \leq 2$, since $\partial\Omega = \mathbf{S}^N$, by means of Lemma 1.6.3 we simplify (1.12) to

$$\begin{aligned} &|\lambda(x_1) - \lambda(x_0)| \\ &\leq \frac{2}{k_U}\left[|t^+(x_1) - t^+(x_0)| + \lambda(x_0)|x_0 - x_1|\right] \\ &\leq \frac{2}{k_U}[|t^+(x_1) - t^+(x_0)| + 2|x_1 - x_0|]. \end{aligned} \tag{1.13}$$

Therefore, by (1.10) and (1.13),

$$\begin{aligned} &|t^-(x_1) - t^-(x_0)| \\ &\leq |t^+(x_1) - t^+(x_0)| + \lambda(x_1)|x_1 - x_0| + |\lambda(x_1) - \lambda(x_0)| \\ &\leq |t^+(x_1) - t^+(x_0)| + 2|x_1 - x_0| + |\lambda(x_1) - \lambda(x_0)| \\ &\leq \left(1 + \frac{2}{k_U}\right)|t^+(x_1) - t^+(x_0)| + 2\left(1 + \frac{2}{k_U}\right)|x_1 - x_0|. \end{aligned} \tag{1.14}$$

Combining (1.14) and $t^+ \in C^\alpha(U)$, we conclude

$$\begin{aligned} &|t^-(x_1) - t^-(x_0)| \\ &\leq C_U^+\left(1 + \frac{2}{k_U}\right)|x_1 - x_0|^\alpha + 2\left(1 + \frac{2}{k_U}\right)|x_1 - x_0|, \end{aligned} \tag{1.15}$$

i.e. t^- is Hölder continuous on S_2 whenever $|x_1 - x_0| < \delta_\epsilon$, with $\epsilon^2 < \frac{k_U}{2}$. We can take $\delta_\epsilon < 1$, so that (1.15) implies

$$\begin{aligned} |t^-(x_1) - t^-(x_0)| &\leq \left(1 + \frac{2}{k_U}\right)\left[C_U^+ + 2\right]|x_1 - x_0|^\alpha \\ &= C_U^-|x_1 - x_0|^\alpha. \qquad \square \end{aligned}$$

Proof of Lemma 1.6.3: Let $z_i = t^-(x_i), i = 1, 2$. Notice that $\nabla h(u) = -\frac{u}{|u|}$. We have $u = sz_1 + (1-s)z_0$ for some $s \in (0,1)$. Hence, there exists $\xi \in (0, \epsilon)$ such that

$$\begin{aligned} x_1 \cdot u &< -k_U s + (1-s)x_1 \cdot z_0 \\ &= -k_U s + (1-s)x_1 \cdot (z_1 + \xi(z_0 - z_1)) \\ &< -k_U + (1-s)\xi\epsilon < -k_U + \epsilon^2. \end{aligned}$$

Using a similar argument for $x_0 \cdot u$, we conclude that if $\epsilon^2 < \frac{k_U}{2}$ then $x_i \cdot \nabla h(u) > \frac{k_U}{2|u|} > \frac{k_U}{2} > 0$, for $i = 1, 2$. $\square$

Remark 1.6.4 *Proposition 1.6.2 admits a converse, i.e. if* $t^- \in C^\alpha_{loc}(S_2)$ *then* $t^+ \in C^\alpha_{loc}(S_2)$*. This can be proved with minor changes in the preceding argument.*

Remark 1.6.5 *By means of Theorem 1.5.3 and Proposition 1.6.2,* t^- *is indeed locally Hölder continuous on* S_2 *with exponent* $\frac{1}{4N-1}$.

The injectivity (ii) of the inverse map $s^- = (t^-)^{-1}$ on T_2, is an immediate consequence of the local Hölder continuity of t^- on S_2, and it has been included in the following proposition.

Proposition 1.6.6 (Quantifying injectivity of s^-) *Let $V \subset T_2$. Under the hypotheses of Theorem 1.6.1 $s^- := (t^-)^{-1}$ satisfies*

$$\forall y_0, y_1 \in V \text{ sufficiently close}, |s^-(y_1) - s^-(y_0)| \geq \hat{C}_V^- |y_1 - y_0|^{4N-1},$$

where

$$\hat{C}_V^- = (C_U^-)^{-1},$$

with $U = s^-(V)$ and $0 < k_V := \min\{-y \cdot s^-(y) \mid y \in V\}$.

Proof: Since $s^- := (t^-)^{-1}$ is uniformly continuous on $\bar{T}_2$, given $\delta_\epsilon > 0$ there exists $\gamma_{\delta_\epsilon} > 0$ such that, if $|y_1 - y_0| < \gamma_{\delta_\epsilon}$, then $|s^-(y_1) - s^-(y_0)| < \delta_\epsilon$. Supposing $|y_1 - y_0| < \gamma_{\delta_\epsilon}$, we can apply Proposition 1.6.2 to $x_1 = s^-(y_1), x_0 = s^-(x_0)$ to get

$$|s^-(y_1) - s^-(y_0)| \geq \frac{1}{C_U^-} |y_1 - y_0|^{4N-1}. \qquad \square$$

We now state an elementary Lemma about vectors in $\mathbf{R}^N$.

Lemma 1.6.7 *Let $u, v \in \mathbf{R}^N$. Suppose the angle between u and v is less than $\frac{\pi}{2} + \alpha$, with $\alpha \in \left[0, \frac{\pi}{2}\right)$. Then $|u + v| \geq |u| \cos \alpha$.*

Proof: Let $\theta_{u,v}$ denote the angle between u and v. Keeping $|u|$ and $|v|$ fixed, $|u + v|$ can be seen as a function of $\theta_{u,v}$ by mean of

$$|u + v|^2 (\theta_{u,v}) = |u|^2 + |v|^2 + 2|u||v| \cos \theta_{u,v},$$

When $\theta_{u,v} \in \left[0, \frac{\pi}{2} + \alpha\right]$, the function $|u + v| (\theta_{u,v})$ reaches its minimum at $\theta_{u,v} = \frac{\pi}{2} + \alpha$. To our purpose we can take $\theta_{u,v} = \frac{\pi}{2} + \alpha$. For simplicity we assume v parallel to $e_1 \in \mathbf{R}^N$. Let us consider the projection p on the hyperplane perpendicular to e_1 and containing the origin. Then $p(u + v) = p(u) = |u| \cos \alpha$. Since $|p(u + v)| \leq |u + v|$, we have the thesis. $\square$

This Lemma turns out to be the key to the proof of step (iii). Under the hypothesis of symmetrically suitable measures, the optimal transportation problem we are studying is symmetric, hence every result that holds for t^+ on $\mathbf{S}^N$ implies an analogous result for s^+ on $\mathbf{S}^N$. In particular, from Lemma 1.6 of [27], for every $y \in T_2$ we can write

$$s^+(y) - s^-(y) = \omega(y) y, \tag{1.16}$$

where ω is a nonnegative function on T_2. Hence

$$|s^+(y_1) - s^+(y_0)| = |s^-(y_1) - s^-(y_0) + \omega(y_1) y_1 - \omega(y_0) y_0|.$$

If we were allowed to apply Lemma 1.6.7 to the right-hand side of the previous equality, with $u = s^-(y_1) - s^-(y_0)$ and $v = \omega(y_1) y_1 - \omega(y_0) y_0$, we would then be able to exploit the regularity of s^- to prove step (iii). Therefore, we need to understand the behaviour of the angle between $s^-(y_1) - s^-(y_0)$ and $\omega(y_1) y_1 - \omega(y_0) y_0$, when y_0 gets close to y_1. From the monotonicity of $\partial \psi$ we have

$$(s^-(y_1) - s^-(y_0)) \cdot (y_1 - y_0) \geq 0 \qquad \forall y_1, y_0 \in T_2,$$

which says that the angle between $s^-(y_1) - s^-(y_0)$ and $y_1 - y_0$ is in $\left[0, \frac{\pi}{2}\right]$. If we can show that the angle between $y_1 - y_0$ and $\omega(y_1) y_1 - \omega(y_0) y_0$ is in $[0, \alpha]$, for a certain $\alpha \in \left[0, \frac{\pi}{2}\right)$, then we can apply Lemma 1.6.7 to get the desired estimate on $|s^+(y_1) - s^+(y_0)|$.

Lemma 1.6.8 *Given $y_0, y_1 \in T_2$ we denote with $\beta(y_0, y_1)$ the angle between $y_1 - y_0$ and $\omega(y_1)y_1 - \omega(y_0)y_0$. If the angle between y_0 and y_1 is equal to γ then*

$$\beta(y_0, y_1) \in \left[0, \frac{\pi - \gamma}{2}\right). \tag{1.17}$$

Proof: The angle between y_1 and $-y_0$ is equal to $\pi - \gamma$, while the angle between y_1 (or $-y_0$) and $y_1 - y_0$ is $\frac{\pi-\gamma}{2}$. Since $\omega(y_0), \omega(y_1) > 0$, $\beta(y_0, y_1) \in \left[0, \frac{\pi-\gamma}{2}\right)$. □

Lemma 1.6.9 (Dichotomy) *Fix $y_1 \in T_2$. For every integer $m > 1$ define*

$$\Theta_m(y_1) := \left\{ y \in T_2 \mid \beta(y, y_1) \in \left[\frac{\pi}{2} - \frac{1}{m}, \frac{\pi}{2}\right]\right\}.$$

Unless $\Theta_m(y_1)$ is empty for m sufficiently large, there exist $m_M > 0$ and $K > 0$ such that

$$|s^+(y_1) - s^+(y)| \geq K|y_1 - y|, \qquad \forall y \in \Theta_m(y_1), \text{ with } m > m_M. \tag{1.18}$$

Proof: We are interested in the sets $\Theta_m(y_1)$ for m large, so hereafter we assume $m > 50$. Define

$$0 < \varpi_m := \inf\{\omega(y) > 0 \mid y \in \Theta_m(y_1)\}$$

and note $\varpi_m \leq \varpi_{m+1}$ since $\Theta_m(y_1) \supset \Theta_{m+1}(y_1)$. By elementary computations, we have

$$\begin{aligned}
|\omega(y_1)y_1 - \omega(y)y| \cos\beta(y, y_1) &= \frac{(\omega(y_1)y_1 - \omega(y)y) \cdot (y_1 - y)}{|y_1 - y|} \\
&= \frac{\omega(y_1)y_1 \cdot (y_1 - y) - \omega(y)y \cdot (y_1 - y)}{|y_1 - y|} \\
&\geq \frac{\varpi_m y_1 \cdot (y_1 - y) - \omega(y)y \cdot (y_1 - y)}{|y_1 - y|} \\
&= \frac{\varpi_m |y_1 - y|^2 + (\varpi_m - \omega(y))y \cdot (y_1 - y)}{|y_1 - y|} \\
&\geq \varpi_m |y_1 - y| \qquad\qquad \forall y \in \Theta_m(y_1),
\end{aligned} \tag{1.19}$$

where we used the definition of ϖ_m and the trivial inequality $y \cdot y_1 \leq 1$ to show that the term $(\varpi_m - \omega(y))y \cdot (y_1 - y)$ is non-negative. Consider now the two vectors $\omega(y_1)y_1 - \omega(y)y$ and $(\omega(y_1) - \omega(y))y_1$, with $y \in T_2$. Their difference is parallel to $y_1 - y$, so they have the same projection on any hyperplane perpendicular to $y_1 - y$. This projection has length $|\omega(y_1)y_1 - \omega(y)y| \sin\beta(y, y_1)$. Therefore

$$|\omega(y_1)y_1 - \omega(y)y| \sin\beta(y, y_1) \leq |\omega(y_1) - \omega(y)| \qquad \forall y \in T_2. \tag{1.20}$$

Putting together (1.19) and (1.20), we obtain an estimate for $\tan\left(\frac{\pi}{2} - \frac{1}{m}\right)$

$$\tan\left(\frac{\pi}{2} - \frac{1}{m}\right) \leq \tan\beta(y, y_1) \leq \frac{|\omega(y_1) - \omega(y)|}{\varpi_m |y_1 - y|} \qquad \forall y \in \Theta_m(y_1).$$

As $m \to +\infty$, $\tan\left(\frac{\pi}{2} - \frac{1}{m}\right) \to +\infty$; then for every $M > 0$ there exists $m_M > 50$ such that

$$|\omega(y_1) - \omega(y)| > M\varpi_m |y_1 - y|, \qquad \forall y \in \Theta_m(y_1), m > m_M. \tag{1.21}$$

From (1.16) we have, for every $y \in T_2$,

$$s^+(y_1) - s^+(y) - \omega(y)(y_1 - y) = s^-(y_1) - s^-(y) + (\omega(y_1) - \omega(y))y_1.$$

We define

$$\begin{aligned} A &:= |s^+(y_1) - s^+(y)| + |\omega(y)(y_1 - y)| \\ &\geq |s^-(y_1) - s^-(y) + (\omega(y_1) - \omega(y))y_1|, \qquad y \in T_2. \end{aligned} \tag{1.22}$$

Using $|v - u| \geq |v| - |u| \quad \forall u, v \in \mathbf{R}^{N+1}$, we get two different estimates for A

$$A \geq |s^-(y_1) - s^-(y)| - |\omega(y_1) - \omega(y)|, \tag{1.23}$$

$$A \geq |\omega(y_1) - \omega(y)| - |s^-(y_1) - s^-(y)|. \tag{1.24}$$

By the symmetry of the problem, using (1.13), we have

$$|\omega(y_1) - \omega(y)| \leq \frac{2}{k'_m}\left[|s^+(y_1) - s^+(y)| + 2|y_1 - y|\right] \qquad \forall y \in \Theta_m(y_1),$$

where $0 < k'_m := \inf\{-y \cdot s^-(y) \mid y \in \Theta_m(y_1)\} \leq k'_{m+1}$. From (1.23) it follows

$$A \geq |s^-(y_1) - s^-(y)| - \frac{2}{k'_m}\left[|s^+(y_1) - s^+(y)| + 2|y_1 - y|\right]. \tag{1.25}$$

On the other hand, combining (1.21) and (1.24)

$$A \geq M\varpi_m|y_1 - y| - |s^-(y_1) - s^-(y)|, \qquad \forall y \in \Theta_m(y_1), m > m_M. \tag{1.26}$$

We can sum (1.25) and (1.26) to get

$$2A \geq M\varpi_m|y_1 - y| - \frac{2}{k'_m}\left[|s^+(y_1) - s^+(y)| + 2|y_1 - y|\right].$$

From the definition (1.22) of A, this becomes

$$2\left(1 + \frac{1}{k'_m}\right)|s^+(y_1) - s^+(y)| \geq \left(M\varpi_m - \frac{4}{k'_m} - 2\omega(y)\right)|y_1 - y|,$$

for every $y \in \Theta_m(y_1), m > m_M$. Since neither ϖ_m nor k'_m is decreasing as a function of m, taking M large enough ensures $M > \left(\frac{4}{k'_{m_M}} + 4\right)\frac{1}{\varpi_{m_M}}$ to yield a positive constant

$$K = \frac{M\varpi_{m_M} - 4(\frac{1}{k'_{m_M}} + 1)}{2\left(1 + \frac{1}{k'_{m_M}}\right)},$$

such that

$$|s^+(y_1) - s^+(y)| \geq K|y_1 - y|, \qquad \forall y \in \Theta_m(y_1), m > m_M.$$

□

The injectivity (iii) of $s^+ = (t^+)^{-1}$ on the bivalent target $T_2 = t^-(S_2)$. follows from Lemma 1.6.7 and Lemma 1.6.9.

Proposition 1.6.10 (Quantifying injectivity of s^+ on the bivalent target) *Let $y_1 \in V \subset T_2$. Under the hypotheses of Theorem 1.6.1, there exists $\hat{C}_V^+ > 0$, depending on $\hat{C}_V^-$, k_V (from Proposition 1.6.6), and $\bar{\theta}(y_1)$ (from Lemma 1.6.9), such that, when y_0 is sufficiently close to y_1,*

$$|s^+(y_1) - s^+(y_0)| \geq \hat{C}_V^+ |y_1 - y_0|^{4N-1}.$$

Proof: When $y_0 \in \Theta(y_1, \theta)$, with $\theta > \bar{\theta}(y_1)$ we apply Lemma 1.6.9 and we are done. Otherwise the angle between $s^-(y_1) - s^-(y_0)$ and $\omega(y_1)y_1 - \omega(y_0)y_0$ is smaller than $\frac{\pi}{2} + \bar{\theta}(y_1)$. Applying Lemma 1.6.7, we obtain

$$\begin{aligned} &|s^+(y_1) - s^+(y_0)| \\ &= \quad |s^-(y_1) - s^-(y_0) + \omega(y_1)y_1 - \omega(y_0)y_0| \\ &\geq \quad |s^-(y_1) - s^-(y_0)| \cos\bar{\theta}(y_1). \end{aligned}$$

Taking y_0, y_1 sufficiently close ($|y_1 - y_0| < \gamma_{\delta_\epsilon}$, from the proof of Proposition 1.6.6), Proposition 1.6.6 implies

$$|s^+(y_1) - s^+(y_0)| \geq \cos\bar{\theta}(y_1)\hat{C}_V^- |y_1 - y_0|^{4N-1}. \qquad \square$$

Proof of Theorem 1.6.1: Define $y_i := t^+(x_i) \in V \subset T_2$. If $y_0 \in \Theta(y_1, \theta)$, with $\theta > \bar{\theta}(y_1)$, we have

$$|y_1 - y_0| < K|x_1 - x_0|.$$

Otherwise, by the uniform continuity of t^+, taking x_0 sufficiently close to x_1, we have $|t^+(x_1) - t^+(x_0)| < \gamma_{\delta_\epsilon}$ and we can apply Proposition 1.6.10 to $y_i = t^+(x_i)$, $i = 1, 0$ to conclude

$$|y_1 - y_0| < \frac{1}{\hat{C}_V^+} |x_1 - x_0|^{\frac{1}{4N-1}}. \qquad \square$$

1.7 On the dependence of the Hölder constant on the distance from $\mathcal{N}$

Theorem 1.7.1 *Suppose (μ, ν) are symmetrically suitable measures on $(\mathbf{S}^N, \mathbf{S}^N)$ (in particular, from Definition 1.2.1, there exists $\epsilon > 0$ such that $\nu > \epsilon \mathrm{dVol}^N$). Consider $x_0, x_1 \in (t^+)^{-1}(T_1) \subset \mathbf{S}^N$ sufficiently close, so that x_0, x_1, $t^+(x_0)$, $t^+(x_1)$ can be represented in local coordinates by means of the same map (π_{x_0} or π_{x_1}). Denote $y_0 = t^+(x_0), y_1 = t^+(x_1)$ and let $M := \max\{|Y_0|, |Y_1|\}$. Then there exists $C > 0$, depending only on ϵ and n, such that*

$$|t^+(x_1) - t^+(x_0)| \leq C \left(1 + \frac{M}{\sqrt{1 - M^2}}\right)^{\frac{1}{2}} |x_1 - x_0|^{\frac{1}{4N-1}}.$$

Remark 1.7.1 *We immediately notice that the Hölder constant in the previous statement tends to infinity, when one approaches the boundary of $\mathcal{N}$.*

The following proposition is the geometrical translation of assumption (As), i.e. of the condition, firstly introduced by Ma, Trudinger and Wang, which implies regularity. It is a sort of maximum principle and constitutes the main step of Loeper's proof of regularity. It corresponds to Proposition 5.1 in [41] and to Proposition B.1 in [35]. We are going to reproduce Loeper's proof (Proposition

5.1, [41]) using local coordinates. We are working locally on the set $\mathcal{N} := \{(x,y) \in \mathbf{S}^N \times \mathbf{S}^N \mid n_{\mathbf{S^N}}(x) \cdot n_{\mathbf{S^N}}(y) > 0\}$. Unlike in Loeper's result, we will find a constant which explodes, when we approach the boundary of $\mathcal{N}$. As a matter of fact, Kim and McCann noticed that the nondegeneracy condition (A2) on the cost, required both in [41] and [34], fails on the boundary of $\mathcal{N}$ (see Example 2.4 of [34]). For this same reason we cannot apply Proposition B.1 of [35] directly to derive the maximum principle.

Proposition 1.7.2 *For $x_0 \in \mathbf{S}^N$ and $y_0, y_1 \in \hat{\mathcal{N}}(x_0)$, let $Y_\theta \in [Y_0, Y_1]$, where the capital letters refer to the projections by π_{x_0}. Let $M := \max\{|Y_0|, |Y_1|\}$ and*

$$\bar{\psi}(X) = \max\{-c(X,Y_0) + c(X_0,Y_0), -c(X,Y_1) + c(X_0,Y_1)\}.$$

There exists $\upsilon > 0$, depending on M, such that

$$\bar{\psi}(X) \geq -c(X,Y_\theta) + c(X_0,Y_\theta) + \frac{\theta(1-\theta)}{2}|Y_1 - Y_0|^2|X - X_0|^2 - \upsilon|X - X_0|^3.$$

The dependence of υ on M implies that $\upsilon \to +\infty$ when $M \to 1$.

Proof Shifting and rotating coordinates, we can assume that $X_0 = 0$ and that $Y_0 - Y_1$ is parallel to $e_1 \in \mathbf{R}^N$. We apply the first part of Proposition A.1.0.2 of the Appendix to the function

$$f : t \to -D^2_{XX}c(0,Y_t) \cdot (X', X')$$

where $X' = (0, X^2, \dots, X^N)$, $Y_t = tY_0 + (1-t)Y_1$. Since $f''(t) \geq 2|Y_1 - Y_0|^2|X'|^2$, we get

$$\begin{gathered} -D^2_{XX}c(0,Y_\theta) \cdot (X', X') \\ \leq -\theta D^2_{XX}c(X_0,Y_1) \cdot (X', X') - (1-\theta)D^2_{XX}c(X_0,Y_0) \cdot (X', X') \\ -\theta(1-\theta)|X'|^2|Y_1 - Y_0|^2. \end{gathered}$$

Notice that, though the condition $f''(t) \geq 2|Y_1 - Y_0|^2|X'|^2$ is written in local coordinates, it does not depend on the choice of the chart. Indeed it derives from condition (A3s) (see [34], in particular Example 2.4). On the other hand, we need to work in local coordinates, in order to maintain the Euclidean form of Loeper's argument. We then apply the second part of Proposition A.1.0.2 to the function

$$g : t \to D^2_{XX}c(0,Y_t) \cdot (X, X) - D^2_{XX}c(0,Y_t) \cdot (X', X'),$$

which satisfies

$$||g''||_{L^\infty[0,1]} = 2|X^1|^2 \frac{M^2}{(1-M^2)^{\frac{3}{2}}}|Y_1 - Y_0|^2.$$

We get

$$\begin{gathered} -D^2_{XX}c(0,Y_\theta) \cdot (X, X) \\ \leq -[\theta D^2_{XX}(0,Y_1) + (1-\theta)D^2_{XX}(0,Y_0)] \cdot (X, X) + \Delta|X_1|^2 - \delta|X|^2, \end{gathered} \tag{1.27}$$

where

$$\Delta = \theta(1-\theta)\frac{M^2}{(1-M^2)^{\frac{3}{2}}}|Y_1 - Y_0|, \qquad \delta = \theta(1-\theta)|Y_1 - Y_0|^2.$$

Then, for all $\theta \in [0,1]$

$$\begin{aligned} &-c(X,Y_\theta)+c(0,Y_\theta) \\ &= -X\cdot D_X c(0,Y_\theta) - \tfrac{1}{2}D^2_{XX}c(0,Y_\theta)\cdot(X,X) + o(|X|^2). \end{aligned} \tag{1.28}$$

Hence, using the general fact that for $f_0, f_1 \in \mathbf{R}$ and $0 \le \theta \le 1$, $\max\{f_0, f_1\} \ge \theta f_1 + (1-\theta) f_0$, we obtain

$$\begin{aligned} \bar\psi(X) &\ge \theta[-c(X,Y_1)+c(0,Y_1)] + (1-\theta)[-c(X,Y_0)+c(0,Y_0)] \\ &= \theta[2X\cdot Y_1 - \frac{1}{2}D^2_{XX}c(0,Y_1)\cdot(X,X)] \\ &+(1-\theta)[2X\cdot Y_0 - \frac{1}{2}D^2_{XX}c(0,Y_0)\cdot(X,X)] + o(|X|^2) \\ &= 2X\cdot Y_\theta - \frac{1}{2}[\theta D^2_{XX}c(0,Y_1) + (1-\theta)D^2_{XX}c(0,Y_0)]\cdot(X,X) + o(|X|^2). \end{aligned}$$

By means of (1.27) we conclude

$$\bar\psi(X) \ge 2X\cdot Y_\theta - \tfrac{1}{2}D^2_{XX}c(0,Y_\theta)\cdot(X,X) - \tfrac{\Delta}{2}|X^1|^2 + \tfrac{\delta}{2}|X|^2 + o(|X|^2). \tag{1.29}$$

In order to eliminate the term $-\Delta|X^1|^2$ in the right-hand side, we write (1.29) for some $\theta' \in [0,1]$ and we then proceed as follows

$$\begin{aligned} \bar\psi(X) &\ge 2X\cdot Y_{\theta'} - \frac{1}{2}D^2_{XX}c(0,Y_{\theta'})\cdot(X,X) - \frac{\Delta'}{2}|X^1|^2 + \frac{\delta'}{2}|X|^2 + o(|X|^2) \\ &= X\cdot Y_\theta - \frac{1}{2}D^2_{XX}c(0,Y_\theta) + \frac{\delta}{2}|X|^2 + o(|X|^2) \\ &+\frac{1}{2}[D^2_{XX}c(0,Y_\theta) - D^2_{XX}c(0,Y_{\theta'})]\cdot(X,X) \\ &+2(\theta'-\theta)X\cdot(Y_1-Y_0) - \frac{\Delta}{2}|X^1|^2 + \frac{(\Delta-\Delta')}{2}|X^1|^2 + \frac{(\delta-\delta')}{2}|X|^2, \end{aligned} \tag{1.30}$$

where $\Delta' = \Delta(\theta'), \delta' = \delta(\theta')$. We now have to control the following three terms

$$\begin{aligned} T_1 &= 2(\theta-\theta')2X\cdot(Y_1-Y_0) - \frac{\Delta}{2}|X^1|^2, \\ T_2 &= \frac{1}{2}[D^2_{XX}c(0,Y_\theta) - D^2_{XX}c(0,Y_{\theta'})]\cdot(X,X), \\ T_3 &= \frac{\delta-\delta'}{2}|X|^2 + \frac{\Delta-\Delta'}{2}|X^1|^2. \end{aligned}$$

Fix $\epsilon \in (0,1)$; taking $\theta \in [\epsilon, 1-\epsilon]$ and restricting to $|X^1| \le \frac{\epsilon|Y_1-Y_0|}{\Delta}$, we can choose

$$\theta' = \theta + \frac{\Delta|X^1|^2}{4|Y_1-Y_0|X^1},$$

so that $T_1 = 0$. After few computations we can write

$$T_2 = (\sqrt{1-|Y_\theta|^2} - \sqrt{1-|Y_{\theta'}|^2})|X|^2.$$

Recalling that $M = \max\{|Y_0|, |Y_1|\}$, we have

$$|\sqrt{1-|Y_\theta|^2} - \sqrt{1-|Y_{\theta'}|^2}| \le \frac{M}{\sqrt{1-M^2}}|Y_1 - Y_0||\theta - \theta'|,$$

which leads to

$$T_2 \le \frac{M}{\sqrt{1-M^2}}|Y_1 - Y_0||\theta - \theta'||X|^2.$$

Notice that the constant $\frac{M}{\sqrt{1-M^2}}$ is very big when Y_0 or Y_1 is close to the boundary of $\hat{\mathcal{N}}(X_0)$. To control T_3, we observe that

$$\begin{aligned}|\Delta - \Delta'| &= \frac{M^2}{(1-M^2)^{\frac{3}{2}}}|Y_1 - Y_0||X^1|^2|\theta(1-\theta) - \theta'(1-\theta')| \\ &\le \frac{M^2}{(1-M^2)^{\frac{3}{2}}}|Y_1 - Y_0||\theta - \theta'|,\end{aligned}$$

and

$$|\delta - \delta'| \le |Y_1 - Y_0|^2|\theta - \theta'|.$$

Using $\theta' = \theta + \frac{\Delta X^1}{4|Y_1 - Y_0|X^1}$, we obtain

$$\begin{aligned}|T_2 + T_3| &\le \frac{M}{\sqrt{1-M^2}}|Y_1 - Y_0||\theta - \theta'||X|^2 \\ &+\frac{1}{2}\frac{M^2}{(1-M^2)^{\frac{3}{2}}}|Y_1 - Y_0||\theta - \theta'||X|^2 + \frac{1}{2}|Y_1 - Y_0|^2|\theta - \theta'||X|^2 \\ &\le \left(\frac{M}{\sqrt{1-M^2}} + \frac{1}{2}\left(1 + \frac{M^2}{(1-M^2)^{\frac{3}{2}}}\right)\Delta|Y_1 - Y_0|\right)|X|^3 \\ &\le \tilde{C}|X|^3,\end{aligned}$$

where $\tilde{C}$ depends only on M and $\tilde{C} \to +\infty$ when $M \to 1$. We can now improve the inequality (1.30) as follow

$$\bar{\psi}(X) \ge X \cdot Y_\theta - \frac{1}{2}D^2_{XX}c(0, Y_\theta) + \frac{\delta}{2}|X|^2 - \tilde{C}|X|^3 + o(|X|^2).$$

Since $c \in C^3(N)$ we can replace $-\tilde{C}|X|^3 + o(|X|^2)$ by $-\upsilon|X|^3$, with $\upsilon > 0$; using (1.28), we conclude

$$\bar{\psi}(X) \ge -c(X, Y_\theta) + c(0, Y_\theta) + \frac{\delta}{2}|X|^2 - \upsilon|X|^3 + o(|X|^2). \qquad \square$$

The Hölder continuity of a function f guarantees that the distance between the images $f(a), f(b)$ can be estimated through a certain power of $|b-a|$. In other words, if a, b are close, their images cannot spread too much. The following proposition provides a preliminary relation between areas in $\mathbf{S}^N = \partial\Lambda$ and images (through t^+) of areas in $\mathbf{S}^N = \partial\Omega$. This result, combined with the hypothesis $\nu_1 > \epsilon\mathcal{H}^N\lfloor_{\mathbf{S}^N}$, will give the desired Hölder continuity of t^+ on $(t^+)^{-1}(T_1) \subset \mathbf{S}^N$.

Let ψ denote Brenier's potential for (1.2), whose subdifferential $\partial\psi$ contains the support of the optimal measure $\gamma \in \Gamma(\mu, \nu)$ which solves (1.2) (see [45] for this characterization). In particular we have that $t^+(x) \in \partial\psi(x)$ for every $x \in \mathbf{S}^N$. The proof of Proposition 1.7.3 below is based on the construction of supporting functions for ψ. Once again we reproduce an argument due to Loeper (Proposition 5.6 of [41]), turned into a local coordinates setting. Many computations are identical to Loeper's ones, so we will skip some of the details. In particular, we will postpone the proofs of Lemma 1.7.5 and Lemma 1.7.6 to the Appendix.

In the following, by '$x_0, x_1 \in \mathbf{S}^N$ sufficiently close' we mean that we can project x_0, x_1, y_0, y_1 into local coordinates using the same map; we will choose π_{x_0} or π_{x_1}.

Proposition 1.7.3 *Let $x_0, x_1 \in \mathbf{S}^N$ be sufficiently close, and $y_0 = t^+(x_0), y_1 = t^+(x_1)$. Let $M := \max\{|Y_0|, |Y_1|\}$. Then there exist constants $C', C'' > 0$ (depending only on $\|c\|_{C^2}$), K (depending on $\|c\|_{C^2}$ and M), and $x_m \in \pi_{x_0}^{-1}[X_0, X_1]$ such that, if $\mathcal{U}_\eta([X_0, X_1]) \subset \{X : |X - X_0| < 1\}$ and*

$$|Y_1 - Y_0| \geq \max\{|X_1 - X_0|, K|X_1 - X_0|^{\frac{1}{5}}\} > 0,$$

then

$$\pi_{x_0}^{-1}\left(\mathcal{U}_\rho\left(\left\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right)\right) \subset t^+(\pi_{x_0}^{-1}(B_\eta(X_m)),$$

where

$$\begin{aligned} \eta &= C'\left(\frac{|X_1 - X_0|}{|Y_1 - Y_0|}\right)^{\frac{1}{2}}, \\ \rho &= C''\eta|Y_1 - Y_0|^2. \end{aligned}$$

Remark 1.7.4 *Should the hypothesis*

$$|Y_1 - Y_0| \geq \max\{|X_1 - X_0|, K|X_1 - X_0|^{\frac{1}{5}}\}$$

fail, we would easily get

$$|t^+(x_1) - t^+(x_0)| < \left(1 + \frac{M}{\sqrt{1 - M^2}}\right)^{\frac{1}{2}} \max\{|x_1 - x_0|, |x_1 - x_0|^{\frac{1}{5}}\}.$$

Proof of Proposition 1.7.3 According to Proposition A.1.0.1 of the Appendix, by subtracting the affine function

$$\phi(x) = \frac{\psi(x_1) - \psi(x_0)}{\sum_{i=1}^{n+1} x_0^i - x_1^i}\left(\sum_{i=1}^{N+1} x^i - x_1^i\right)$$

from the cost function c, we will not modify the solution to the optimal transportation problem, and the potential ψ will be changed into $\psi + \phi$. Notice that $(\psi + \phi)(x_0) = (\psi + \phi)(x_1)$, hence we can assume, without loss of generality, that $\psi(x_0) = \psi(x_1)$. Suppose we choose to project by means of π_{x_0}. In local coordinates we write $Y_0 = t^+(X_0)$ and $Y_1 = t^+(X_1)$. Hereafter we will continue to write X_0 even though $X_0 = 0$. From this assumption we have

$$\begin{aligned} -c(X, Y_0) + c(X_0, Y_0) + \psi(X_0) \leq \psi(X), \\ -c(X, Y_1) + c(X_0, Y_1) + \psi(X_1) \leq \psi(X), \end{aligned}$$

for all X in a neighbourhood of X_0. Since $\psi(X_0) = \psi(X_1)$, the difference between the supporting functions

$$X \mapsto -c(X, Y_0) + c(X_0, Y_0) + \psi(X_0),$$
$$X \mapsto -c(X, Y_1) + c(X_0, Y_1) + \psi(X_1),$$

will vanish at some point X_m in the segment $[X_0, X_1]$. Without loss of generality, we can add a constant to ψ so that at this point both supporting functions are equal to 0, i.e.

$$-c(X_m, Y_0) + c(X_0, Y_0) + \psi(X_0) = 0, \tag{1.31}$$

$$-c(X_m, Y_1) + c(X_0, Y_1) + \psi(X_1) = 0. \tag{1.32}$$

This implies

$$\psi(X) \geq \max\{-c(X, Y_0) + c(X_m, Y_0), -c(X, Y_1) + c(X_m, Y_1)\}.$$

We apply Proposition 1.7.2 centered at X_m and we get

$$\begin{aligned} \psi(X) \quad &\geq \quad -c(X, Y_\theta) + c(X_m, Y_\theta) \\ &+ \quad \theta(1-\theta)|Y_1 - Y_0|^2 |X - X_m|^2 - \upsilon |X - X_m|^3 \\ &=: \quad \Psi(X). \end{aligned} \tag{1.33}$$

To proceed we also need an estimate on ψ from above. We modify Lemma 5.7 of [41] as follows

Lemma 1.7.5 *Under the assumptions made above, and assuming moreover*

$$|Y_1 - Y_0| \geq |X_1 - X_0|,$$

we have, for all X in the segment $[X_0, X_1]$,

$$\psi(X) \leq C_3 |X_1 - X_0| |Y_1 - Y_0|,$$

where C_3 depends only on $||c(.,.)||_{C^2(\mathbf{S}^N \times \mathbf{S}^N)}$.

Lemma 1.7.6 *Let X_m, Y_0, Y_1 be defined as above. For $Y \in \mathbf{R}^N$, consider the function*

$$f_Y(X) = -c(X, Y) + c(X_m, Y) + \psi(X_m).$$

Under the assumptions made above, if

$$|Y_1 - Y_0| \geq K |X_1 - X_0|^{\frac{1}{5}}, \qquad K = \left(16^3 C_3 \upsilon^2\right)^{\frac{1}{5}} \tag{1.34}$$

there exist η and ρ such that for all $Y \in \mathcal{U}_\mu\left(\left\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right)$

$$\psi - f_Y \geq 0 \qquad \text{on } \partial B_\eta(X_m).$$

At this point we can prove Proposition 1.7.3. By construction we have $f_Y(X_m) = \psi(X_m)$. Applying Lemma 1.7.6 we have $\psi \geq f_Y$ on $\partial B_\eta(X_m)$, then $\psi - f_Y$ will have a local minimum inside $B_\eta(X_m)$ at some point $X \in B_\eta(X_m)$. Going back to $\mathbf{S}^N$ with the ordinary coordinates of $\mathbf{R}^N$, $\psi - f_y$ will have a local minimum at some point $x \in \pi_{x_0}^{-1}(B_\eta(X_m))$, where $f_y(x) = -c(x, y) + c(x_m, y) + \psi(x_m)$. This will imply $-\nabla_x c(x, y) \in \partial\psi(x)$ which is equivalent to $y \in \partial\psi(x)$. From Lemma 1.7.6 we deduce that $y \cdot x > 0$, then $y = t^+(x) \subset t^+(\pi_{x_0}^{-1}(B_\eta(X_m))$. $\square$

We now state a general result.

Lemma 1.7.7 *Let $\Sigma \subset \mathbf{R}^N$ be a convex set and $Y_0, Y_1 \in \Sigma$. There exist $\sigma > 0$, $\rho_0 > 0$, depending only on Σ and n, such that, for all $\rho \in (0, \rho_0)$,*

$$Vol^N(\mathcal{U}_\rho([Y_0, Y_1]) \cap \Sigma) \geq \sigma \, Vol^N(\mathcal{U}_\rho([Y_0, Y_1])).$$

Let us define $\Sigma := \pi_{x_0}(\hat{\mathcal{N}}(x_0)$. Applying Proposition A.1.0.3 of the Appendix and Lemma 1.7.7, we can write

$$\begin{aligned} &\mathcal{H}^N(\pi_{x_0}^{-1}(\mathcal{U}_\rho(\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\}) \cap \Sigma) \\ \geq\ & \mathcal{H}^N(\mathcal{U}_\rho(\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\}) \cap \Sigma) \\ \geq\ & \sigma \frac{2^N}{\omega_N} \mathrm{Vol}^N(\mathcal{U}_\rho(\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\})) \\ =\ & \frac{\sigma}{2} \frac{2^N}{\omega_N} |Y_1 - Y_0| \rho^{N-1} \\ =\ & C_5 \eta^{N-1} |Y_1 - Y_0|^{2N-1}, \end{aligned} \tag{1.35}$$

where $C_5 = \frac{\sigma}{2} \frac{2^N}{\omega_N} \left(\frac{1}{16C_4}\right)^{N-1}$ and $\omega_N = \mathrm{Vol}^N(B_1(0))$. On the other hand, as a consequence of Proposition 1.7.3,

$$\begin{aligned} &\mathcal{H}^N(\pi_{x_0}^{-1}(\mathcal{U}_\rho(\{Y_\theta, \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\}) \cap \Sigma) \\ &\qquad \leq \mathcal{H}^N(t^+(\pi_{x_0}^{-1}(B_\eta(X_m))). \end{aligned} \tag{1.36}$$

Lemma 1.7.8 *Since $(t^+)_\sharp \mu_1 = \nu_1$ and $\nu_1 \geq \epsilon \mathcal{H}^N$ on $\mathbf{S}^N \setminus T_2$ then, for all $A \subset \mathbf{S}^N \setminus (t^+)^{-1}(T_2)$*

$$\mu_1(A) \geq \epsilon \mathcal{H}^N(t^+(A)),$$

and hence

$$(t^+)^\sharp d\mathcal{H}^N \leq \frac{1}{\epsilon} \mu_1.$$

Using the definition of push-forward measure we have

$$\mu_1(A) = \mu_1((t^+)^{-1} t^+(A)) = \nu_1(t^+(A)) \geq \epsilon \mathrm{Vol}^N(t^+(A)). \qquad \square$$

Since we are proving a local of Hölder regularity result, we can suppose $B_\eta(X_m) \subset \left\{X : |X| < \frac{1}{2}\right\} =: B$. On B the function $\pi_{x_0}^{-1}$ is Lipschitz, and the Lipschitz constant is $L = \left(\frac{7}{4}\right)^{\frac{1}{2}}$. Using Lemma 1.7.8 and Proposition A.1.0.3 of the Appendix, we get

$$\begin{aligned} &\mathcal{H}^N(t^+(\pi_{x_0}^{-1}(B_\eta(X_m)))) \leq \frac{1}{\epsilon} \mu_1(\pi_{x_0}^{-1}(B_\eta(X_m))) \\ \leq\ & \frac{1}{\epsilon^2} \mathcal{H}^N(\pi_{x_0}^{-1}(B_\eta(X_m))) \leq C_6 \mathrm{Vol}^N(B_\eta(X_m)) = C_7 \eta^N, \end{aligned} \tag{1.37}$$

where C_6, C_7 depend only on ϵ, L, and N. Combining (1.35), (1.36), and (1.37) we obtain

$$|Y_1 - Y_0|^{2N-1} \leq \frac{C_7}{C_5} \eta = \frac{C_7 C'}{C_5} \left(\frac{|X_1 - X_0|}{|Y_1 - Y_0|}\right)^{\frac{1}{2}},$$

which becomes, after few computations,

$$|Y_1 - Y_0| \leq C|X_1 - X_0|^{\frac{1}{4N-1}}, \qquad C = \left(\frac{C_7 C'}{C_5}\right)^{\frac{2}{4N-1}}.$$

Since $M = \max\{|Y_0|, |Y_1|\}$, and

$$|y_1 - y_0| \leq \left(1 + \frac{M}{\sqrt{1-M^2}}\right)^{\frac{1}{2}} |Y_1 - Y_0|,$$

we conclude

$$|t^+(x_1) - t^+(x_0)| \leq C\left(1 + \frac{M}{\sqrt{1-M^2}}\right)^{\frac{1}{2}} |x_1 - x_0|^{\frac{1}{4N-1}}. \qquad \square$$

1.8 Bi-Lipschitz estimates for t^+ when $N = 1$

Theorem 1.8.1 *Consider two bounded, strictly convex, planar domains $\Omega, \Lambda \subset \mathbf{R}^2$ with symmetrically suitable measures μ on $\partial\Omega$, ν on $\partial\Lambda$. Suppose that the boundaries $\partial\Omega, \partial\Lambda$ are C^2. Then there exists two positive constants L_1, L_2 such that for all $x_0 \in \partial\Omega \setminus S_0$ there exist $x_1, x_2 \in \partial\Omega$ sufficiently close to x_0 such that $[x_1, x_2]_{\partial\Omega} \ni x_0$ and*

$$L_1 \mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}) < \nu([t^+(x_1), t^+(x_2)]_{\partial\Lambda}) < L_2 \mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}),$$

where $[x_1, x_2]_{\partial\Omega}$ ($[y_1, y_2]_{\partial\Lambda}$) represents the shortest portion of the boundary curve $\partial\Omega$ ($\partial\Lambda$) joining x_1 and x_2 (y_1 and y_2).

We can analyze this problem in three distinct cases

(1) $x_0 \in S_1$, $t^+(x_0) \in T_1$;

(2) $x_0 \in S_2$, $t^+(x_0) \in T_1$;

(3) $x_0 \in S_1$, $t^+(x_0) \in T_2$.

We recall that if $x_0 \in S_2$ then $t^+(x_0) \in T_1$, as observed in Remark 3.10 of [27].

In case **(1)**, since both S_1 and T_1 are open, there exist $x_1, x_2 \in \partial\Omega$, sufficiently close to x_0, such that $x_0 \in [x_1, x_2]_{\partial\Omega} \subset S_1$ and $t^+(x_0) \in [t^+(x_1), t^+(x_2)]_{\partial\Lambda} \subset T_1$. We have

$$\begin{aligned} \nu[t^+(x_1), t^+(x_2)]_{\partial\Lambda} &= \gamma[\{x, t^+(x)\} \mid t^+(x) \in [t^+(x_1), t^+(x_2)]_{\partial\Lambda}] \\ &= \mu[x_1, x_2]. \end{aligned}$$

Since μ, ν are symmetrically suitable, there exist $\epsilon, \epsilon' > 0$ such that

$$\epsilon' \mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}) < \mu([x_1, x_2]_{\partial\Omega}) < \frac{1}{\epsilon} \mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}).$$

Taking $L_1 = \epsilon'$ and $L_2 = \frac{1}{\epsilon}$ proves Theorem 1.8.1 in case (1).

In case **(2)**, since T_1 is open, there exist $x_1, x_2 \in \partial\Omega$, sufficiently close to x_0, such that $t^+(x_0) \in [t^+(x_1), t^+(x_2)]_{\partial\Lambda} \subset T_1$. We have

$$\begin{aligned}\nu[t^+(x_1), t^+(x_2)]_{\partial\Lambda} &= \gamma[\{x, t^+(x)\} \mid t^+(x) \in [t^+(x_1), t^+(x_2)]_{\partial\Lambda}] \\ &< \mu[x_1, x_2]_{\partial\Omega};\end{aligned}$$

on the other hand

$$\begin{aligned}\nu[t^+(x_1), t^+(x_2)]_{\partial\Lambda} &= \nu_1[t^+(x_1), t^+(x_2)]_{\partial\Lambda} \\ &= \mu_1[x_1, x_2]_{\partial\Omega}, \end{aligned} \tag{1.38}$$

where $\mu_1 = (s^+)_\sharp \nu_1$, with $\nu_1 = \nu\lfloor_{T_0 \cup T_1}$. Hence we only need to prove the left-hand inequality of Theorem 1.8.1.

In case **(3)**, since S_1 is open, there exist $x_1, x_2 \in \partial\Omega$, sufficiently close to x_0, such that $x_0 \in [x_1, x_2]_{\partial\Omega} \subset S_1$. We have

$$\begin{aligned}\nu[t^+(x_1), t^+(x_2)]_{\partial\Lambda} &> \gamma[\{x, t^+(x)\} \mid t^+(x) \in [t^+(x_1), t^+(x_2)]_{\partial\Lambda}] \\ &= \mu[x_1, x_2]_{\partial\Omega}.\end{aligned}$$

Hence we only need to prove the right-hand inequality of Theorem 1.8.1.

Remark 1.8.2 *Since there exist* $\epsilon, \epsilon' > 0$ *s.t.* $\epsilon\mathcal{H}\lfloor_{\partial\Lambda} < \nu < \frac{1}{\epsilon'}\mathcal{H}\lfloor_{\partial\Lambda}$ *and* $\nu_1 = \nu\lfloor_{dom\nabla\phi}, \nu_2 = \nu - \nu_1$ *(which implies* $\operatorname{spt}\nu_1 \cap \operatorname{spt}\nu_2 = \emptyset$*), we have*

$$\begin{aligned}\epsilon\mathcal{H}\lfloor_{\partial\Lambda\cap\operatorname{spt}\nu_1} &< \nu_1\lfloor_{\operatorname{spt}\nu_1} < \tfrac{1}{\epsilon'}\mathcal{H}\lfloor_{\partial\Lambda\cap\operatorname{spt}\nu_1} \\ \epsilon\mathcal{H}\lfloor_{\partial\Lambda\cap\operatorname{spt}\nu_2} &< \nu_2\lfloor_{\operatorname{spt}\nu_2} < \tfrac{1}{\epsilon'}\mathcal{H}\lfloor_{\partial\Lambda\cap\operatorname{spt}\nu_2}\end{aligned}$$

Remark 1.8.3 *For what concerns* μ_1 *and* μ_2*, we cannot state* $\operatorname{spt}\mu_1 \cap \operatorname{spt}\mu_2 = \emptyset$*, but it holds true that* $\operatorname{spt}\mu_1 \cap \operatorname{spt}\mu_2 \subset S_2$*. Notice that*

$$\begin{aligned}\mu_1([x_1, x_2]_{\partial\Omega}) &= \mu_1([x_1, x_2]_{\partial\Omega} \cap (S_0 \cup S_1)) + \mu_1([x_1, x_2]_{\partial\Omega} \cap S_2) \\ &= \mu([x_1, x_2]_{\partial\Omega} \cap (S_0 \cup S_1)) + \mu_1([x_1, x_2]_{\partial\Omega} \cap S_2) \\ &> \epsilon'\mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega} \cap (S_0 \cup S_1)) + \mu_1([x_1, x_2]_{\partial\Omega} \cap S_2).\end{aligned}$$

Therefore, in case (2), we can suppose $[x_1, x_2]_{\partial\Omega} \subset S_2$*; indeed, from the previous computation, if* $[x_1, x_2]_{\partial\Omega} \cap (S_0 \cup S_1) \neq \emptyset$*, the only term that we actually need to estimate from below is* $\mu_1([x_1, x_2]_{\partial\Omega} \cap S_2)$*.*

Remark 1.8.4 *In order to obtain the bi-Lipschitz estimates for case* ***(2)*** *it is sufficient to prove that there exists* $\tilde{\epsilon} > 0$ *such that, for every* $S_2 \supset [x_1, x_2]_{\partial\Omega} \ni x_0$*,*

$$\tilde{\epsilon}\mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}) < \mathcal{H}\lfloor_{\partial\Lambda}[t^+(x_1), t^+(x_2)]_{\partial\Lambda}). \tag{1.39}$$

Indeed (1.39) *would imply*

$$\begin{aligned}\mu_1([x_1, x_2]_{\partial\Omega}) &= \nu_1([t^+(x_1), t^+(x_2)]_{\partial\Lambda}) \\ &> \mathcal{H}\lfloor_{\partial\Lambda}([t^+(x_1), t^+(x_2)]_{\partial\Lambda}) \\ &\overset{(1.39)}{>} \epsilon\tilde{\epsilon}\mathcal{H}\lfloor_{\partial\Omega}([x_1, x_2]_{\partial\Omega}),\end{aligned}$$

which, combined with (1.38)*, gives the desired estimate.*

Since we are dealing with strictly convex, planar domains $\Omega, \Lambda \subset \mathbf{R}^2$, following Definition 3.2.3 of [1], we can introduce an *angular parametrization* of the boundaries $\partial\Omega, \partial\Lambda$. The *angular parameter* ϕ (or θ) denotes points on $[0, 2\pi] \equiv \mathbf{R}/2\pi\mathbf{Z} \equiv \mathbf{T}^1$, parametrizing the Gauss circle $\mathbf{S}^1$ so that $\hat{n}(\phi) := (\cos\phi, \sin\phi) \in \mathbf{S}^1$. Under this parametrization, the points on the domain boundaries, $\partial\Omega$ and $\partial\Lambda$, can be represented by

$$x(\phi) \in \arg\max_{x\in\partial\Omega} \quad x \cdot \hat{n}(\phi) \qquad \text{and} \qquad y(\theta) \in \arg\max_{y\in\partial\Lambda} \quad y \cdot \hat{n}(\theta).$$

This implies

$$n_\Omega(x(\phi)) = \hat{n}(\phi) \qquad \text{and} \qquad n_\Lambda(y(\theta)) = \hat{n}(\theta).$$

Let us write the cost function in terms of the angular parametrization

$$c(\phi, \theta) = |x(\phi) - y(\theta)|^2.$$

Let us assume that $\partial\Omega$ and $\partial\Lambda$ are differentiable with respect to ϕ and θ. Then

$$\frac{\partial c}{\partial \phi}(\phi, \theta) = 2(x(\phi) - y(\theta)) \cdot \dot{x}(\phi);$$

$$\begin{aligned}
\frac{\partial^2 c}{\partial\phi\partial\theta}(\phi, \theta) &= -2|\dot{y}(\theta)||\dot{x}(\phi)|n_\Lambda(y(\theta)) \cdot n_\Omega(x(\phi)) \\
&= -2|\dot{y}(\theta)||\dot{x}(\phi)|\hat{n}(\theta) \cdot \hat{n}(\phi) \\
&= -2|\dot{y}(\theta)||\dot{x}(\phi)| \cos(\theta - \phi).
\end{aligned}$$

From Lemma 5.2.1 of [1] we know that

$$\int_{t^+(\phi)}^{t^-(\phi)} |\dot{y}(\theta)||\dot{x}(\phi)| \cos(\theta - \phi) d\theta = 0 \qquad \forall \phi \in S_2. \tag{1.40}$$

Equality (1.40) of [1] holds true if and only if

$$(x(\phi) - y(t^+(\phi))) \cdot \dot{x}(\phi) = (x(\phi) - y(t^-(\phi))) \cdot \dot{x}(\phi) \qquad \forall \phi \in S_2$$

i.e. if

$$\begin{aligned}
&[y(t^+(\phi)) - y(t^-(\phi))] \cdot \dot{x}(\phi) \\
&\quad = |(y(t^+(\phi)) - y(t^-(\phi)))| n_\Omega(\phi) \cdot \dot{x}(\phi) \\
&\quad = 0 \qquad \forall \phi \in S_2.
\end{aligned} \tag{1.41}$$

Let us define

$$F(\phi) := [y(t^+(\phi)) - y(t^-(\phi))] \cdot \dot{x}(\phi).$$

Thanks to (1.41) we have $F(\phi) = 0$ for all $\phi \in \mathbf{T}^1$. In particular, for every $\bar{\phi}, \bar{\phi} + h \in \mathbf{T}^1$

$$\Delta_h F(\phi) = \frac{F(\bar{\phi} + h) - F(\bar{\phi})}{h} = 0.$$

Fix $n \in \mathbf{N}$, and let $\phi_1^n, \phi_2^n := \phi_1^n + h_n \in S_2$, for a certain $h_n > 0$; introduce

$$\begin{aligned}\Delta^n F(\phi_1^n) &= \frac{[y(t^+(\phi_2^n)) - y(t^-(\phi_2^n))] \cdot \dot{x}(\phi_2^n)}{\phi_2^n - \phi_1^n} \\ &- \frac{[y(t^+(\phi_1^n)) - y(t^-(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n}.\end{aligned}$$

Summing and subtracting the following term

$$\frac{[y(t^+(\phi_2^n)) - y(t^-(\phi_2^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n},$$

we get

$$\begin{aligned}\Delta^n F(\phi_1^n) &= \frac{[y(t^+(\phi_2^n)) - y(t^+(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \\ &- \frac{[y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \\ &+ [y(t^+(\phi_2^n)) - y(t^-(\phi_2^n))] \cdot \Big(\frac{\dot{x}(\phi_2^n) - \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n}\Big).\end{aligned} \tag{1.42}$$

Now, let n vary in $\mathbf{N}$ and construct two sequences $\{\phi_1^n\}, \{\phi_2^n\} \subset S_2$ such that

$$\phi^\infty = \lim_{n\to\infty} \phi_1^n = \lim_{n\to\infty} \phi_2^n \in S_2 \qquad \text{(i.e. } h_n \to 0).$$

Then

$$\Delta^n F(\phi_1^n) = 0 \quad \forall n \qquad \text{and} \qquad \lim_{n\to\infty} \Delta^n F(\phi_1^n) = 0.$$

Since $x = x(\phi)$ is differentiable on S_2, we can define

$$v_\Omega^M := \max_{\bar{\phi} \in \bar{S}_2} \Big| \frac{dx}{d\phi}(\bar{\phi}) \Big| > 0.$$

Let us estimate the first term on the right-hand side of (1.42). Since $|\phi_2^n - \phi_1^n| v_\Omega^M \geq \mathcal{H}^1[\phi_1^n, \phi_2^n]_{\partial\Omega}$

$$\begin{aligned}&\Big| \frac{[y(t^+(\phi_2^n)) - y(t^+(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \Big| \leq \frac{|y(t^+(\phi_2^n)) - y(t^+(\phi_1^n))| |\dot{x}(\phi_1^n)|}{|\phi_2^n - \phi_1^n|} \\ &\leq v_\Omega^M \frac{\mathcal{H}^1[y(t^+(\phi_2^n)), y(t^+(\phi_1^n))]_{\partial\Lambda}}{\mathcal{H}^1[\phi_2^n, \phi_1^n]_{\partial\Omega}} |\dot{x}(\phi_1^n)| \\ &\leq (v_\Omega^M)^2 \frac{\mathcal{H}^1[y(t^+(\phi_2^n)), y(t^+(\phi_1^n))]_{\partial\Lambda}}{\mathcal{H}^1[\phi_2^n, \phi_1^n]_{\partial\Omega}}.\end{aligned} \tag{1.43}$$

By contradiction, let us assume that (1.39) is false. Then, the previous estimate implies

$$\Big| \frac{[y(t^+(\phi_2^n)) - y(t^+(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_1^n - \phi_1^n} \Big| \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 1.8.5 *Let $[\phi_1^n, \phi_2^n]_{\partial\Omega}$ be in S_2 for every $n \in \mathbf{N}$. There exists $C > 0$ such that for every $n \in \mathbf{N}$*

$$C|\phi_2^n - \phi_1^n| \le |t^-(\phi_2^n) - t^-(\phi_1^n)|.$$

Proof of the Claim By contradiction, assume

$$\frac{|t^-(\phi_2^n) - t^-(\phi_1^n)|}{|\phi_2^n - \phi_1^n|} \to 0 \quad \text{as} \quad n \to \infty;$$

then

$$\frac{\mathcal{H}^1[y(t^-(\phi_2^n)), y(t^-(\phi_1^n))]_{\partial\Lambda}}{\mathcal{H}^1[\phi_2^n, \phi_1^n]_{\partial\Omega}} \le \frac{v_\Lambda^M}{v_\Omega^m} \frac{|t^-(\phi_2^n) - t^-(\phi_1^n)|}{|\phi_2^n - \phi_1^n|} \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$v_\Lambda^M = \max_{\bar\theta \in \partial\Lambda \cap \mathcal{V}} \left|\frac{dy}{d\theta}(\bar\theta)\right| > 0, \qquad \mathcal{V} \text{ neighbourhood of } t^-(\phi^\infty),$$
$$v_\Omega^m = \min_{\bar\phi \in \partial\Omega \cap \mathcal{U}} \left|\frac{dx}{d\phi}(\phi^\infty)\right| > 0, \qquad \mathcal{U} \text{ neighbourhood of } \phi^\infty.$$

Since (1.39) is false, this implies

$$\frac{\mathcal{H}^1[y(t^-(\phi_2^n)), y(t^-(\phi_1^n))]_{\partial\Lambda} + \mathcal{H}^1[y(t^+(\phi_2^n)), y(t^+(\phi_1^n))]_{\partial\Lambda}}{\mathcal{H}^1[\phi_2^n, \phi_1^n]_{\partial\Omega}} \to 0 \quad \text{as} \quad n \to \infty;$$

but

$$\begin{aligned} &\frac{\mathcal{H}^1[y(t^-(\phi_2^n)), y(t^-(\phi_1^n))]_{\partial\Lambda} + \mathcal{H}^1[y(t^+(\phi_2^n)), y(t^+(\phi_1^n))]_{\partial\Lambda}}{\mathcal{H}^1[\phi_2^n, \phi_1^n]_{\partial\Omega}} \\ &> (\epsilon')^2 \frac{\nu[y(t^-(\phi_2^n)), y(t^-(\phi_1^n))]_{\partial\Lambda} + \nu[y(t^+(\phi_2^n)), y(t^+(\phi_1^n))]_{\partial\Lambda}}{\mu[\phi_2^n, \phi_1^n]_{\partial\Omega}} \\ &> (\epsilon')^2 \frac{\mu[\phi_2^n, \phi_1^n]_{\partial\Omega}}{\mu[\phi_2^n, \phi_1^n]_{\partial\Omega}} = (\epsilon')^2 > 0, \end{aligned}$$

and we get a contradiction. □

Thanks to the Claim we can estimate the second term on the right-hand side of (1.42) as follows

$$\begin{aligned} & - \frac{[y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \\ = & -\left|\frac{y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))}{\phi_2^n - \phi_1^n}\right| |\dot{x}(\phi_1^n)| \cos\alpha_n \\ \le & -C \frac{|y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))|}{|t^-(\phi_2^n) - t^-(\phi_1^n)|} |\dot{x}(\phi_1^n)| \cos\alpha_n, \end{aligned}$$

where α_n is defined to be the angle between the vectors

$$\frac{y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))}{\phi_2^n - \phi_1^n} \qquad \text{and} \qquad \dot{x}(\phi_1^n).$$

Since $\phi_2^n > \phi_1^n$ and t^- is locally non-increasing on S_2, as proved by Ahmad in Proposition 3.2.5 of [1], α_n is also the angle between the vectors

$$\frac{y(t^-(\phi_2^n)) - y(t^-(\phi_1^n))}{t^-(\phi_1^n) - t^-(\phi_2^n)} \qquad \text{and} \qquad \dot{x}(\phi_1^n).$$

As $n \to \infty$, $\alpha_n \to \alpha$, where α is the angle between the vectors

$$-\dot{y}(t^-(\phi^\infty)) \qquad \text{and} \qquad \dot{x}(\phi^\infty).$$

Since $\phi^\infty \in S_2$, we know that

$$-\dot{y}(t^-(\phi^\infty)) \cdot \dot{x}(\phi^\infty) = -n_\Lambda(t^-(\phi^\infty)) \cdot n_\Omega(\phi^\infty) > 0,$$

which means that $\cos\alpha > 0$. Hence, taking n sufficiently large, $\cos\alpha_n > 0$, and

$$\begin{aligned} & - \frac{[y(t^-(\phi_2^n)) - y(t^-(\phi_2^n))] \cdot \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \\ \leq \; & -C \frac{|y(t^-(\phi_2^n)) - y(t^-(\phi_2^n))|}{|t^-(\phi_2^n) - t^-(\phi_1^n)|} |\dot{x}(\phi_1^n)| \cos\alpha_n < 0 \end{aligned} \tag{1.44}$$

We conclude that, as $n \to \infty$ the second term on the right-hand side of (1.42) tends to a negative real number equal to $Cn_\Lambda(t^-(\phi^\infty)) \cdot n_\Omega(\phi^\infty)$.

Finally, we estimate the third term on the right-hand side of (1.42). We have that

$$[y(t^+(\phi_2^n)) - y(t^-(\phi_2^n))] \to [y(t^+(\phi^\infty)) - y(t^-(\phi^\infty))].$$

Let us assume that $x = x(\phi)$ is C^2 in ϕ^∞, then

$$\frac{\dot{x}(\phi_2^n) - \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \to \ddot{x}(\phi^\infty).$$

Denoting with $K_\Omega(\phi)$ the curvature of $\partial\Omega$ at ϕ, we can write

$$\ddot{x}(\phi^\infty) = -|\dot{x}(\phi^\infty)| K_\Omega(\phi^\infty) n_\Omega(\phi^\infty).$$

Hence, as $n \to \infty$,

$$\begin{aligned} & [y(t^+(\phi_2^n)) - y(t^-(\phi_2^n))] \cdot \Big(\frac{\dot{x}(\phi_2^n) - \dot{x}(\phi_1^n)}{\phi_2^n - \phi_1^n} \Big) \\ & \quad \to [y(t^+(\phi^\infty)) - y(t^-(\phi^\infty))] \cdot \ddot{x}(\phi^\infty) \\ & \quad = -|\dot{x}(\phi^\infty)| K_\Omega(\phi^\infty) [y(t^+(\phi^\infty)) - y(t^-(\phi^\infty))] \cdot n_\Omega(\phi^\infty) < 0. \end{aligned} \tag{1.45}$$

Recall that $\Delta^n F(\phi_1^n) = 0$ for every n. On the other hand, combining the estimates (1.43)-(1.44)-(1.45), for n sufficiently large, we have that the right-hand side of (1.42) must be negative. This contradicts the assumption that (1.39) is false, and Theorem 1.8.1 has been proved for case **(2)**.

Remark 1.8.6 *In order to obtain the bi-Lipschitz estimates for case **(3)** it is sufficient to prove that there exists $\hat{\epsilon} > 0$ such that for every $[y_1, y_2]_{\partial\Lambda} \subset T_2$, with $y_1 \neq y_2$, $[y_1, y_2]_{\partial\Lambda} \ni t^+(x_0)$,*

$$\hat{\epsilon} \mathcal{H}\lfloor_{\partial\Lambda}([y_1, y_2]_{\partial\Lambda}) < \mathcal{H}\lfloor_{\partial\Omega}[s^+(y_1), s^+(y_2)]_{\partial\Omega}).$$

Thanks to the symmetry of the problem, one can prove this result following the argument used to prove (1.39).

Chapter 2

Preliminary results for the proof of some Harnack estimates

2.1 Introduction

Consider an open set $E_T \subset \mathbf{R}^N$, $T > 0$, and quasi-linear parabolic differential equations

$$u_t - \mathrm{div} A(x,t,u,Du) = B(x,t,u,Du) \tag{2.1}$$

in $E_T = E \times (0,T]$. The function $A : E_T \times \mathbf{R}^{N+1} \to \mathbf{R}^N, B : E_T \times \mathbf{R}^{N+1} \to \mathbf{R}$ are assumed to be measurable and subject to the structure conditions

$$m > 1 : \quad \begin{cases} A(x,t,u,\eta) \cdot \eta \geq C_0 |u|^{m-1} |\eta|^p - C^p \\ |A(x,t,u,\eta)| \leq C_1 |u|^{m-1} |\eta|^{p-1} + C^{p-1} |u|^{\frac{m-1}{p}} \\ |B(x,t,u,\eta)| \leq C |u|^{m-1} |\eta|^{p-1} + C^p |u|^{\frac{m-1}{p}} \end{cases}, \tag{2.2}$$

$$m < 1 : \quad \begin{cases} A(x,t,u,\eta) \cdot \eta \geq C_0 |u|^{m-1} |\eta|^p - C^p |u|^{m+p-1} \\ |A(x,t,u,\eta)| \leq C_1 |u|^{m-1} |\eta|^{p-1} + C^{p-1} |u|^{m+p-2} \\ |B(x,t,u,\eta)| \leq C |u|^{m-1} |\eta|^{p-1} + C^p |u|^{m+p-2} \end{cases}, \tag{2.3}$$

for almost all $(x,t) \in E_T$, for all $u \in \mathbf{R}$ and $\eta \in \mathbf{R}^N$, with $p + m > 2$, C_0, C_1 positive constants, and C non-negative constant. The prototype of equations (2.1)–(2.2) is

$$u_t - \mathrm{div}(|u|^{m-1} |Du|^{p-2} Du) = 0, \quad m \geq 1, p \geq 2, \tag{2.4}$$

which models the filtration of a polytropic non-Newtonian fluid in a porous medium. Equations of this type are classified as doubly nonlinear and include the standard porous media equation ($p = 2$), and the parabolic p–Laplacian ($m = 1$). From a theoretical point of view, it is interesting to see how much of the regularity properties of solutions to the two model equations is preserved in this more general case.

The aim of the following chapters consists in proving some Harnack estimates for non-negative local weak solutions to (2.1) both in the degenerate ($m+p>3$) and the singular case ($m+p<3$). As for the case $m+p=3$, we limited ourselves to study the stability of the constants as $m+p\to 3$. This chapter is devoted to the introduction of some preliminary results. In Section 2.2 we prove proper energy estimates, which are a direct consequence of the structure conditions (2.2)-(2.3). Sections 2.3-2.4 contain some DeGiorgi-type lemmas. In the last three sections we prove some estimates on certain Sobolev norms of weak solutions to (2.1).

In the following we denote by γ positive constants which depend only on the data, namely N, p, m, C_0, C_1. We will not distinguish these constants by subscripts, but provide that they can be enlarged without invalidating the inequalities considered. We say that the constant γ, depending only on the data $\{N, p, m, C_0, C_1\}$, is "stable" as $m+p\to 3$ if

$$\lim_{m+p\to 3}\gamma(m,p,N,C_0,C_1)$$

is finite.

Finally, throughout the following chapters, u denotes a non-negative local weak solution to (2.1) and, if $k\in\mathbf{R}_+$, We set

$$(u-k)_+=\max\{u-k,0\},\qquad (u-k)_-=\max\{-(u-k),0\}.$$

2.2 Weak solutions and energy estimates

A function $u: E_T\to\mathbf{R}$ is said to be a local weak solution of (2.1) if

$$u\in C(0,T;L^2_{\mathrm{loc}}(E)),\qquad |u|^{\frac{m+p-2}{p-1}}\in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(E)), \tag{2.5}$$

and

$$\int_K u\psi dx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_K[-u\psi_t+A(x,t,u,Du)\cdot D\psi]dxdt$$
$$=\int_{t_1}^{t_2}\int_K B(x,t,u,Du)\psi\,dxdt, \tag{2.6}$$

for every compact set $K\subset E$, for every sub-interval $[t_1,t_2]\subset(0,T]$ and for every test function

$$\psi\in W^{1,2}_{\mathrm{loc}}(0,T;L^2(K))\cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_0(K)).$$

In (2.5) we require integrability hypothesis on u so that the integrals in (2.6) are well defined. We could distinguish the cases $m>1$ and $m<1$ to have the sharp integrability hypothesis on u. For simplicity, we prefer to maintain a univalent definition.

We denote by $K_\rho(y)$ the cube of $\mathbf{R}^N$ centered at y with edge 2ρ. If $y=0$, we simply write K_ρ instead of $K_\rho(0)$. For $\theta>0$, We set

$$Q^-_\rho(\theta)=K_\rho\times(-\theta\rho^p,0],$$
$$Q^+_\rho(\theta)=K_\rho\times(0,\theta\rho^p].$$

Definition 2.2.1 *The partial differential equation* (2.1) *is* parabolic *if it satisfies the structure conditions* (2.2)-(2.3) *and, in addition, for every weak, local sub(super)-solution u, the truncations $+(u-k)_+$, $(u-k)_-$, for all $k \in \mathbf{R}$, are weak, local sub(super)-solutions of* (2.1)*, with $A(x,t,u,Du)$ and $B(x,t,u,Du)$ replaced respectively by*

$$\begin{aligned} &A(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm),\\ &B(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm). \end{aligned}$$

Proposition 2.2.2 *If $m>1$, there exist two positive constants ϖ, γ, depending only on N, p, C_0, C_1, such that for every cylinder $(y,s)+Q_\rho^-(\theta) \subset E_T$, $k \in \mathbf{R}_+$ and every piecewise smooth cutoff function ζ vanishing on the boundary of $K_\rho(y)$, with $\zeta_t \geq 0$, it holds*

$$\begin{aligned} \sup_{s-\theta\rho^p<t\leq s} &\int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x,t)dx - \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x,s-\theta\rho^p)dx \\ &\qquad + \varpi \iint_{(y,s)+Q_\rho^-(\theta)} u^{m-1}|D(u-k)_\pm|^p \zeta^p dxdt \\ \leq \gamma &\iint_{(y,s)+Q_\rho^-(\theta)} (u-k)_\pm^2 \zeta^{p-1}\zeta_t dxdt \\ + \gamma &\iint_{(y,s)+Q_\rho^-(\theta)} u^{m-1}(u-k)_\pm^p |D\zeta|^p dxdt \\ + \gamma &\iint_{(y,s)+Q_\rho^-(\theta)} \Big(C^p u^{m-1}(u-k)_\pm^p + C^p \chi_{\{(u-k)_\pm>0\}}\Big) \zeta^p dxdt. \end{aligned} \tag{2.7}$$

Analogous estimates hold in the cylinder $(y,s)+Q_\rho^+(\theta) \subset E_T$. The constants ϖ and γ are stable as $m+p \to 3$.

Proof We prove (2.7) for $(u-k)_-$. We proceed formally, multiplying both sides of (2.1) by $-(u-k)_-\zeta^p$ and integrating on $K_\rho(y) \times (s-\theta\rho^p, \tau]$, where $s-\theta\rho^p < \tau \leq s$. As in general u_t does not make sense for a weak solution, to give a rigorous proof of (2.7) we need to introduce the Steklov averages of u. We refer the reader to Proposition 3.1 of Chapter II in [16] for details. We obtain

$$\begin{aligned} -&\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} u_t (u-k)_- \zeta^p dxdt \\ &= -\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} A(x,t,u,Du)\cdot Du\, \chi_{\{(u-k)_->0\}} \zeta^p dxdt \\ &\quad + p \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} A(x,t,u,Du)\cdot D\zeta\, (u-k)_-\, \zeta^{p-1} dxdt \\ &\quad - \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} B(x,t,u,Du)\zeta^p\, (u-k)_- dxdt. \end{aligned}$$

Concerning the left-hand side, we have

$$
\begin{aligned}
-\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} & u_t(u-k)_-\zeta^p dxdt \\
&= \frac{1}{2}\int_{K_\rho(y)}\int_{s-\theta\rho^p}^{\tau}[(u-k)_-^2]_t\zeta^p dtdx \\
&\geq \frac{1}{2}\int_{K_\rho(y)}(u-k)_-^2\zeta^p(x,\tau)dx \\
&-\frac{1}{2}\int_{K_\rho(y)}(u-k)_-^2\zeta^p(x,s-\theta\rho^p)dx \\
&-\frac{p}{2}\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_-^2\zeta^{p-1}\zeta_t\,dxdt.
\end{aligned}
$$

On the other hand, from the first condition in (2.2) it follows that

$$
\begin{aligned}
-\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} & A(x,t,u,Du)\cdot Du\,\chi_{\{(u-k)_->0\}}\zeta^p dxdt \\
&\leqslant -C_0\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{m-1}|D(u-k)_-|^p\zeta^p dxdt \\
&+C^p\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}\zeta^p\,\chi_{\{(u-k)_->0\}}dxdt
\end{aligned}
$$

and from the second condition in (2.2) and Young inequality it follows that

$$
\begin{aligned}
\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} & |A(x,t,u,Du)|\,|D\zeta|\,(u-k)_-\,\zeta^{p-1} \\
&\leqslant C_1\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{m-1}|D(u-k)_-|^{p-1}\zeta^{p-1}|D\zeta|(u-k)_- \\
&+C^{p-1}\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{\frac{m-1}{p}}\zeta^{p-1}|D\zeta|(u-k)_- \\
&\leqslant \varepsilon\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{m-1}|D(u-k)_-|^p\zeta^p \\
&+C_1^pC_\varepsilon\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{m-1}|D\zeta|^p(u-k)_-^p \\
&+\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}u^{m-1}|D\zeta|^p(u-k)_-^p \\
&+C^p\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]}\zeta^p\chi_{\{(u-k)_->0\}}.
\end{aligned}
$$

Finally, the third condition of (2.2) implies

$$\begin{aligned}\iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} & |B(x,t,u,Du)|\,(u-k)_-\,\zeta^p dxdt \\ & \leqslant C^p C_\varepsilon \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} u^{m-1}(u-k)_-^p\zeta^p dxdt \\ & \quad + \varepsilon \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} u^{m-1}|D(u-k)_-|^p\zeta^p dxdt \\ & \quad + C^p \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} u^{m-1}(u-k)_-^p\zeta^p\, dxdt \\ & \quad + C^p \iint_{K_\rho(y)\times(s-\theta\rho^p,\tau]} \zeta^p\chi_{\{(u-k)_->0\}} dxdt.\end{aligned}$$

Combining all the estimates so far, choosing ε small enough, and then taking the supremum over τ we obtain (2.7). By the same argument, we deduce estimate (2.7) with $(u-k)_+$ instead of $(u-k)_-$.
□

Remark 2.2.3 By a simple computation, it is possible to rewrite estimate (2.7) with a slight change in the third integral on the left-hand side, namely

$$\begin{aligned}\sup_{s-\theta\rho^p<t\leq s} \int_{K_\rho(y)} (u-k)_\pm^2\zeta^p(x,t)dx & - \int_{K_\rho(y)} (u-k)_\pm^2\zeta^p(x,s-\theta\rho^p)dx \\ & + \varpi \iint_{(y,s)+Q_\rho^-(\theta)} u^{m-1}|D[(u-k)_\pm\zeta]|^p dxdt \\ \leq \gamma \iint_{(y,s)+Q_\rho^-(\theta)} & (u-k)_\pm^2\zeta^{p-1}\zeta_t dxdt \\ + \gamma \iint_{(y,s)+Q_\rho^-(\theta)} & u^{m-1}(u-k)_\pm^p|D\zeta|^p dxdt \\ + \gamma \iint_{(y,s)+Q_\rho^-(\theta)} & \Big(C^p u^{m-1}(u-k)_\pm^p + C^p\chi_{\{(u-k)_\pm>0\}}\Big)\zeta^p dxdt.\end{aligned} \tag{2.8}$$

possibly for different values of the constants $\bar{\gamma}$, ϖ.

Proposition 2.2.4 *If $m<1$, there exist two positive constants ϖ,γ, depending only on N,p,m,C_0,C_1, such that for every cylinder $(y,s)+Q_\rho^-(\theta)\subset E_T$, $k\in\mathbf{R}_+$ and every piecewise smooth cutoff function*

ζ vanishing on the boundary of $K_\rho(y)$, with $\zeta_t \geq 0$, it holds

$$\begin{aligned}
&\sup_{s-\theta\rho^p<t\leq s}\int_{K_\rho(y)}(u-k)_-^2\zeta^p(x,t)dx-\frac{k}{l}\int_{K_\rho(y)}(u-k)_-\zeta^p(x,s-\theta\rho^p)dx\\
&\quad+\varpi\, k^{m-1}\iint_{(y,s)+Q_\rho^-(\theta)}|D[(u-k)_-\zeta]|^p dxdt\\
&\quad\leq\gamma\bigg(k^2\iint_{(y,s)+Q_\rho^-(\theta)}\chi_{[u<k]}\zeta\zeta_t dxdt\\
&\quad+k^{m+p-1}\iint_{(y,s)+Q_\rho^-(\theta)}\chi_{[u<k]}(C^p\zeta^p+\zeta^p+|D\zeta|^p)dxdt\bigg),
\end{aligned}\tag{2.9}$$

where

$$l=\frac{m+p-2}{p-1}.$$

Analogous estimates hold in the cylinder $(y,s)+Q_\rho^+(\theta)\subset E_T$. The constants ϖ and γ are stable as $m+p\to 3$.

Proof Suppose $(y,s)=0$, and fix $k>0$. In the weak formulation (2.6) take the test function

$$\varphi=-(u^l-k^l)_-\zeta^p$$

over the cylinder $Q_t=K_\rho\times(-\theta\rho^p,t]$, for $-\theta\rho^p<t\leq 0$. Since $l\in(0,1)$, we have

$$\begin{aligned}
\textstyle\int_u^k(k^l-s^l)_+ds &\geq \tfrac{l}{2}k^{l-1}(u-k)_-^2,\\
\textstyle\int_u^k(k^l-s^l)_+ds &\leq k^l(u-k)_-,
\end{aligned}$$

and we estimate

$$\begin{aligned}
- &\iint_{Q_t}(u^l-k^l)_-\zeta^p u_\tau dxd\tau\\
= &\int_{K_\rho}\int_u^k(k^l-s^l)_+ds\zeta^p(x,t)dx-\int_{K_\rho}\int_u^k(k^l-s^l)_+ds\zeta^p(x,-\theta\rho^p)dx\\
- &\ p\iint_{Q_t}\int_u^k(k^l-s^l)_+ds\zeta^{p-1}\zeta_\tau dxd\tau\\
\geq &\ \frac{l}{2}k^{l-1}\int_{K_\rho}(u-k)_-^2\zeta^p(x,t)dx-k^l\int_{K_\rho}(u-k)_-ds\zeta^p(x,-\theta\rho^p)dx\\
- &\ 2k^l\iint_{Q_t}(u-k)_-\zeta^{p-1}\zeta_\tau dxd\tau.
\end{aligned}$$

Applying the structure conditions (2.3) and Young's inequality, we estimate the term containing the function A

$$
\begin{aligned}
&\iint_{Q_t} A(x,\tau,u,Du)\cdot D(-(u^l-k^l)_-\zeta^p)dxd\tau \\
&\quad \geq lC_0 \iint_{Q_t} u^{m+l-2}\chi_{[u<k]}\zeta^p|Du|^p dxd\tau \\
&\quad -lC^p \iint_{Q_t} u^{m+p+l-2}\chi_{[u<k]}\zeta^p dxd\tau \\
&\quad -pC_1 \iint_{Q_t} u^{m-1}(u^l-k^l)_-\zeta^{p-1}|D\zeta||Du|^{p-1}dxd\tau \\
&\quad -pC^{p-1} \iint_{Q_t} u^{m+p-2}(u^l-k^l)_-\zeta^{p-1}|D\zeta|dxd\tau \\
&\quad \geq \frac{3}{4}C_0 l^{p-1} \iint_{Q_t} \chi_{[u<k]}\zeta^p|Du^l|^p dxd\tau \\
&\quad -k^{lp} \iint_{Q_t} \chi_{[u<k]}(C^p\zeta^p+|D\zeta|^p)dxd\tau.
\end{aligned}
$$

Applying the structure conditions (2.3) and Young's inequality, we estimate also the term containing the function B

$$
\begin{aligned}
&\iint_{Q_t} |B(x,\tau,u,Du)|(u^l-k^l)_-\zeta^p dxd\tau \\
&\quad \leq C \iint_{Q_t} u^{m-1}|Du|^{p-1}(u^l-k^l)_-\zeta^p dxd\tau \\
&\quad +C^p \iint_{Q_t} u^{m+p-2}(u^l-k^l)_-\zeta^p dxd\tau \\
&\quad \leq \frac{C_0}{2} l^{p-1} \iint_{Q_t} \chi_{[u<k]}\zeta^p|Du^l|^p dxd\tau + \gamma C^p k^{lp} \iint_{Q_t} \chi_{[u<k]}\zeta^p dxd\tau.
\end{aligned}
$$

Notice that, since $l \in (0,1)$,

$$
\iint_{Q_\rho^-(\theta)} |D(u^l-k^l)_-|\zeta^p dxd\tau \geq l^p k^{l-1} \iint_{Q_\rho^-(\theta)} |D(u-k)_-|^p\zeta^p dxd\tau;
$$

moreover

$$
|D[(u-k)_-\zeta]|^p \leq p[|D(u-k)_-|^p\zeta^p + (u-k)_-^p|D\zeta|^p].
$$

Combining all the previous estimates, and dividing everything by k^{l-1}, we obtain the thesis of the proposition. $\square$

Proposition 2.2.5 *If $m<1$, there exist two positive constants ϖ, γ, depending only on N, p, m, C_0, C_1, such that for every cylinder $(y,s)+Q_\rho^-(\theta) \subset E_T$, $k \in \mathbf{R}_+$ and every piecewise smooth cutoff function*

ζ vanishing on the boundary of $K_\rho(y)$, with $\zeta_t \geq 0$, it holds

$$\sup_{s-\theta\rho^p<t\leq s}\int_{K_\rho(y)} u^{l-1}(u-k)_+^2\zeta^p(x,t)dx$$
$$-\frac{k}{l}\int_{K_\rho(y)}(u-k)_+\zeta^p(x,s-\theta\rho^p)dx$$
$$+\varpi\, k^{m-1}\iint_{(y,s)+Q_\rho^-(\theta)} u^{(l-1)p}|D[(u-k)_+\zeta]|^p dxdt$$
$$\leq \gamma\int_{K_\rho} u^{l+1}\chi_{[u>k]}\zeta^p(x,-\theta\rho^p)dx$$
$$+\gamma\iint_{(y,s)+Q_\rho^-(\theta)} u^{l+1}\chi_{[u>k]}\zeta^{p-1}\zeta_\tau dxd\tau$$
$$+\gamma\iint_{(y,s)+Q_\rho^-(\theta)} u^{lp}(C^p\zeta^p+|D\zeta|^p)\chi_{[u>k]}dxd\tau$$
$$+\gamma\iint_{(y,s)+Q_\rho^-(\theta)}(u^l-k^l)_+^p(C^p\zeta^p+|D\zeta|^p)dxd\tau, \tag{2.10}$$

where

$$l=\frac{m+p-2}{p-1}.$$

Analogous estimates hold in the cylinder $(y,s)+Q_\rho^+(\theta)\subset E_T$. The constants ϖ and γ are stable as $m+p\to 3$.

Proof Suppose $(y,s)=0$, and fix $k>0$. In the weak formulation (2.6) take the test function

$$\varphi=(u^l-k^l)_+\zeta^p$$

over the cylinder $Q_t=K_\rho\times(-\theta\rho^p,t]$, for $-\theta\rho^p<t\leq 0$. We proceed as in the Proof of Proposition 2.2.4. By means of

$$\int_k^u(s^l-k^l)_+ds\geq \tfrac{l}{2}u^{l-1}(u-k)_+^2,$$
$$\int_k^u(s^l-k^l)_+ds\leq u^{l+1}\chi_{[u>k]},$$

the term containing u_τ can be estimated as

$$\iint_{Q_t}(u^l-k^l)_+\zeta^p u_\tau dxd\tau$$
$$=\int_{K_\rho}\int_k^u(s^l-k^l)_+ds\zeta^p(x,t)dx$$
$$-\int_{K_\rho}\int_k^u(s^l-k^l)_+ds\zeta^p(x,-\theta\rho^p)dx$$
$$-p\iint_{Q_t}\int_k^u(s^l-k^l)_+ds\zeta^{p-1}\zeta_\tau dxd\tau$$
$$\geq\frac{l}{2}\int_{K_\rho}u^{l-1}(u-k)_+^2\zeta^p(x,t)dx-\int_{K_\rho}u^{l+1}\chi_{[u>k]}\zeta^p(x,-\theta\rho^p)dx$$
$$-p\iint_{Q_t}\int_k^u(s^l-k^l)_+ds\zeta^{p-1}\zeta_\tau dxd\tau.$$

We split the term containing the function A into

$$\begin{aligned}&\iint_{Q_t} A(x,\tau,u,Du)\cdot D[(u^l-k^l)_+\zeta^p]dxd\tau\\&\quad=\iint_{Q_t} A(x,\tau,u,Du)\cdot D(u^l-k^l)_+\zeta^p dxd\tau\\&\quad=\iint_{Q_t} A(x,\tau,u,Du)\cdot D(\zeta^p)(u^l-k^l)_+ dxd\tau,\end{aligned}$$

and we estimate the two terms on the right-hand side using the structure conditions (2.3) and the Young inequality. We obtain

$$\begin{aligned}&\iint_{Q_t} A(x,\tau,u,Du)\cdot D(u^l-k^l)_+\zeta^p dxd\tau\\&\quad\geq l^{1-p}C_0\iint_{Q_t}|D(u^l-k^l)_+|^p\zeta^p dxd\tau - lC^p\iint_{Q_t} u^{pl}\chi_{[u>k]}\zeta^p dxd\tau,\end{aligned}$$

and

$$\begin{aligned}&\iint_{Q_t} A(x,\tau,u,Du)\cdot D(\zeta^p)(u^l-k^l)_+ dxd\tau\\&\quad\leq l^{1-p}\frac{C_0}{4}\iint_{Q_t}|D(u^l-k^l)_+|^p\zeta^p dxd\tau\\&\quad+\gamma\left(C^p\iint_{Q_t} u^{lp}\chi_{[u>k]}\zeta^p dxd\tau+\iint_{Q_t}(u^l-k^l)_+^p|D\zeta|^p dxd\tau\right).\end{aligned}$$

Applying the structure conditions (2.3) and the Young inequality we estimate also the term containing the function B

$$\begin{aligned}&\iint_{Q_t} B(x,\tau,u,Du)(u^l-k^l)_+\zeta^p dxd\tau\\&\quad\leq l^{1-p}\frac{C_0}{4}\iint_{Q_t}|D(u^l-k^l)_+|^p\zeta^p dxd\tau\\&\quad+\gamma C^p\left(\iint_{Q_t}(u^l-k^l)_+^p\zeta^p dxd\tau+\iint_{Q_t} u^{lp}\chi_{[u>k]}\zeta^p dxd\tau\right).\end{aligned}$$

Combining the previous estimates we obtain the thesis. □

2.3 A DeGiorgi-type lemma

Denote

$$\mu_+\geq \operatorname*{ess\,sup}_{[(y,s)+Q^-_{2\rho}(\theta)]} u,\quad \mu_-\leq \operatorname*{ess\,inf}_{([y,s)+Q^-_{2\rho}(\theta)]} u,\quad \omega=\mu_+-\mu_-.$$

Since the singularity occurs at $u=0$, we will assume at the outset that $\mu_-=0$ so that $\omega=\mu_+$.

Lemma 2.3.1 (DeGiorgi-type lemma) *Let u be a non-negative, locally bounded, local, weak solution to equation (2.1) in E_T. Let $\xi, a \in (0,1)$. Then the following two assertions hold.*

(i) There exists a positive number ν_-, depending upon θ, ω, ξ, a, and the data $\{m, p, N, C_0, C_1\}$, such that if

$$|[u \le \xi\omega] \cap [(y,s) + Q_{2\rho}^-(\theta)]| \le \nu_- |Q_{2\rho}^-(\theta)|$$

then either

$$(C\rho)^p > \min\{1, (\xi\omega)^{m+p-1}\}$$

or

$$u \ge a\xi\omega \qquad a.e.\ in\ (y,s) + Q_\rho^-(\theta).$$

(ii) There exists a positive number ν_+, depending upon $\mu^+, \omega, \theta, \xi, a$, and the data $\{m, p, N, C_0, C_1\}$, such that if

$$|[u \ge \mu^+ - \xi\omega] \cap [(y,s) + Q_{2\rho}^-(\theta)]| \le \nu_+ |Q_{2\rho}^-(\theta)|$$

then either

$$(C\rho)^p > \min\{1, \mu_+^{m-1}(\xi\omega)^p\}$$

or

$$u \le \mu_+ - a\xi\omega \qquad a.e.\ in\ (y,s) + Q_\rho^-(\theta).$$

The numbers ν_- and ν_+ are stable as $m + p \to 3$.

***Proof of Lemma 2.3.1 for* $\mathbf{m > 1, p \ge 2}$.** From now on we assume $C\rho \le 1$. We limit ourselves to proving (i) in the case when $(y,s) = (0,0)$. This is always possible by using a translation. To keep u away from 0, we define

$$v = \max\{u, a\xi\omega\}.$$

We set

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad K_n = K_{\rho_n}, \qquad Q_n = K_n \times (-\theta\rho_n^p, 0], \tag{2.11}$$

$$k_n = \xi_n\omega, \quad \text{where} \quad \xi_n = a\xi + \frac{1-a}{2^n}\xi, \tag{2.12}$$

for $n = 0, 1, 2, \ldots$ and we choose $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ as a cutoff function on Q_n such that

$$\zeta_1 = \begin{cases} 1 & \text{in} \quad K_{n+1} \\ 0 & \text{in} \quad \mathbf{R}^N \setminus K_n \end{cases} \qquad |D\zeta_1| \le \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho}, \tag{2.13}$$

and

$$\zeta_2 = \begin{cases} 0 & \text{if} \quad t \le -\theta\rho_n^p \\ 1 & \text{if} \quad t \ge -\theta\rho_{n+1}^p \end{cases} \qquad 0 \le (\zeta_2)_t \le \frac{2^{p(n+1)}}{\theta\rho^p}. \tag{2.14}$$

We also set

$$A_n = [u < k_n] \cap Q_n \qquad \text{and} \qquad Y_n = \frac{|A_n|}{|Q_n|}. \tag{2.15}$$

We apply the energy estimates (2.8) on Q_n, for $(u-k_n)_-$ and ζ defined as above, getting

$$\begin{aligned}\sup_{-\theta\rho_n^p<t\leq 0}\int_{K_n}(u-k_n)_-^2\zeta^p(x,t)dx+\varpi\iint_{Q_n}u^{m-1}|D[(u-k_n)_-\zeta]|^p dxd\tau\\ \leq\bar\gamma\iint_{Q_n}(u-k_n)_-^2\zeta^{p-1}\zeta_t dxd\tau+\bar\gamma\iint_{Q_n}u^{m-1}(u-k_n)_-^p|D\zeta|^p dxd\tau\\ +\bar\gamma\iint_{Q_n}\Big(C^p u^{m-1}(u-k_n)_-^p+C^p\chi_{\{(u-k_n)_->0\}}\Big)\zeta^p dxd\tau.\end{aligned}\tag{2.16}$$

At this point, we need to estimate the left-hand side from below and the right-hand side from above. As $\xi\omega\geq(u-k_n)_-\geq(v-k_n)_-$ and $p\geqslant 2$, we easily have

$$\begin{aligned}\int_{K_n}(u-k_n)_-^2\zeta^p(x,t)dx &\geq \int_{K_n}(v-k_n)_-^2\zeta^p(x,t)dx\\ &\geq (\xi\omega)^{2-p}\int_{K_n}(v-k_n)_-^p\zeta^p(x,t)dx.\end{aligned}$$

Moreover,

$$\begin{aligned}\gamma_1(a\xi\omega)^{m-1}\iint_{Q_n}|D[(v-k_n)_-\zeta]|^p dxd\tau &\leq\iint_{Q_n}v^{m-1}|D[(v-k_n)_-\zeta]|^p dxd\tau\\ &=\iint_{Q_n\cap\{u=v\}}u^{m-1}|D[(u-k_n)_-\zeta]|^p dxd\tau\\ &\quad+\iint_{Q_n\cap\{u<v\}}(a\xi\omega)^{m-1}(v-k_n)_-^p|D\zeta|^p dxd\tau\\ &\leq\iint_{Q_n}u^{m-1}|D[(u-k_n)_-\zeta]|^p dxd\tau\\ &\quad+\iint_{Q_n}\gamma_2(a\xi\omega)^{m-1}(u-k_n)_-^p|D\zeta|^p dxd\tau.\end{aligned}$$

Using (2.13), (2.14) and noticing that $u\leq\xi\omega$, when $(u-k_n)_->0$, by (2.16) and the previous estimates we get

$$\begin{aligned}&\sup_{-\theta\rho_n^p<t\leq 0}(\xi\omega)^{2-p}\int_{K_n}(v-k_n)_-^p\zeta^p(x,t)dx\\ &\qquad+\varpi\gamma_1(a\xi\omega)^{m-1}\iint_{Q_n}|D[(v-k_n)_-\zeta]|^p dxd\tau\\ &\leq\gamma\frac{2^{p(n+1)}}{\rho^p}(\xi\omega)^p|A_n|\left(\frac{1}{\theta(\xi\omega)^{p-2}}+(\xi\omega)^{m-1}+(C\rho)^p(\xi\omega)^{m-1}+\frac{(C\rho)^p}{(\xi\omega)^p}\right).\end{aligned}$$

Note that, by the definition of v, there holds $A_n=\{v<k_n\}\cap Q_n$, for every n. Assuming

$$(C\rho)^p\leqslant(\xi\omega)^{p+m-1},$$

and recalling that $C\rho \leqslant 1$, we can estimate

$$\sup_{-\theta\rho_n^p<t\leq 0}(\xi\omega)^{2-p}\int_{K_n}(v-k_n)_-^p\zeta^p(x,t)dx$$
$$+\varpi\gamma_1(a\xi\omega)^{m-1}\iint_{Q_n}|D[(v-k_n)_-\zeta]|^p dxd\tau$$
$$\leq \gamma\frac{2^{p(n+1)}}{\rho^p}(\xi\omega)^p|A_n|\left(\frac{1}{\theta(\xi\omega)^{p-2}}+(\xi\omega)^{m-1}\right). \tag{2.17}$$

Applying Hölder inequality and recalling that $\zeta = 1$ on Q_{n+1}, it turns out that

$$\left(\frac{1-a}{2^{n+1}}\right)^p(\xi\omega)^p|A_{n+1}| \leq \iint_{Q_{n+1}}(v-k_n)_-^p dxd\tau$$
$$\leq \left(\iint_{Q_n}[(v-k_n)_-\zeta]^{p\frac{N+p}{N}}dxd\tau\right)^{\frac{N}{N+p}}|A_n|^{\frac{p}{N+p}}. \tag{2.18}$$

From Proposition B.3.1, it follows that the right-hand side of (2.18) can be estimated by

$$\gamma\left(\iint_{Q_n}|D[(v-k_n)_-\zeta]|^p dxd\tau\right)^{\frac{N}{N+p}}$$
$$\times\left(\sup_{-\theta\rho_n^p<t\leq 0}\int_{K_n}|(v-k_n)_-\zeta|^p(x,t)dx\right)^{\frac{p}{N+p}}|A_n|^{\frac{p}{N+p}}, \tag{2.19}$$

where γ depends only upon N, p. Now, combining estimates (2.18), (2.19) and (2.17) we find

$$|A_{n+1}| \leqslant \gamma\frac{2^{2np}}{(1-a)^p\rho^p}a^{\frac{-m+1N}{N+p}}(\xi\omega)^{\frac{(1-m)N+(p-2)p}{N+p}}\left(\frac{1}{\theta(\xi\omega)^{p-2}}+(\xi\omega)^{m-1}\right)$$
$$\times|A_n|^{1+\frac{p}{N+p}}.$$

Recalling that $Y_n = \frac{|A_n|}{|Q_n|}$, the last inequality can be rewritten as

$$Y_{n+1} \quad\leq\quad \gamma\frac{2^{2np}}{a^{\frac{(m-1)N}{N+p}}(1-a)^p}\frac{\left(1+\theta(\xi\omega)^{p+m-3}\right)}{\left(\theta(\xi\omega)^{p+m-3}\right)^{\frac{N}{N+p}}}Y_n^{1+\frac{p}{N+p}}.$$

If $|A_0| \leq \nu_-|Q_0|$, where

$$\nu_- = \gamma a^{\frac{(m-1)N}{p}}(1-a)^{N+p}\frac{\left(\theta(\xi\omega)^{p+m-3}\right)^{\frac{N}{p}}}{\left(1+\theta(\xi\omega)^{p+m-3}\right)^{\frac{N+p}{p}}}, \tag{2.20}$$

then Lemma B.4.1 implies that $Y_n \to 0$. This means that

$$u \geq a\xi\omega \quad \text{in } Q_\rho^-(\theta),$$

which is the thesis. □

Proof of Lemma 2.3.1 for $\mathbf{m > 1, p < 2}$. Maintaining the definitions (2.11)–(2.15), for $p < 2$ we have $\zeta^p \ge \zeta^2$. Applying the energy estimates (2.8) we get

$$\begin{aligned} &\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} (v-k_n)_-^2 \zeta^2(x,t)dx \\ &\qquad + \varpi\gamma_1(a\xi\omega)^{m-1} \iint_{Q_n} |D[(v-k_n)_-\zeta]|^p dxd\tau \\ &\qquad \le \gamma \frac{2^{p(n+1)}}{\rho^p}(\xi\omega)^p|A_n| \left(\frac{1}{\theta(\xi\omega)^{p-2}} + (\xi\omega)^{m-1}\right). \end{aligned}$$

Applying Hölder inequality and recalling that $\zeta = 1$ on Q_{n+1}, it turns out that

$$\begin{aligned} &\left(\frac{1-a}{2^{n+1}}\right)^p (\xi\omega)^p |A_{n+1}| \le \iint_{Q_{n+1}} (v-k_n)_-^p dxd\tau \\ &\le \left(\iint_{Q_n} [(v-k_n)_-\zeta]^{p\frac{N+2}{N}} dxd\tau\right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}}. \end{aligned} \tag{2.21}$$

From Proposition B.3.1, it follows that the right-hand side of (2.21) can be estimated by

$$\begin{aligned} &\gamma\left(\iint_{Q_n} |D[(v-k_n)_-\zeta]|^p dxd\tau\right)^{\frac{N}{N+2}} \\ &\times\left(\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} |(v-k_n)_-\zeta|^2(x,t)dx\right)^{\frac{p}{N+2}} |A_n|^{\frac{2}{N+2}}, \end{aligned} \tag{2.22}$$

where γ depends only upon N, p. Combining the estimates (2.21)–(2.22) we get

$$|A_{n+1}| \le \gamma \frac{2^{np(1+\frac{N+p}{N+2})}}{(1-a)^p} \frac{a^{(1-m)\frac{N}{N+2}}}{(\rho^p\theta)^{\frac{N+p}{N+2}}} \frac{\left(1+\theta(\xi\omega)^{p+m-3}\right)^{\frac{N+p}{N+2}}}{(\xi\omega)^{\frac{N(p+m-3)}{N+2}}} |A_n|^{1+\frac{p}{N+2}}$$

Recalling that $Y_n = \frac{|A_n|}{|Q_n|}$, the last inequality can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{np(1+\frac{N+p}{N+2})}}{(1-a)^p a^{(m-1)\frac{N}{N+2}}} \frac{\left(1+\theta(\xi\omega)^{p+m-3}\right)^{\frac{N+p}{N+2}}}{\left(\theta(\xi\omega)^{p+m-3}\right)^{\frac{N}{N+2}}} |Y_n|^{1+\frac{p}{N+2}}$$

If $|A_0| \le \nu_-|Q_0|$, where

$$\nu_- = \gamma a^{\frac{(m-1)N}{p}} (1-a)^{N+2} \frac{\left(\theta(\xi\omega)^{p+m-3}\right)^{\frac{N}{p}}}{\left(1+\theta(\xi\omega)^{p+m-3}\right)^{\frac{N+p}{p}}}, \tag{2.23}$$

then Lemma B.4.1 implies that $Y_n \to 0$. This means that

$$u \ge a\xi\omega \quad \text{in } Q_\rho^-(\theta),$$

which is the thesis. □

Remark 2.3.2 *Summarizing, when $m>1$, the number ν_- of Lemma 2.3.1 is given by*

$$\nu_- = \gamma a^{\frac{(m-1)N}{p}}(1-a)^{N+\max\{2,p\}} \frac{(\theta(\xi\omega)^{m+p-3})^{\frac{N}{p}}}{(1+\theta(\xi\omega)^{m+p-3})^{\frac{N+p}{p}}}.$$

***Proof of Lemma 2.3.1 for* m < 1**. We limit ourselves to proving (i) in the case when $(y,s)=(0,0)$. This is always possible modulo a translation. We maintain the definitions (2.11)–(2.15). We write the energy estimates (2.9) on Q_n, for $(u-k_n)_-$. The second term on the left-hand side vanishes because of the choice of the cutoff function ζ. The first term on the right-hand side is majorized by

$$\gamma \frac{2^{pn}}{\theta\rho^p} k_n^2 |A_n|.$$

Either $C\rho>1$ or, taking into account that $k_n \le \xi\omega$, the energy estimates give

$$\begin{aligned}
&\sup_{s-\theta\rho^p<t\le s} \int_{K_\rho(y)} (u-k)_-^2 \zeta^p(x,t)dx \\
&\quad + \varpi\, k^{m-1} \iint_{(y,s)+Q_\rho^-(\theta)} |D(u-k)_-|^p \zeta^p dxdt \\
&\quad \le \gamma \frac{2^{pn}}{\rho^p}(\xi\omega)^{p+m-1} \frac{(1+\theta k_n^{m+p-3})}{\theta k_n^{m+p-3}} |A_n|. \qquad (2.24)
\end{aligned}$$

Let us **first suppose p ≥ 2**, then

$$(u-k_n)_-^2\zeta^p \ge k_n^{2-p}(u-k_n)_-^p\zeta^p \ge (\xi\omega)^{2-p}(u-k_n)_-^p\zeta^p,$$

and the first term in the previous inequality can be estimated from below

$$\begin{aligned}
&\sup_{s-\theta\rho^p<t\le s} \int_{K_\rho(y)} (u-k)_-^2 \zeta^p(x,t)dx \\
&\ge (\xi\omega)^{2-p} \sup_{s-\theta\rho^p<t\le s} \int_{K_\rho(y)} (u-k)_-^p \zeta^p(x,t)dx
\end{aligned}$$

Applying Hölder inequality and Proposition B.3.1, we have

$$\begin{aligned}
\left(\frac{1-a}{2^{n+1}}\right)^p (\xi\omega)^p |A_{n+1}| &\le \left(\iint_{Q_n} [(u-k_n)_-\zeta]^{p\frac{N+p}{N}} dxd\tau\right)^{\frac{N}{N+p}} |A_n|^{\frac{p}{N+p}} \\
&\le \gamma \left(\iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dxd\tau\right)^{\frac{N}{N+p}} \\
&\quad \times \left(\operatorname*{ess\,sup}_{-\theta\rho_n^p\le t\le 0} \int_{K_n} (u-k_n)_-^p\zeta^p dx\right)^{\frac{p}{N+p}} |A_n|^{\frac{p}{N+p}}. \qquad (2.25)
\end{aligned}$$

Combining (2.24) and (2.25) we deduce

$$|A_{n+1}| \le \gamma \frac{2^{np}}{(1-a)^p\rho^p} (\xi\omega)^{(m+p-3)\frac{p}{N+p}} \left(1+\frac{1}{\theta(\xi\omega)^{m+p-3}}\right) |A_n|^{1+\frac{p}{N+p}};$$

recalling that $Y_n = \frac{|A_n|}{|Q_n|}$, the previous inequality becomes

$$Y_{n+1} \leq \frac{2^{np}}{(1-a)^p} (\theta(\xi\omega)^{m+p-3})^{\frac{p}{N+p}} \left(\frac{1+\theta(\xi\omega)^{m+p-3}}{\theta(\xi\omega)^{m+p-3}} \right) |A_n|^{1+\frac{p}{N+p}}.$$

By Lemma B.4.1, $Y_n \to 0$ as $n \to \infty$, provided

$$Y_0 = \frac{A_0}{Q_0} \leq \gamma(1-a)^{N+p} \frac{(\theta(\xi\omega)^{m+p-3})^{\frac{N}{p}}}{(1+\theta(\xi\omega)^{m+p-3})^{\frac{N+p}{p}}} =: \nu_-. \tag{2.26}$$

If $\mathbf{p < 2}$, the energy estimates (2.9) give

$$\begin{aligned} &\sup_{s-\theta\rho^p < t \leq s} \int_{K_\rho(y)} (u-k)_-^2 \zeta^2(x,t) dx \\ &\quad + \varpi\, k^{m-1} \iint_{(y,s)+Q_\rho^-(\theta)} |D(u-k)_-|^p \zeta^p dx dt \\ &\quad \leq \gamma \frac{2^{pn}}{\rho^p} (\xi\omega)^{p+m-1} \frac{(1+\theta k_n^{m+p-3})}{\theta k_n^{m+p-3}} |A_n|. \end{aligned} \tag{2.27}$$

Applying Hölder inequality, Proposition B.3.1, and (2.27), We obtain

$$Y^{n+1} \leq \gamma \frac{2^{2n}}{(1-a)^p} \frac{(1+\theta(\xi\omega)^{m+p-3})}{(\theta(\xi\omega)^{m+p-3})^{\frac{N}{N+2}}} |Y_n|^{1+\frac{p}{N+2}}.$$

Lemma B.4.1 leads to the thesis with

$$\nu_- := \gamma(1-a)^{N+2} \frac{(\theta(\xi\omega)^{m+p-3})^{\frac{N}{p}}}{(1+\theta(\xi\omega)^{m+p-3})^{\frac{N+2}{p}}}. \qquad \square$$

Remark 2.3.3 *Summarizing, when $m < 1$, the number ν_- of Lemma 2.3.1 is given by*

$$\nu_- = \gamma(1-a)^{N+\max\{2,p\}} \frac{(\theta(\xi\omega)^{m+p-3})^{\frac{N}{p}}}{(1+\theta(\xi\omega)^{m+p-3})^{\frac{N+\max\{N,p\}}{p}}}.$$

Remark 2.3.4 (Proof of (ii)) *The second part of the statement can be proved more easily, since we do not need to introduce any truncation of u. It suffices to replace k_n with $\tilde{k}_n = \mu_+ - \xi_n\omega$ and to apply the energy estimates (2.8) when $m > 1$ ((2.10) when $m < 1$) for $(u - \tilde{k}_n)_+$. For the sequel, we just need to know the explicit expression of ν_+ which is given by*

$$\nu_+ = \gamma(1-a)^{N+\max\{2,p\}} \left(\frac{\xi\omega}{\mu_+} \right)^{m+p+N-1} \frac{\left(\theta(\xi\omega)^{p+m-3}\right)^{\frac{N}{p}}}{\left(1+\theta(\xi\omega)^{p+m-3}\right)^{\frac{N+p}{p}}}. \tag{2.28}$$

2.4 A variant of DeGiorgi-type lemma

Assume now that some information is available on the "initial data" relative to the cylinder $(y,s)+Q^+_{2\rho}(\theta)\subset E_T$. Say for example

$$u(x,s)\geq \xi M \qquad \text{for a.e. } x\in K_{2\rho}(y), \tag{2.29}$$

for some $M>0$ and $\xi\in(0,1]$. Then the following lemma applies.

Lemma 2.4.1 (Variant of DeGiorgi-type lemma) *Let u be a non-negative, locally bounded, local, weak solution to the equation (2.1) in E_T. Let $a\in(0,1)$ and suppose that (2.29) holds true. Then there exists $\nu_0\in(0,1)$, depending only upon a and the data $\{N,m,p,C_0,C_1\}$, such that, if*

$$|[u\leq \xi M]\cap[(y,s)+Q^+_{2\rho}(\theta)]|\leq \frac{\nu_0}{\theta(\xi M)^{p+m-3}}|Q^+_{2\rho}(\theta)|, \tag{2.30}$$

then either

$$(C\rho)^p>\max\{1,(\xi M)^{p+m-1}\}$$

or

$$u\geq a\xi M \qquad \text{in } K_\rho(y)\times(s,s+\theta(2\rho)^p]. \tag{2.31}$$

The number ν_0 is stable as $m+p\to 3$.

***Proof of Lemma 2.4.1 for* $\mathbf{m>1}$** Let us consider

$$\rho_n=\rho+\frac{\rho}{2^n},\quad K_n=K_{\rho_n},\quad \widetilde{Q}_n=K_n\times(0,\theta(2\rho)^p],$$
$$\xi_n=a\xi+\frac{1-a}{2^n}\xi, \tag{2.32}$$

and a cutoff function $\zeta(x,t)=\zeta(x)$ independent of t and satisfying (2.13). We define

$$\tilde{v}=\max\{a\xi M,u\}.$$

Let us **first assume $\mathbf{p\geq 2}$**. From the energy estimates (2.8) for $(u-\xi_nM)_-$, $\widetilde{Q}_n$ and ζ, arguing as in Lemma 2.3.1, it follows that

$$\begin{aligned}&\sup_{0<t\leq\theta(2\rho)^p}(\xi M)^{2-p}\int_{K_n}(\tilde{v}-\xi_nM)^p_-(x,t)\zeta^p(x)dx\\&+\varpi\gamma_1(a\xi M)^{m-1}\iint_{\widetilde{Q}_n}|D[(\tilde{v}-\xi_nM)_-\zeta]|^p dxd\tau\\&\leq\gamma\frac{2^{p(n+1)}}{\rho^p}(\xi M)^p\left((\xi M)^{m-1}+C^p\rho^p(\xi M)^{m-1}+\frac{C^p\rho^p}{(\xi M)^p}\right)|\widetilde{A}_n|,\end{aligned}$$

where $\widetilde{A}_n\overset{\text{def}}{=}\{u<\xi_nM\}\cap\widetilde{Q}_n=\{\tilde{v}<\xi_nM\}\cap\widetilde{Q}_n$, $n=0,1,2,\ldots$. Note that the integral on the lower side of the cylinder $\widetilde{Q}_n$ vanishes as a consequence of assumption (2.29) and the fact that $\xi_n\leq\xi$.

At this point, assuming $(C\rho)^p \leqslant (\xi M)^{p+m-1}$, we get

$$\sup_{0<t\leq\theta(2\rho)^p} (\xi M)^{2-p}\int_{K_n} (\tilde{v}-\xi_n M)_-^p(x,t)\zeta^p(x)dx + \varpi\gamma_1(a\xi M)^{m-1}\iint_{\widetilde{Q}_n} |D[(\tilde{v}-\xi_n M)_-\zeta]|^p dxd\tau \leq \gamma\frac{2^{p(n+1)}}{\rho^p}(\xi M)^{p+m-1}|\widetilde{A}_n|.$$

After some computations, which are completely similar to those in the proof of Lemma 2.3.1, we obtain

$$|\widetilde{A}_{n+1}| \leq \gamma\frac{2^{2np}}{a^{\frac{(m-1)N}{N+p}}(1-a)^p\rho^p}(\xi M)^{(p+m-3)\frac{p}{N+p}}|\widetilde{A}_n|^{1+\frac{p}{N+p}}. \tag{2.33}$$

Setting $\widetilde{Y}_n = \frac{|\widetilde{A}_n|}{|\widetilde{Q}_n|}$, (2.33) yields to

$$\widetilde{Y}_{n+1} \leq \gamma\frac{2^{2np}}{a^{\frac{(m-1)N}{N+p}}(1-a)^p}\left(\theta(\xi M)^{(p+m-3)}\right)^{\frac{p}{N+p}}\widetilde{Y}_n^{1+\frac{p}{N+p}}.$$

The thesis follows from Lemma B.4.1, provided $\widetilde{Y}_0 \leq \nu$, with

$$\nu = \frac{\nu_0}{\theta(\xi M)^{p+m-3}},$$

where

$$\nu_0 = \gamma\, a^{\frac{(m-1)N}{p}}(1-a)^{N+p}.$$

If p < 2, from the energy estimates (2.8) for $(u-\xi_n M)_-$, $\widetilde{Q}_n$ and ζ, arguing as in Lemma 2.3.1, it follows that

$$\sup_{0<t\leq\theta(2\rho)^p}\int_{K_n} (\tilde{v}-\xi_n M)_-^2(x,t)\zeta^2(x)dx + \varpi\gamma_1(a\xi M)^{m-1}\iint_{\widetilde{Q}_n} |D[(\tilde{v}-\xi_n M)_-\zeta]|^p dxd\tau \leq \gamma\frac{2^{p(n+1)}}{\rho^p}(\xi M)^{p+m-1}|\widetilde{A}_n|.$$

After some computations, which are completely similar to those in the proof of Lemma 2.3.1, we obtain

$$\tilde{Y}_{n+1} \leq \gamma\frac{2^{np}}{a^{(m-1)\frac{N}{N+2}}(1-a)^p}(\theta(\xi M)^{p+m-3})^{\frac{p}{N+2}}\tilde{Y}_n^{1+\frac{p}{N+2}}.$$

The thesis follows from Lemma 4.1, provided $\tilde{Y}_0 \leq \nu$, with

$$\nu = \frac{\nu_0}{\theta(\xi M)^{p+m-3}},$$

where

$$\nu_0 = \gamma\, a^{\frac{(m-1)N}{p}}(1-a)^{N+2}. \qquad \square$$

Remark 2.4.2 *Summarizing, when $m > 1$, the number ν_0 of Lemma 2.4.1 is given by*

$$\nu_0 = \gamma a^{\frac{(m-1)N}{p}} (1-a)^{N+\max\{2,p\}}.$$

***Proof of Lemma 2.4.1 for* $\mathbf{m} < \mathbf{1}$** Once more we consider the sequences (2.32) and a cutoff function $\zeta(x,t) = \zeta(x)$ independent of t and satisfying (2.13). Applying the energy estimates (2.9) to $(u - k_n)-$, with $k_n = \xi_n M$, over the cylinder $\tilde{Q}_n$ and the indicated choice of ζ, we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<t<\theta(2\rho_n)^p} \int_{K_n} (u-k_n)_-^2(x,t)\zeta^p(x)dx \\ &+\varpi C_0 k_n^{m-1} \iint_{\tilde{Q}_n} |D[(u-k_n)_-\zeta]|^p dx d\tau \\ &\leq \gamma k_n^{m+p-1} \iint_{\tilde{Q}_n} \chi_{[u<k]} (\phi + \zeta^p + |D\zeta|^p) dx dt. \end{aligned}$$

Either $C\rho > 1$ or

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<t<\theta(2\rho_n)^p} \int_{K_n} (u-k_n)_-^2(x,t)\zeta^p dx \\ &+\varpi C_0 k_n^{m-1} \iint_{\tilde{Q}_n} |D[(u-k_n)_-\zeta]|^p dx d\tau \\ &\leq \gamma \frac{2^{pn}}{\rho^p} (\xi M)^{p+m-1} |\tilde{A}_n|. \end{aligned}$$

Starting from this inequality, we proceed as in the proof of Lemma 2.3.1 to obtain

$$\tilde{Y}_{n+1} \leq \gamma \frac{2^{np}}{(1-a)^p} (\theta(\xi M)^{p+m-3})^{\frac{p}{N+\max\{2,p\}}} \tilde{Y}_n^{1+\frac{p}{N+\max\{2,p\}}}.$$

The thesis follows from Lemma B.4.1, provided $\tilde{Y}_0 \leq \nu$, with

$$\nu = \frac{\nu_0}{\theta(\xi M)^{p+m-3}},$$

where

$$\nu_0 = \gamma (1-a)^{N+\max\{2,p\}}. \qquad \square$$

2.5 A $L^1_{loc}-$ form of the Harnack Inequality for $2 < m + p < 3$

Proposition 2.5.1 *Let u be a non-negative, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T. There exists a positive constant γ depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, either*

$$C\rho > \min\{1, \epsilon^{\frac{p+m-1}{p}}\}, \qquad \text{where} \qquad \epsilon = \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}},$$

or

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \\ \leq \gamma \inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx + \gamma \left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}$$

where $\lambda = N(p+m-3)+p$*. The constant* $\gamma = \gamma(p,m) \to \infty$ *as either* $m+p \to 3, 2$*.*

Lemma 2.5.2 *Let* u *be a non-negative, local,weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in* E_T*. Assume* $m > 1$*. There exist positive constants* γ, ϖ*, depending only upon the data* $\{p, m, N, C_0, C_1\}$*, such that for all cylinders* $K_\rho(y) \times [s,t] \subset E_T$ *and all* $\sigma \in (0,1)$*, either*

$$C\rho > \min\{1, \epsilon^{\frac{p+m-1}{p}}\}, \qquad \textit{where} \qquad \epsilon = \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}},$$

or

$$\int_s^t \int_{K_{\sigma\rho}(y)} \tau^{\frac{1}{p}} (u+\epsilon)^{\frac{m-3}{p}} u^{m-1} |Du|^p \zeta^p dx d\tau \\ +\varpi \iint_Q \tau^{\frac{1}{p}-1} (u+\epsilon)^{1+\frac{p+m-3}{p}}(x,\tau)\zeta^p dx d\tau \\ \leq \gamma \frac{\rho}{(1-\sigma)^p} \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{p}} (S+\epsilon\rho^N)^{\frac{2p+m-3}{p}},$$

where

$$S = \sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx, \qquad \lambda = N(p+m-3)+p.$$

The constant $\gamma = \gamma(p,m) \to \infty$ *as either* $m+p \to 3$ *or* $m+p \to 2$*.*

Proof Assume $(y,s) = (0,0)$, fix $\sigma \in (0,1)$, and let $x \to \zeta(x)$ be a non-negative, piecewise smooth cutoff function in K_ρ that vanishes outside K_ρ, equals one on $K_{\sigma\rho}$ and such that

$$|D\zeta| \leq \frac{1}{(1-\sigma)\rho^p}.$$

In the weak formulation, over the cylinder $Q = K_\rho \times [0,t]$, take a test function

$$\varphi = -\tau^{\frac{1}{p}} (u+\epsilon)^{\frac{p+m-3}{p}} \zeta^p, \qquad \text{for some } \epsilon > 0,$$

modulo a Steklov averaging process. We obtain

$$\iint_Q u_\tau \varphi dx d\tau + \iint_Q A(x,\tau,u,Du) \cdot D\varphi dx d\tau \\ = \iint_Q B(x,\tau,u,Du)\varphi dx d\tau.$$

We estimate each term separately

$$\iint_Q u_\tau \varphi\, dxd\tau = -\frac{p}{2p+m-3}\int_{K_\rho} \tau^{\frac{1}{p}}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,\tau)\zeta^p dxd\tau$$
$$+\frac{1}{2p+m-3}\iint_Q \tau^{\frac{1}{p}-1}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,\tau)\zeta^p dxd\tau; \tag{2.34}$$

applying the structure conditions (2.2)

$$\begin{aligned}
&\iint_Q A(x,\tau,u,Du)\cdot D\varphi dxd\tau\\
&\quad\geq \frac{3-m-p}{p}C_0\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}u^{m-1}|Du|^p\zeta^p dxd\tau\\
&\quad -\frac{3-m-p}{p}C^p\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}\zeta^p dxd\tau\\
&\quad -pC_1\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{m-1}|Du|^{p-1}|D\zeta|\zeta^{p-1}dxd\tau\\
&\quad -p\iint_Q C^{p-1}\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{\frac{m-1}{p}}|D\zeta|\zeta^{p-1}dxd\tau;
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
&\iint_Q B(x,\tau,u,Du)\left(-\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}\zeta^p\right)dxd\tau\\
&\quad\leq C\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{m-1}|Du|^{p-1}\zeta^p dxd\tau\\
&\quad +C\iint_Q C^{p-1}\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{\frac{m-1}{p}}\zeta^p dxd\tau.
\end{aligned} \tag{2.36}$$

Combining (2.34), (2.35), and (2.36) we obtain

$$\begin{aligned}
&\frac{3-m-p}{p}\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}u^{m-1}|Du|^p\zeta^p dxd\tau\\
&\quad+\frac{1}{2p+m-3}\iint_Q \tau^{\frac{1}{p}-1}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,\tau)\zeta^p dxd\tau dxd\tau\\
&\quad\leq \frac{p}{2p+m-3}t^{\frac{1}{p}}\int_{K_\rho}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,t)\zeta^p dx\\
&\quad+\frac{3-m-p}{p}\iint_Q C^p\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}\zeta^p dxd\tau\\
&\quad+\iint_Q (pC_1\zeta^{p-1}|D\zeta|+C\zeta^p)\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{m-1}|Du|^{p-1}dxd\tau\\
&\quad+\iint_Q C^{p-1}(p\zeta^{p-1}|D\zeta|+C\zeta^p)\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{p+m-3}{p}}u^{\frac{m-1}{p}}dxd\tau.
\end{aligned}$$

From this, applying Young's inequality

$$\begin{aligned}
&\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}u^{m-1}|Du|^p\zeta^p dxd\tau \\
&\quad +\varpi\iint_Q \tau^{\frac{1}{p}-1}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,\tau)\zeta^p dxd\tau dxd\tau \\
&\quad \le \gamma t^{\frac{1}{p}}\int_{K_\rho}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,t)\zeta^p dx \\
&\quad +\gamma\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{p+\frac{m-3}{p}}u^{m-1}(|D\zeta|^p+C\zeta^p)dxd\tau \\
&\quad +\gamma\iint_Q C^p\tau^{\frac{1}{p}}(u+\epsilon)^{\frac{m-3}{p}}\zeta^p dxd\tau,
\end{aligned}$$

where $\gamma=\gamma(\text{data})$ tends to ∞ as either $m+p\to 3$ or as $m+p\to 2$. By Hölder's inequality

$$\begin{aligned}
&\gamma t^{\frac{1}{p}}\int_{K_\rho}(u+\epsilon)^{1+\frac{p+m-3}{p}}(x,t)\zeta^p dx \\
&\quad \le \gamma t^{\frac{1}{p}}\left(\int_{K_\rho}(u+\epsilon)(x,t)dx\right)^{\frac{2p+m-3}{p}}|K_\rho|^{\frac{3-m-p}{p}} \\
&\quad \le \gamma t^{\frac{1}{p}}\left(\sup_{0\le\tau\le t}\int_{K_\rho}u(x,\tau)dx+\epsilon(2\rho)^N\right)^{\frac{2p+m-3}{p}}\rho^{N\frac{(3-m-p)}{p}} \\
&\quad \le \gamma\rho\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{p}}(S+\epsilon\rho^N)^{\frac{2p+m-3}{p}}.
\end{aligned}$$

Next

$$\begin{aligned}
&\gamma\iint_Q \tau^{\frac{1}{p}}(u+\epsilon)^{p+\frac{m-3}{p}}u^{m-1}(|D\zeta|^p+C\zeta^p)dxd\tau \\
&\quad \le \gamma\frac{1+C^p\rho^p}{(1-\sigma)^p\rho^p}\int_0^t\int_{K_\rho}\tau^{\frac{1}{p}}(u+\epsilon)^{p+m-3}(u+\epsilon)^{\frac{2p+m-3}{p}}dxd\tau \\
&\quad \le \gamma\frac{1+C^p\rho^p}{(1-\sigma)^p\rho^p}\epsilon^{p+m-3}t^{1+\frac{1}{p}}\sup_{0\le\tau\le t}\int_{K_\rho}(u+\epsilon)^{\frac{2p+m-3}{p}}dx \\
&\quad \le \gamma\rho\frac{1+C^p\rho^p}{(1-\sigma)^p}\left(\frac{t}{\rho^p}\right)\epsilon^{p+m-3}\left(\frac{t}{\rho^p}\right)^{\frac{1}{p}}(S+\epsilon\rho^N)^{\frac{2p+m-3}{p}}.
\end{aligned}$$

Finally

$$\begin{aligned}
\gamma \iint_Q & C^p \tau^{\frac{1}{p}} (u+\epsilon)^{\frac{m-3}{p}} \zeta^p dx d\tau \\
& \le \gamma \frac{C^p}{\epsilon^p} \iint_Q \tau^{\frac{1}{p}} (u+\epsilon)^{p+\frac{m-3}{p}} \zeta^p dx d\tau \\
& \le \gamma \frac{C^p}{\epsilon^{p+m-1}} \iint_Q \tau^{\frac{1}{p}} (u+\epsilon)^{p+m-1+\frac{m-3}{p}} \zeta^p dx d\tau \\
& \le \gamma \rho C^p \rho^p \frac{t}{\rho^p} \frac{\epsilon^{p+m-3}}{\epsilon^{p+m-1}} \left(\frac{t}{\rho^p}\right)^{\frac{1}{p}} (S+\epsilon\rho^N)^{\frac{2p+m-3}{p}}.
\end{aligned}$$

Combining the previous estimates

$$\begin{aligned}
\iint_Q & \tau^{\frac{1}{p}} (u+\epsilon)^{\frac{m-3}{p}} u^{m-1} |Du|^p \zeta^p dx d\tau \\
& + \varpi \iint_Q \tau^{\frac{1}{p}-1} (u+\epsilon)^{1+\frac{p+m-3}{p}} (x,\tau) \zeta^p dx d\tau \\
& \le \gamma \frac{\rho}{(1-\sigma)^p} \left[1 + (1+C^p\rho^p) \left(\frac{t}{\rho^p}\right) \epsilon^{p+m-3} + \frac{C^p\rho^p}{\epsilon^{p+m-1}} \left(\frac{t}{\rho^p}\right) \epsilon^{p+m-3} \right] \\
& \qquad \times \left(\frac{t}{\rho^p}\right)^{\frac{1}{p}} (S+\epsilon\rho^N)^{\frac{2p+m-3}{p}}.
\end{aligned}$$

Choose $\epsilon = \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}}$. Either $C\rho > \min\{1, \epsilon^{\frac{p+m-1}{p}}\}$ or we are done. $\square$

Lemma 2.5.3 *Let u be a non-negative, local,weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T. Assume $m > 1$. There exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$ and all $\sigma \in (0,1)$, either*

$$C\rho > \min\{1, \epsilon^{\frac{p+m-1}{p}}\}, \qquad \textit{where} \qquad \epsilon = \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}}, \tag{2.37}$$

or

$$\frac{1}{\rho} \int_s^t \int_{K_{\sigma\rho}} u^{m-1} |Du|^{p-1} dx d\tau \le \delta S + \frac{\gamma}{\delta^{\frac{2p+m-3}{3-m-p}} (1-\sigma)^{\frac{p^2}{3-m-p}}} \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}$$

for all $\delta \in (0,1)$. The constant $\gamma = \gamma(p,m) \to \infty$ as either $m+p \to 3$ or $m+p \to 2$.

Proof Continue to assume that $(y,s) = (0,0)$ and that C violates (2.37). Applying Hölder's inequality and Lemma 2.5.2

$$\begin{aligned}
&\int_0^t \int_{K_{\sigma\rho}} u^{m-1}|Du|^{p-1} dx d\tau \\
&\quad = \int_0^t \int_{K_{\sigma\rho}} \left[\tau^{\frac{p-1}{p^2}} (u+\epsilon)^{\frac{(m-3)}{p}\frac{(p-1)}{p}} u^{(m-1)\frac{(p-1)}{p}} |Du|^{p-1}\right] \\
&\quad\times \left[\tau^{-\frac{p-1}{p^2}} (u+\epsilon)^{\frac{(3-m)}{p}\frac{(p-1)}{p}} u^{\frac{m-1}{p}}\right] dx d\tau \\
&\quad \le \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1}{p}} (u+\epsilon)^{\frac{(m-3)}{p}} u^{(m-1)} |Du|^p dx d\tau\right)^{\frac{p-1}{p}} \\
&\quad\times \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1-p}{p}} (u+\epsilon)^{(3-m)\frac{(p-1)}{p}} u^{m-1} dx d\tau\right)^{\frac{1}{p}} \\
&\quad \le \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1}{p}} (u+\epsilon)^{\frac{(m-3)}{p}} u^{(m-1)} |Du|^p dx d\tau\right)^{\frac{p-1}{p}} \\
&\quad\times \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1-p}{p}} (u+\epsilon)^{\frac{(2p+m-3)}{p}} dx d\tau\right)^{\frac{1}{p}} \\
&\quad \le \gamma \frac{\rho}{(1-\sigma)^p} \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{p}} (S+\epsilon\rho^N)^{\frac{2p+m-3}{p}}.
\end{aligned}$$

Finally, applying Young's inequality to the right-hand side, we obtain the claim. □

***Proof of Proposition 2.5.1, case* m > 1** Assume $(y,s) = (0,0)$ and for $n = 0, 1, 2, \dots$ set

$$\rho_n = \sum_{j=1}^n \frac{\rho}{2^j}, \quad K_n = K_{\rho_n}, \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2}, \quad \tilde{K}_n = K_{\tilde{\rho}_n},$$

and let $x \to \zeta_n(x)$ be a non-negative, piecewise smooth, cutoff function in $\tilde{K}_n$ that equals one on K_n, and such that $|D\zeta_n| \le \frac{2^{n+2}}{\rho}$. In the weak formulation (2.6) take ζ_n as a test function over the cylinder $\tilde{K}_n \times [\tau_1, \tau_2]$, with $\tau_1, \tau_2 \in [0,t]$; after few computations we obtain

$$\begin{aligned}
\int_{\tilde{K}_n} u(x,\tau_1)\zeta_n dx &\le \int_{K_{2\rho}} u(x,\tau_2) dx \\
&+ \frac{2^{n+2}}{\rho}(C_1+\rho C) \left|\int_{\tau_1}^{\tau_2} \int_{\tilde{K}_n} u^{m-1}|Du|^{p-1} dx d\tau\right| \\
&+ \frac{2^{n+2}}{\rho} C^{p-1}(1+\rho C) \left|\int_{\tau_1}^{\tau_2} \int_{\tilde{K}_n} u^{\frac{m-1}{p}} dx d\tau\right|. \qquad (2.38)
\end{aligned}$$

The integral of the second term in the right-hand side can be estimated by means of Hölder's inequality as follows

$$
\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} u^{\frac{m-1}{p}}dxd\tau\right| \leq \\
&\quad\leq \left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} u\,dxd\tau\right|^{\frac{m-1}{p}} \left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} dxd\tau\right|^{\frac{p-m+1}{p}} \\
&\quad\leq tS_{n+1}^{\frac{m-1}{p}}(2\rho)^{N\frac{p-m+1}{p}},
\end{aligned}
$$

where

$$S_n = \sup_{0\leq\tau\leq t}\int_{K_n} u(x,\tau)dx.$$

Hence, applying Young's inequality

$$
\begin{aligned}
&\frac{2^{n+2}}{\rho}C^{p-1}(1+\rho C)\left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} u^{\frac{m-1}{p}}dxd\tau\right| \\
&\quad\leq \frac{2^{n+2}}{\rho^p}\left(\frac{C\rho}{\epsilon^{\frac{p+m-1}{p}}}\right)^{p-1}\left(\frac{t}{\rho^p}\right)^{\frac{(p+m-1)(p-1)}{p(3-m-p)}}(1+\rho C)tS_{n+1}^{\frac{m-1}{p}}(2\rho)^{N\frac{p-m+1}{p}} \\
&\quad\leq \gamma 2^{n+2+N}\left(\frac{C\rho}{\epsilon^{\frac{p+m-1}{p}}}\right)^{p-1}(1+\rho C)\frac{t^{\frac{p-m+1}{p(3-m-p)}}}{\rho^{\frac{p-m+1}{3-m-p}-N\frac{(p-m+1)}{p}}}S_{n+1}^{\frac{m-1}{p}} \\
&\quad\leq \gamma 2^{n+2+N}\left(\frac{C\rho}{\epsilon^{\frac{p+m-1}{p}}}\right)^{p-1}(1+\rho C)\left[\delta S_{n+1}+\frac{1}{\delta^{\frac{m-1}{p-m+1}}}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}\right].
\end{aligned}
$$

Suppose C violates (2.37); combining the previous estimates we get

$$
\begin{aligned}
\int_{\tilde{K}_n} u(x,\tau_1)\zeta_n dx &\leq \int_{K_{2\rho}} u(x,\tau_2)dx \\
&\quad+\gamma\frac{2^{n+2}}{\rho}\left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} u^{m-1}|Du|^{p-1}dxd\tau\right| \\
&\quad+\gamma 2^{n+2+N}\left[\delta S_{n+1}+\frac{1}{\delta^{\frac{m-1}{p-m+1}}}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}\right].
\end{aligned}
\tag{2.39}
$$

As time level τ_2 take one for which

$$\int_{K_{2\rho}} u(x,\tau_2)dx = \inf_{0\leq\tau\leq t}\int_{K_{2\rho}} u(x,\tau)dx =: I.$$

Since $\tau_1 \in [0,t]$ is arbitrary, inequality (2.39) yields

$$
\begin{aligned}
S_n &\leq I + \gamma\frac{2^{n+2}}{\rho}\left|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n} u^{m-1}|Du|^{p-1}dxd\tau\right| \\
&\quad+\gamma 2^{n+2+N}\left[\delta S_{n+1}+\frac{1}{\delta^{\frac{m-1}{p-m+1}}}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}\right].
\end{aligned}
$$

The term involving $|Du|$ is estimated above by applying Lemma 2.5.3 over the pair of cubes $\tilde{K}_n \subset K_{n+1}$, for which $(1-\sigma) = 2^{-(n+2)}$, and for $\delta = \frac{\epsilon_0}{2\gamma 2^{n+2}}$, where $\epsilon_0 \in (0,1)$ is to be chosen. For these choices

$$\gamma \frac{2^{n+2}}{\rho} \left| \int_{\tau_1}^{\tau_2} \int_{\tilde{K}_n} u^{m-1} |Du|^{p-1} dx d\tau \right|$$
$$\leq \frac{\epsilon_0}{2} S_{n+1} + \gamma(\text{data}, \epsilon_0) b^n \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{3-m-p}},$$

where $b = 2^{\frac{p(p+1)}{3-m-p}}$. Combining these remarks we obtain the recursive inequality

$$S_n \leq \epsilon_0 S_{n+1} + \gamma(\text{data}, \epsilon_0) \left[I + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{3-m-p}} \right] b^n,$$

where $b = \max\{2^{\frac{p(p+1)}{3-m-p}}, 2^{\frac{p}{p-m+1}}\}$. From this, by iteration

$$S_0 \leq \epsilon_0^n S_n + \gamma(\text{data}, \epsilon_0) \left[I + \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{3-m-p}} \right] \sum_{i=0}^{n-1} (\epsilon_0 b)^i.$$

Choose ϵ_0 so that the last term is majorized by a convergent series, and let $n \to \infty$. $\square$

The number $0 < m < 1$ being fixed, choose

$$\alpha = \begin{cases} -\frac{1}{2}(p+m-2) & \text{if } 0 < m+p-2 < \frac{2}{3}, \\ -\frac{1}{2}(3-m-p) & \text{if } \frac{1}{3} < m+p-2 < 1. \end{cases}$$

Notice that $0 < p+m-2 < 1$, and that $\alpha \to 0$ as either $m+p \to 3$ or $m+p \to 2$. One verifies that for such α, the numbers $(p+m+\alpha-2), (1+\alpha)$ and $(p+m-\alpha-2)$ are all in $(0,1)$.

Lemma 2.5.4 *Let u be a non-negative, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T. Assume $m < 1$. There exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$ and all $\sigma \in (0,1)$ such that $K_{(1+\sigma)\rho}(y) \subset E$, either $C\rho > 1$, or*

$$\int_s^t \int_{K_\rho(y)} u^{m-1} u^{\alpha-1} |Du|^p \zeta^p dx d\tau$$
$$\leq \frac{\gamma(\alpha)}{\sigma^p \rho^p} S_\sigma^{p+m+\alpha-2} (t-s) \rho^{N(3-m-p-\alpha)} + \gamma(\alpha) S_\sigma^{1+\alpha} \rho^{-\alpha N},$$

where

$$S_\sigma = \sup_{s \leq \tau \leq t} \int_{K_{(1+\sigma)\rho}(y)} u(\cdot, \tau) dx.$$

The constant $\gamma(p,m) \to \infty$ as either $m+p \to 3, 2$.

Proof Assume $(y,s) = (0,0)$, fix $\sigma \in (0,1)$ and let $x \to \zeta(x)$ be a non-negative, piecewise smooth, cutoff function in $K_{(1+\sigma)\rho}$ that vanishes outside $K_{(1+\sigma)\rho}$, equals one on K_ρ, and such that

$$|D\zeta| \leq \frac{1}{\sigma\rho}.$$

In the weak formulation take the test function $u^\alpha\zeta^p$ and integrate over $Q = K_{(1+\sigma)\rho\times(0,t]}$, to formally obtain

$$0 = \frac{1}{1+\alpha}\iint_Q \frac{\partial}{\partial\tau}(u^{\alpha+1}\zeta^p)dxd\tau + \iint_Q A(x,\tau,u,Du)\cdot D(u^\alpha\zeta^p)dxd\tau$$
$$-\iint_Q B(x,\tau,u,Du)u^\alpha\zeta^p dxd\tau = I_1 + I_2 + I_3. \tag{2.40}$$

Assume momentarily that $u^\alpha\zeta^p$ is an admissible test function, and proceed to estimate the various terms formally. Since $0 < 1+\alpha < 1$, estimate

$$\begin{aligned}|I_1| &\le \frac{1}{1+\alpha}\left(\left|\int_{K_{(1+\sigma)\rho}} u^{\alpha+1}(x,t)dx\right| + \left|\int_{K_{(1+\sigma)\rho}} u^{\alpha+1}(x,0)dx\right|\right)\\
&\le \frac{1}{1+\alpha}\left|\left(\int_{K_{(1+\sigma)\rho}} u(x,t)dx\right)^{\alpha+1} |K_{(1+\sigma)\rho}|^{-\alpha}\right|\\
&+\frac{1}{1+\alpha}\left|\left(\int_{K_{(1+\sigma)\rho}} u(x,0)dx\right)^{\alpha+1} |K_{(1+\sigma)\rho}|^{-\alpha}\right|\\
&\le \frac{2}{1+\alpha}[(1+\sigma)\rho]^{-\alpha N} S_\sigma^{\alpha+1}.\end{aligned}$$

Applying the structure conditions (2.3) and Young's inequality

$$\begin{aligned}|I_2| &\ge \left|\iint_Q |\alpha| u^{\alpha-1}\zeta^p A(x,t,u,Du)\cdot Dudxd\tau\right|\\
&- \iint_Q p\zeta^{p-1}u^\alpha |A(x,\tau,u,Du)||D\zeta|dxd\tau\\
&\ge |\alpha| C_0 \iint_Q u^{\alpha-1}u^{m-1}|Du|^p\zeta^p dxd\tau - |\alpha|C^p \iint_Q u^{p+m+\alpha-2}\zeta^p dxd\tau\\
&-p\iint_Q (C_1 u^{\alpha+m-1}|Du|^{p-1} + C^{p-1}u^{p+m+\alpha-2})|D\zeta|\zeta^{p-1}dxd\tau\\
&\ge |\alpha|\frac{C_0}{2}\iint_Q u^{\alpha-1}u^{m-1}|Du|^p\zeta^p dxd\tau - |\alpha|C^p\iint_Q u^{p+m+\alpha-2}\zeta^p dxd\tau\\
&-\iint_Q (\gamma(\alpha,C_0,C_1)|D\zeta|^p + pC^{p-1}|D\zeta|)u^{p+m+\alpha-2}dxd\tau\\
&\ge |\alpha|\frac{C_0}{2}\iint_Q u^{\alpha-1}u^{m-1}|Du|^p\zeta^p dxd\tau\\
&-\left(\frac{|\alpha|}{\rho^p} + \frac{\gamma(\alpha,C_0,C_1)}{\sigma^p\rho^p} + \frac{p}{\sigma\rho^p}\right)\iint_Q u^{p+m+\alpha-2}dxd\tau\\
&\ge |\alpha|\frac{C_0}{2}\iint_Q u^{\alpha-1}u^{m-1}|Du|^p\zeta^p dxd\tau\\
&-\frac{\gamma(\alpha)}{\sigma^p\rho^p}S_\sigma^{p+m+\alpha-2}t[(1+\sigma)\rho]^{N(3-m-p-\alpha)},\end{aligned}$$

where the conditions $C\rho \le 1$ and $0 < p + m + \alpha - 2 < 1$ have been enforced. Applying again the structure conditions (2.3) and Young's inequality

$$\begin{aligned}|I_3| &\le \iint_Q (Cu^{\alpha+m-1}|Du|^{p-1}\zeta^p + C^p u^{p+m+\alpha-2}\zeta^p)dxd\tau \\ &\le \frac{C_0}{4}|\alpha| \iint_Q u^{\alpha+m-2}|Du|^p\zeta^p dxd\tau \\ &+\gamma(C_0,\alpha)C^p \iint_Q u^{p+m+\alpha-2}\zeta^p dxd\tau \\ &\le \frac{C_0}{4}|\alpha| \iint_Q u^{\alpha+m-2}|Du|^p\zeta^p dxd\tau \\ &+\frac{\gamma(C_0,\alpha)}{\sigma^p\rho^p} S_\sigma^{p+m+\alpha-2}\{t[(1+\sigma)\rho]^N\}^{3-m-p-\alpha}.\end{aligned}$$

Since $|I_2| \le |I_1| + |I_3|$, combining the previous estimates we get the claim. The use of $u^\alpha\zeta^p$ as a test function can be justified using $(u+\epsilon)^\alpha\zeta^p$ and then letting $\epsilon \to 0$. □

Corollary 2.5.5 *Let u be a non-negative, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T. Assume $m < 1$. There exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$ and all $\sigma \in (0,1)$ such that $K_{(1+\sigma)\rho}(y) \subset E$, either $C\rho > 1$, or*

$$\begin{aligned}&\frac{1}{\rho}\int_s^t \int_{K_\rho(y)} (|A(x,\tau,u,Du)| + |B(x,\tau,u,Du)|\rho)dxd\tau \\ &\quad\le \frac{\gamma}{\sigma} S_\sigma^{p+m-2}\left(\frac{t-s}{\rho^\lambda}\right) + \gamma S_\sigma^{\frac{p+m-2}{p}+\frac{p-1}{p}}\left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{p}}.\end{aligned}$$

Proof Assume $(y,s) = (0,0)$, and let $Q = K_\rho \times (0,t]$. By the structure conditions (2.3), and enforcing the requirement $C\rho \le 1$

$$\begin{aligned}&\frac{1}{\rho}\int_s^t \int_{K_\rho(y)} (|A(x,\tau,u,Du)| + |B(x,\tau,u,Du)|\rho)dxd\tau \\ &\quad\le \frac{\gamma}{\rho}\iint_Q u^{m-1}|Du|^{p-1}dxd\tau + \frac{\gamma}{\rho^p}\iint_Q u^{p+m-2}dxd\tau.\end{aligned}$$

Estimate

$$\begin{aligned}\frac{\gamma}{\rho^p}\iint_Q u^{p+m-2}dxd\tau &\le \frac{\gamma}{\rho^p}\int_0^t \left(\int_{K_\rho} udx\right)^{p+m-2} |K_\rho|^{3-m-p} \\ &\le \gamma S_\sigma^{p+m-2}\left(\frac{t}{\rho^\lambda}\right).\end{aligned}$$

Next, by the Lemma 2.5.4

$$
\begin{aligned}
&\frac{\gamma}{\rho}\iint_Q u^{m-1}|Du|^{p-1}dxd\tau \\
&\quad\le \frac{\gamma}{\rho}\left(\iint_Q u^{m-1}u^{\alpha-1}|Du|^p dxd\tau\right)^{\frac{p-1}{p}}\left(\iint_Q u^{p+m-2+(1-p)\alpha}\right)^{\frac{1}{p}} \\
&\quad\le \frac{\gamma}{\rho}\left(\frac{\gamma(\alpha)}{\sigma^p\rho^p}S_\sigma^{p+m+\alpha-2}t\rho^{N(3-m-p-\alpha)}+\gamma(\alpha)S_\sigma^{1+\alpha}\rho^{-\alpha N}\right)^{\frac{p-1}{p}} \\
&\qquad\times\quad t^{\frac{1}{p}}S_\sigma^{\frac{p+m-2+(1-p)\alpha}{p}}\rho^{N\frac{3-m-p-(1-p)\alpha}{p}}.
\end{aligned}
$$

Combining the previous estimates we obtain the claim. □

Proof of Proposition 2.5.1, case $\mathbf{m<1}$ Assume $(y,s)=(0,0)$ and for $n=0,1,2,\dots$ set

$$
\rho_n=\sum_{j=1}^{n}\frac{\rho}{2^j},\quad K_n=K_{\rho_n},\quad \tilde\rho_n=\frac{\rho_n+\rho_{n+1}}{2},\quad \tilde K_n=K_{\tilde\rho_n},
$$

and let $x\to\zeta_n(x)$ be a non-negative, piecewise smooth, cutoff function in $\tilde K_n$ that equals one on K_n, and such that $|D\zeta_n|\le\frac{2^{n+2}}{\rho}$. In the weak formulation take ζ_n as a test function over the cylinder $\tilde K_n\times[\tau_1,\tau_2]$, with $\tau_1,\tau_2\in[0,t]$; enforcing $C\rho\le 1$, since $\tilde K_n\subset K_{n+1}$, by means of Corollary 2.5.5 we have

$$
\begin{aligned}
&\int_{\tilde K_n}u\zeta_n(x,\tau_1)dx=\int_{\tilde K_n}u\zeta_n(x,\tau_2)dx \\
&+\int_{\tau_1}^{\tau_2}\int_{\tilde K_n}A(x,t,u,Du)\cdot D\zeta_n dxd\tau-\int_{\tau_1}^{\tau_2}\int_{\tilde K_n}B(x,\tau,u,Du)\zeta_n dxd\tau \\
&\le\int_{\tilde K_n}u\zeta_n(x,\tau_2)dx \\
&+\frac{2^{n+2}}{\rho}\left|\int_{\tau_1}^{\tau_2}\int_{\tilde K_n}(|A(x,\tau,u,Du)|+|B(x,\tau,u,Du)|\rho)dxd\tau\right| \\
&\le\int_{\tilde K_n}u\zeta_n(x,\tau_2)dx \\
&+2^{n+2}\left(\frac{\gamma}{\sigma}S_\sigma^{p+m-2}\left(\frac{t-s}{\rho^\lambda}\right)+\gamma S_\sigma^{\frac{p+m-2}{p}+\frac{p-1}{p}}\left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{p}}\right) \\
&\le\int_{\tilde K_n}u\zeta_n(x,\tau_2)dx+4^n\gamma S_{n+1}^{p+m-2}\left(\frac{t}{\rho}\right)+2^n\gamma S_{n+1}^{\frac{p+m-2}{p}+\frac{p-1}{p}}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{p}},
\end{aligned}
$$

where $S_n=\sup_{0\le\tau\le t}\int_{K_n}u(\cdot,\tau)dx$. Since the time levels τ_1,τ_2 are arbitrary in $[0,t]$ choose τ_2 one for which

$$
\int_{K_{2\rho}}u(x,\tau_2)dx=\inf_{0\le\tau\le t}\int_{K_{2\rho}}u(x,\tau)dx=:I.
$$

With this notation, the previous inequality leads to

$$
S_n\le I+\gamma 4^n S_{n+1}^{p+m-2}\left(\frac{t}{\rho^\lambda}\right)+\gamma 2^n S_{n+1}^{\frac{p+m-2}{p}+\frac{p-1}{p}}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{p}}. \tag{2.41}
$$

By Young's inequality, for all $\epsilon_0 \in (0,1)$

$$\begin{aligned} S_n &\le \epsilon_0 S_{n+1} + \gamma(\text{data}, \epsilon_0)\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}} (4^{\frac{n}{3-m-p}} + 2^{\frac{np}{3-m-p}}) + I \\ &\le \epsilon_0 S_{n+1} + \gamma(\text{data}, \epsilon_0) b^n \left[I + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}\right], \end{aligned}$$

where $b = 2^{\frac{\max\{2,p\}}{3-m-p}}$. From this, by iteration

$$S_0 \le \epsilon_0^n S_n + \gamma(\text{data}, \epsilon_0)\left[I + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{3-m-p}}\right] \sum_{i=1}^{n-1} (\epsilon_0 b)^i.$$

Choose ϵ_0 so that the last term is majorized by a convergent series, and let $n \to \infty$. □

2.6 $L^r_{loc} - L^\infty_{loc}$ Estimates in the range $2 < m + p < 3$

Proposition 2.6.1 *Let u be a locally bounded, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T, and let $r \ge 1$ such that $\lambda_r = N(p+m-3)+rp > 0$. There exists a positive constant γ depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders $K_\rho(y) \times [2s-t, t] \subset E_T$, either*

$$C\rho > \min\{1, \epsilon^{\frac{p+m-1}{p}}\}, \qquad \textit{where} \qquad \epsilon = \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}}$$

or

$$\begin{aligned} \sup_{K_{\frac{1}{2}\rho}(y)\times[s,t]} u_\pm \le \gamma \left(\frac{\rho^p}{t-s}\right)^{\frac{N}{\lambda_r}} \left(\frac{1}{\rho^N(t-s)} \int_{2s-t}^t \int_{K_\rho(y)} u_\pm^r \, dx d\tau\right)^{\frac{p}{\lambda_r}} \\ + \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}} \end{aligned}$$

We limit ourselves to giving the proof for positive solutions. Assume $(y, s) = (0, 0)$, and for fixed $\sigma \in (0, 1)$ and $n = 0, 1, 2 \ldots$ set

$$\rho_n = \sigma\rho + \tfrac{1-\sigma}{2^n}\rho, \qquad K_n = K_{\rho_n},$$

$$t_n = -\sigma t - \tfrac{1-\sigma}{2^n} t, \quad Q_n = K_n \times (t_n, t).$$

This is a family of nested and shrinking cylinders with common "vertex" at $(0, t)$, and by construction

$$Q_0 = K_\rho \times (-t, t), \qquad Q_\infty = K_{\sigma\rho} \times (-\sigma t, t).$$

Having assumed that u is locally bounded in E_T, set

$$M := \operatorname*{ess\,sup}_{Q_0} \max\{u, 0\}, \qquad M_\sigma := \operatorname*{ess\,sup}_{Q_\infty} \max\{u, 0\}.$$

We first find a relationship between M and M_σ. Denote by ζ a non-negative, piecewise smooth, cutoff function in Q_n, that equals one on Q_{n+1}, and has the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in} \quad K_{n+1} \\ 0 & \text{in} \quad \mathbf{R}^N \setminus K_n \end{cases}, \qquad |D\zeta_1| \le \tfrac{2^{n+1}}{(1-\sigma)\rho},$$

$$\zeta_2 = \begin{cases} 0 & \text{for} \quad \tau \le t_n \\ 1 & \text{for} \quad \tau \ge t_{n+1} \end{cases}, \qquad 0 \le \zeta_{2,\tau} \le \tfrac{2^{n+1}}{(1-\sigma)t}.$$

Introduce the increasing sequence of levels $k_n = k - 2^{-(n+1)}k$, where $k > 0$ is to be chosen.

Proof of Proposition 2.6.1, case $\mathbf{m > 1}$ Notice that $m > 1$ and $p + m < 3$ imply $p < 2$. In the weak formulation take the test function $(u - k_{n+1})_+ \zeta^p$; the energy estimates (2.8) give

$$\begin{aligned}
&\sup_{t_n < \tau \le t} \int_{K_n} (u - k_{n+1})_+^2 \zeta^2(x,t)\,dx \\
&\qquad + \varpi \iint_{Q_n} u^{m-1} |D[(u-k_{n+1})_+ \zeta]|^p\,dxd\tau \\
&\quad \le \gamma \frac{2^{n+1}}{(1-\sigma)t} \iint_{Q_n} (u - k_{n+1})_+^2 \zeta^{p-1}\,dxd\tau \\
&\quad + \gamma \frac{2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q_n} u^{m-1} (u - k_{n+1})_+^p\,dxd\tau \\
&\quad + \gamma \iint_{Q_n} \Big(C^p u^{m-1} (u-k_{n+1})_+^p + C^p \chi_{\{(u-k_{n+1})_+ > 0\}} \Big) \zeta^p\,dxd\tau.
\end{aligned} \tag{2.42}$$

First assume $\mathbf{p > \frac{N(3-m)}{N+2}}$, and set $\tilde{k} = \frac{k_n + k_{n+1}}{2}$; it is easy to check that $k_n < \tilde{k} < k_{n+1}$. Estimate

$$\begin{aligned}
&\iint_{Q_n} (u - \tilde{k})_+^{p+m-1}\,dxd\tau \\
&\quad \ge \iint_{Q_n} u^{m-1} \left(1 - \frac{\tilde{k}}{k_{n+1}}\right)^{m-1} (u - k_{n+1})_+^p \chi_{[u > k_{n+1}]}\,dxd\tau \\
&\quad \ge \frac{\gamma}{2^{n(m-1)}} \iint_{Q_n} u^{m-1} (u - k_{n+1})_+^p\,dxd\tau,
\end{aligned}$$

$$\iint_{Q_n} (u - \tilde{k})_+^{p+m-1}\,dxd\tau \ge \gamma \frac{k^{p+m-1}}{2^{n(p+m-1)}} \iint_{Q_n} \chi_{[u > k_{n+1}]}\,dxd\tau,$$

$$\iint_{Q_n} (u - k_{n+1})_+^2\,dxd\tau \le \iint_{Q_n} (u - k_n)_+^2\,dxd\tau,$$

$$\iint_{Q_n} (u - \tilde{k})_+^{p+m-1}\,dxd\tau \le \frac{2^{n(3-m-p)}}{k^{3-m-p}} \iint_{Q_n} (u - k_n)_+^2\,dxd\tau,$$

where we have taken into account that $2 < p + m < 3$. Applying the previous results to the energy estimates above, since $u > k_{n+1} > k_0 = \frac{k}{2}$, we obtain

$$
\begin{aligned}
&\sup_{t_n<\tau\le t}\int_{K_n}(u-k_{n+1})_+^2\zeta^2(x,t)dx\\
&+\varpi\frac{k^{m-1}}{2^{m-1}}\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dxd\tau\\
&\le\gamma\frac{2^{n+1}}{(1-\sigma)t}\iint_{Q_n}(u-k_{n+1})_+^2\zeta^{p-1}dxd\tau\\
&+\gamma\frac{2^{2n}}{(1-\sigma)^p\rho^p}\left[(1+C^p\rho^p)\Big(\frac{1}{k^{3-m-p}}\Big)+\frac{C^p\rho^p}{k^2}\right]\iint_{Q_n}(u-k_n)_+^2dxd\tau\\
&\le\gamma\frac{2^{n+1}}{(1-\sigma)t}\left(1+\frac{t}{\rho^p k^{3-m-p}}+\Big(\frac{t}{\rho^p}\Big)^{\frac{2}{3-m-p}}\frac{1}{k^2}\right)\iint_{Q_n}(u-k_n)_+^2dxd\tau,
\end{aligned}
$$

where we enforced $(C\rho)^p \le \left(\frac{t}{\rho^p}\right)^{\frac{p+m-1}{3-m-p}}$. If we now assume

$$
k > \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}},
$$

we reduce to

$$
\begin{aligned}
&\sup_{t_n<\tau\le t}\int_{K_n}(u-k_{n+1})_+^2\zeta^2(x,t)dx\\
&\quad+\varpi\frac{k^{m-1}}{2^{m-1}}\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dxd\tau\\
&\quad\le\gamma\frac{2^{2n}}{(1-\sigma)^p t}\iint_{Q_n}(u-k_n)_+^2dxd\tau.
\end{aligned}
$$

By Hölder's inequality and the embedding Proposition B.3.1

$$
\begin{aligned}
&\iint_{Q_{n+1}} (u-k_{n+1})_+^2 dxd\tau \\
&\le \left(\iint_{Q_n} [(u-k_{n+1})_+\zeta]^{p\frac{N+2}{N}\frac{2}{3-m}} dxd\tau\right)^{\frac{N(3-m)}{p(N+2)}} \\
&\quad\times \left(\iint_{Q_n} \chi_{[u>k_{n+1}]} dxd\tau\right)^{1-\frac{N(3-m)}{p(N+2)}} \\
&\le M^{m-1} \left(\iint_{Q_n} [(u-k_{n+1})_+\zeta]^{p\frac{N+2}{N}} dxd\tau\right)^{\frac{N(3-m)}{p(N+2)}} \\
&\quad\times \left(\iint_{Q_n} \chi_{[u>k_{n+1}]} dxd\tau\right)^{1-\frac{N(3-m)}{p(N+2)}} \\
&\le \gamma M^{m-1} \left(\iint_{Q_n} |D[(u-k_{n+1})_+\zeta]|^p dxd\tau\right)^{\frac{N(3-m)}{p(N+2)}} \\
&\quad\times \left(\sup_{t_n\le\tau\le t}\int_{K_n} [(u-k_{n+1})_+\zeta]^2(x,\tau)dx\right)^{\frac{3-m}{N+2}} \\
&\quad\times \left(\iint_{Q_n} \chi_{[u>k_{n+1}]} dxd\tau\right)^{1-\frac{N(3-m)}{p(N+2)}} \\
&\le \gamma M^{m-1} \left(\frac{2^{m-1}}{k^{m-1}}\frac{2^{2n}}{(1-\sigma)^p t}\iint_{Q_n} (u-k_{n+1})_+^2 dxd\tau\right)^{\frac{N(3-m)}{p(N+2)}} \\
&\quad\times \left(\frac{2^{2n}}{(1-\sigma)^p t}\iint_{Q_n} (u-k_{n+1})_+^2 dxd\tau\right)^{\frac{3-m}{N+2}} \\
&\quad\times \left(\frac{2^{2n}}{k^2}\iint_{Q_n} (u-k_n)_+^2\right)^{1-\frac{N(3-m)}{p(N+2)}} \\
&\le \gamma M^{m-1} \frac{2^{2n\left(1+\frac{3-m}{N+2}\right)}}{(1-\sigma)^{(3-m)\frac{N+p}{N+2}} t^{\frac{3-m}{p}\frac{N+p}{N+2}}} \frac{1}{k^{\frac{N(3-m)(m-3)}{p(N+2)}+2}} \\
&\quad\times \left(\iint_{Q_n} (u-k_n)_+^2\right)^{1+\frac{3-m}{N+2}}.
\end{aligned}
$$

Now set

$$
Y_n = \frac{1}{|Q_n|}\iint_{Q_n} (u-k_n)_+^2 dxd\tau.
$$

Then

$$
Y_{n+1} \le \gamma M^{m-1} \frac{2^{2n\left(1+\frac{3-m}{N+2}\right)}}{(1-\sigma)^{(3-m)\frac{N+p}{N+2}}} \frac{1}{k^{2-\frac{N(3-m)^2}{p(N+2)}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{p}\frac{3-m}{N+2}} Y_n^{1+\frac{3-m}{N+2}}. \tag{2.43}
$$

By Lemma B.4.1 , $Y_n \to 0$ as $n \to \infty$, provided k is chosen from

$$\fint\!\!\!\fint_{Q_0} \Big(u - \frac{k}{2}\Big)^2_+ dxd\tau \le \fint\!\!\!\fint_{Q_0} u^2 dxd\tau = \gamma(1-\sigma)^{N+p}\Big(\frac{t}{\rho^p}\Big)^{\frac{N}{p}} \frac{k^{\frac{2p(N+2)-N(3-m)^2}{p(3-m)}}}{M^{(m-1)\frac{N+2}{3-m}}}$$

From this choice

$$\begin{aligned} M_\sigma &\le \gamma \frac{M^{\frac{p(m-1)(N+2)}{2p(N+2)-N(3-m)^2}}}{(1-\sigma)^{\frac{(N+p)p(3-m)}{2p(N+2)-N(3-m)^2}}} \Big(\frac{\rho^p}{t}\Big)^{\frac{N(3-m)}{2p(N+2)-N(3-m)^2}} \\ &\quad \times \Big(\fint\!\!\!\fint_{Q_0} u^2 dxd\tau\Big)^{\frac{p(3-m)}{2p(N+2)-N(3-m)^2}} \\ &\le \frac{\gamma}{(1-\sigma)^{\frac{(N+p)p(3-m)}{2p(N+2)-N(3-m)^2}}} \Big(\frac{\rho^p}{t}\Big)^{\frac{N(3-m)}{2p(N+2)-N(3-m)^2}} M^{1-\frac{(3-m)\lambda_r}{2p(N+2)-N(3-m)^2}} \\ &\quad \times \Big(\fint\!\!\!\fint_{Q_0} u^r dxd\tau\Big)^{\frac{p(3-m)}{2p(N+2)-N(3-m)^2}}, \end{aligned}$$

where $\lambda_r = N(p+m-3)+rp$. From this, by Lemma B.4.2 and taking into account the previous assumption on k

$$\sup_{K_{\frac{1}{2}\rho}\times(0,t)} u \le \gamma\Big(\frac{\rho^p}{t}\Big)^{\frac{N}{\lambda_r}} \Big(\fint\!\!\!\fint_{Q_0} u^r dxd\tau\Big)^{\frac{p}{\lambda_r}} + \gamma\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{3-m-p}}, \tag{2.44}$$

The assumption $p > \frac{N(3-m)}{N+2}$ is equivalent to $N(p+m-3)+2p > 0$, which amounts to saying that $\lambda_r > 0$ with $r \in [1,2]$.

Now let us assume that $\lambda_r > 0$ with $r > 2$. This means that $\mathbf{1 < p \le \frac{(3-m)N}{N+2} < \frac{2N}{N+2}}$. Then we need to go back to the inequality (2.42) and estimate all the terms in a different way, namely

$$\iint_{Q_n} (u-k_{n+1})^2_+ dxd\tau \le \gamma \frac{2^{n(r-2)}}{k^{r-2}} \iint_{Q_n} (u-k_n)^r_+ dxd\tau,$$

$$\iint_{Q_n} (u-\tilde{k})^{p+m-1}_+ \le \gamma \frac{2^{n(r-(p+m-1))}}{k^{r-(p+m-1)}} \iint_{Q_n} (u-k_n)^r_+ dxd\tau.$$

Therefore (2.42) becomes

$$\begin{aligned} &\sup_{t_n\le\tau\le t} \int_{K_n} [(u-k_{n+1})^+\zeta]^2(x,\tau)dx + \frac{k^{m-1}}{2^{m-1}} \iint_{Q_n} |D[(u-k_{n+1})^+\zeta]|^p dxd\tau \\ &\le \gamma \frac{2^{n(r-1)}}{(1-\sigma)tk^{r-2}} \iint_{Q_n} (u-k_n)^r_+ dxd\tau \\ &+\gamma \frac{2^{nr}}{(1-\sigma)^p\rho^p}\Big(\frac{1}{k^{r-(p+m-1)}} + \frac{(C\rho)^p}{k^r}\Big) \iint_{Q_n} (u-k_n)^r_+ dxd\tau \\ &\le \gamma \frac{2^{nr}}{(1-\sigma)^p t}\Big[\frac{1}{k^{r-2}} + \Big(\frac{t}{\rho^p}\Big)\Big(\frac{1}{k^{r-(p+m-1)}} + \frac{(C\rho)^p}{k^r}\Big)\Big] \iint_{Q_n} (u-k_n)^r_+ dxd\tau. \end{aligned}$$

Assuming as before

$$k > \left(\frac{t}{\rho^p}\right)\frac{1}{3-m-p}$$

and recalling that $(C\rho)^p < \left(\frac{t}{\rho^p}\right)^{\frac{p+m-1}{3-m-p}}$, estimate

$$\left(\frac{t}{\rho^p}\right)\left(\frac{1}{k^{r-(p+m-1)}} + \frac{(C\rho)^p}{k^r}\right) < \left(\frac{t}{\rho^p}\right)\frac{\gamma}{k^{r-(p+m-1)}} < \frac{\gamma}{k^{r-2}},$$

and reduce to

$$\begin{aligned}\sup_{t_n\le\tau\le t}\int_{K_n}[(u-k_{n+1})^+\zeta]^2(x,\tau)dx + \frac{k^{m-1}}{2^{m-1}}\iint_{Q_n}|D[(u-k_{n+1})^+\zeta]|^p dxd\tau \\ \le \gamma\frac{2^{nr}}{(1-\sigma)^p t}\,\frac{1}{k^{r-2}}\iint_{Q_n}(u-k_n)_+^r dxd\tau. \qquad (2.45)\end{aligned}$$

Now set

$$Y_n = \frac{1}{|Q_n|}\iint_{Q_n}(u-k_n)_+^r dxd\tau$$

and $q = p\frac{N+2}{N}$. Notice that $r > 2 > q$. Estimate

$$\begin{aligned}\iint_{Q_{n+1}}(u-k_{n+1})_+^r dxd\tau &= \iint_{Q_{n+1}}(u-k_{n+1})_+^{r-q}(u-k_{n+1})_+^q dxd\tau \\ &\le M^{r-q}\iint_{Q_{n+1}}(u-k_{n+1})_+^q dxd\tau \\ &\le \gamma M^{r-q}\left(\sup_{t_n\le\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx\right)^{\frac{p}{N}}\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dxd\tau \\ &\le \gamma\left(\frac{2^{np}}{(1-\sigma)^p t}\,\frac{1}{k^{r-2}}\iint_{Q_n}(u-k_n)_+^r dxd\tau\right)^{\frac{p}{N}} \\ &\qquad\times\left(\frac{2^{np}}{(1-\sigma)^p t}\,\frac{1}{k^{r-2}}\,\frac{2^{m-1}}{k^{m-1}}\iint_{Q_n}(u-k_n)_+^r dxd\tau\right).\end{aligned}$$

Hence

$$\begin{aligned}\iint_{Q_{n+1}}(u-k_{n+1})_+^r dxd\tau \le \gamma M^{r-q}\frac{2^{nr(1+\frac{p}{N})}}{(1-\sigma)^{p(1+\frac{p}{N})}t^{(1+\frac{p}{N})}} \\ \times\frac{1}{k^{(r-2)\frac{N+p}{N}+m-1}}\left[\iint_{Q_n}(u-k_n)_+^r\right]^{1+\frac{p}{N}},\end{aligned}$$

which leads to

$$Y_{n+1} \le \gamma M^{r-q}\left(\frac{\rho^p}{t}\right)\frac{2^{nr(1+\frac{p}{N})}}{(1-\sigma)^{\frac{p}{N}(N+p)}}\,\frac{1}{k^{(r-2)\frac{N+p}{N}+m-1}}Y_n^{1+\frac{p}{N}}.$$

Once more $Y_n \to 0$ provided

$$\begin{aligned}\fint\!\!\fint_{Q_0}\left(u-\frac{k}{2}\right)_+^r dxd\tau \le \fint\!\!\fint_{Q_0}u^r dxd\tau \\ = \gamma k^{\frac{(r-2)(N+p)+N(m-1)}{p}}(1-\sigma)^{N+p}\left(\frac{t}{\rho^p}\right)^{\frac{N}{p}}M^{(r+q)\frac{N}{p}},\end{aligned}$$

which yields

$$M_\sigma \le \gamma \frac{M^{(r-q)\frac{N}{(r-2)(N+p)+N(m-1)}}}{(1-\sigma)^{\frac{(N+p)p}{(r-2)(N+p)+N(m-1)}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{(r-2)(N+p)+N(m-1)}}$$
$$\times \left(\iint\!\!\!\!\!\!-_{Q_0} u^r dxd\tau\right)^{\frac{p}{(r-2)(N+p)+N(m-1)}}.$$

By interpolation Lemma B.4.2

$$\sup_{K_{\frac{\rho}{2}}\times(0,t)} \le \gamma\Big(\frac{\rho^p}{t}\Big)^{\frac{N}{\lambda_r}} \left(\iint\!\!\!\!\!\!-_{Q_0} u^r dxd\tau\right)^{\frac{p}{\lambda_r}} + \gamma\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{3-m-p}}. \qquad \Box$$

Proof of Proposition 2.6.1, case $\mathbf{m < 1}$ In the weak formulation take the testing function $\varphi = (u^l - k^l_{n+1})_+\zeta^p$ over Q_n, where $l = \frac{p+m-2}{p-1} \in (0,1)$. By means of the structure conditions (2.3) and the Young's inequality, proceeding as for the energy estimates (2.10), we get

$$\iint_{Q_n} u_\tau (u^l - k^l_{n+1})_+\zeta^p dxd\tau + \varpi \iint_{Q_n} |Du^l|^p \zeta^p \chi_{[u>k_{n+1}]} dxd\tau$$
$$\le \gamma \iint_{Q_n} (|D\zeta|^p + C^p\zeta^p)(u^l - k^l_{n+1})^p_+ dxd\tau$$
$$+\gamma C^p \iint_{Q_n} u^{pl}\zeta^p \chi_{[u>k_{n+1}]} dxd\tau.$$

We estimate the first integral on the left-hand side as follows

$$\iint_{Q_n} u_\tau (u^l - k^l_{n+1})_+\zeta^p dxd\tau$$
$$= \int_{K_\rho}\int_k^u (s^l - k^l_{n+1})_+ ds\zeta^p(x,t)dx - p\iint_{Q_n}\int_k^u (s^l - k^l_{n+1})_+ ds\zeta^{p-1}\zeta_\tau dxd\tau$$
$$\ge \frac{1}{l+1}\int_{K_\rho} (u^l - k^l_{n+1})_+^{\frac{l+1}{l}} \zeta^p(x,t)dx - \frac{p}{l}\iint_{Q_n} (u^l - k^l_{n+1})_+ u\zeta^{p-1}\zeta_\tau dxd\tau.$$

Since $|Du^l|^p\chi_{[u>k_{n+1}]} = l^p u^{(l-1)p}|D(u-k_{n+1})_+|^p$ and

$$|D[(u^l - k^l_{n+1})_+\zeta]|^p \le 2^p[|D(u^l - k^l_{n+1})_+|^p\zeta^p + (u^l - k^l_{n+1})^p_+|D\zeta|^p],$$

we estimate the second term on the left-hand side

$$\iint_{Q_n} |Du^l|^p\zeta^p\chi_{[u>k_{n+1}]} dxd\tau$$
$$\ge \frac{1}{2^p}\iint_{Q_n} |D[(u^l - k^l_{n+1})_+\zeta]|^p dxd\tau - \iint_{Q_n} u^{lp}\chi_{[u>k_{n+1}]}|D\zeta|^p dxd\tau.$$

Hence we obtain

$$
\begin{aligned}
&\sup_{t_n\le\tau\le t}\int_{K_\rho}(u^l-k^l_{n+1})_+^{\frac{l+1}{l}}\zeta^p(x,t)dx+\varpi\iint_{Q_n}|D[(u^l-k^l_{n+1})_+\zeta]|^p dxd\tau\\
&\le\gamma\iint_{Q_n}(u^l-k^l_{n+1})_+u\zeta_\tau dxd\tau+\gamma\iint_{Q_n}(|D\zeta|^p+C^p\zeta^p)(u^l-k^l_{n+1})_+^p dxd\tau\\
&\quad+\gamma\iint_{Q_n}C^p\zeta^p u^{pl}\chi_{[u>k_{n+1}]}dxd\tau\\
&\le\gamma\frac{2^{n+1}}{(1-\sigma)t}\iint_{Q_n}u^{l+1}\chi_{[u>k_{n+1}]}dxd\tau\\
&\quad+\gamma\Big(\frac{2^{p(n+1)}}{(1-\sigma)^p\rho^p}+C^p\Big)\iint_{Q_n}(u^l-k^l_{n+1})_+^p dxd\tau\\
&\quad+\gamma\iint_{Q_n}C^p\zeta^p u^{pl}\chi_{[u>k_{n+1}]}dxd\tau.
\end{aligned}
$$

Estimate

$$\iint_{Q_n}u^{l+1}\chi_{[u>k_{n+1}]}dxd\tau\le\gamma 2^{\frac{l+1}{l}}\iint_{Q_n}(u^l-k^l_n)_+^{\frac{l+1}{l}}dxd\tau,$$

$$\iint_{Q_n}(u^l-k^l_{n+1})_+^p dxd\tau\le\gamma k^{p+m-3}\iint_{Q_n}(u^l-k^l_n)_+^{\frac{l+1}{l}}dxd\tau,$$

$$\iint_{Q_n}u^{pl}\chi_{[u>k_{n+1}]}dxd\tau\le\gamma k^{p+m-3}\iint_{Q_n}(u^l-k^l_n)_+^{\frac{l+1}{l}}dxd\tau.$$

Enforcing $C\rho\le 1$ and stipulating

$$k\ge\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{3-m-p}},$$

the previous estimates yield to

$$
\begin{aligned}
&\sup_{t_n\le\tau\le t}\int_{K_\rho}[(u^l-k^l_{n+1})_+\zeta]^{\frac{l+1}{l}}(x,t)dx+\varpi\iint_{Q_n}|D[(u^l-k^l_{n+1})_+\zeta]|^p dxd\tau\\
&\quad\le\gamma\frac{2^{2(\frac{l+1}{l})n}}{(1-\sigma)^p t}\Big[1+\Big(\frac{t}{\rho^p}\Big)k^{p+m-3}\Big]\iint_{Q_n}(u^l-k^l_n)_+^{\frac{l+1}{l}}dxd\tau\\
&\quad\le\gamma\frac{2^{2(\frac{l+1}{l})n}}{(1-\sigma)^p t}\iint_{Q_n}(u^l-k^l_n)_+^{\frac{l+1}{l}}dxd\tau.
\end{aligned}
\tag{2.46}
$$

Let us first assume $\mathbf{l} > \frac{(\mathbf{N}-\mathbf{p})_+}{\mathbf{N}(\mathbf{p}-\mathbf{1})+\mathbf{p}}$; this amounts to taking $\lambda_r > 0$ for $r < l+1$. By the Hölder inequality and the embedding Proposition B.3.1

$$\begin{aligned}
\iint_{Q_{n+1}} (u^l - k^l_{n+1})_+^{\frac{l+1}{l}} dx d\tau &\leq \iint_{Q_n} [(u^l - k^l_{n+1})_+ \zeta]^{\frac{l+1}{l}}(x,t) dx d\tau \\
&\leq \left(\iint_{Q_n} [(u^l - k^l_{n+1})_+ \zeta]^{\frac{p(lN+l+1)}{lN}}(x,t) dx d\tau \right)^{\frac{N(l+1)}{p(lN+l+1)}} \\
&\quad \times \left(\iint_{Q_n} \chi_{[u > k_{n+1}]} dx d\tau \right)^{1-\frac{N(l+1)}{p(lN+l+1)}} \\
&\leq \gamma \left(\sup_{t_n \leq \tau \leq t} \int_{K_\rho} [(u^l - k^l_{n+1})_+ \zeta]^{\frac{l+1}{l}}(x,t) dx \right)^{\frac{l+1}{lN+l+1}} \\
&\quad \times \left(\iint_{Q_n} |D[(u^l - k^l_{n+1})_+ \zeta]|^p dx d\tau \right)^{\frac{N(l+1)}{p(lN+l+1)}} \\
&\quad \times \left(\frac{2^{\frac{l+1}{l}n}}{k^{l+1}} \iint_{Q_n} (u^l - k^l_n)_+^{\frac{l+1}{l}} dx d\tau \right)^{1-\frac{N(l+1)}{p(lN+l+1)}} .
\end{aligned}$$

Now set

$$Y_n = \frac{1}{|Q_n|} \iint_{Q_n} (u^l - k^l_n)_+^{\frac{l+1}{l}} dx d\tau.$$

With this notation, the previous inequality becomes

$$Y_{n+1} \leq \gamma \frac{b^n}{(1-\sigma)^{\frac{(N+p)(l+1)}{(lN+l+1)}}} \frac{1}{k^{(l+1)(1-\frac{N(l+1)}{p(lN+l+1)})}} \left(\frac{\rho^p}{t} \right)^{\frac{N(l+1)}{p(lN+l+1)}} Y_n^{1+\frac{l+1}{lN+l+1}},$$

where $b = 2^{2(\frac{l+1}{l})(1+\frac{l+1}{lN+l+1})}$. Now, $Y_n \to 0$ as $n \to \infty$, provided k is chosen such that

$$Y_0 = \iint\!\!\!\!\!\!-\,_{Q_0} u^{l+1} dx d\tau = \gamma (1-\sigma)^{N+p} \left(\frac{t}{\rho^p} \right)^{\frac{N}{p}} k^{\frac{p(lN+l+1) - N(l+1)}{p}} .$$

With this choice

$$\begin{aligned}
M_\sigma \quad &\leq \frac{\gamma}{(1-\sigma)^{\frac{(N+p)p}{(p-1)Nl - N + pl + p}}} \left(\frac{\rho^p}{t} \right)^{\frac{N}{(p-1)Nl - N + pl + p}} \\
&\quad \times \left(\iint\!\!\!\!\!\!-\,_{Q_0} u^{l+1} dx d\tau \right)^{\frac{p}{(p-1)Nl - N + pl + p}} .
\end{aligned} \tag{2.47}$$

Set

$$\rho_n = \sigma\rho + (1-\sigma)\rho \sum_{i=1}^n 2^{-i}, \qquad t_n = \sigma t - (1-\sigma) t \sum_{i=1}^n 2^{-i},$$

$$Q_n = K_{\rho_n} \times (t_n, t], \qquad Q_\infty = K_\rho \times (-t, t], \qquad Q_0 = K_{\sigma\rho} \times (-\sigma t, t].$$

Recall that we are restricting to $r < l+1$; writing (2.47) over the pair of cubes Q_n and Q_{n+1} gives

$$M_n \leq \frac{\gamma}{(1-\sigma)^{\frac{(N+p)p}{(p-1)Nl-N+pl+p}}}\left(\frac{\rho^p}{t}\right)^{\frac{N}{(p-1)Nl-N+pl+p}} \times \left(\iint_{Q_0} u^r dxd\tau\right)^{\frac{p}{(p-1)Nl-N+pl+p}} M_{n+1}^{\frac{p(l+1-r)}{(p-1)Nl-N+pl+p}},$$

where $M_n := \operatorname{ess\,sup}_{Q_n} \max\{u, 0\}$. Notice that $(p-1)Nl - N + rp = N(p+m-3) + rp = \lambda_r$. By Lemma B.4.2, we conclude that

$$\sup_{K_{\sigma\rho}\times(-\sigma t,t]} u \leq \frac{\gamma}{(1-\sigma)^{\frac{(N+p)p}{\lambda_r}}}\left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda_r}}\left(\iint_{Q_0} u^r dxd\tau\right)^{\frac{p}{\lambda_r}}.$$

Now, assume $\mathbf{l} \leq \frac{(\mathbf{N}-\mathbf{p})_+}{\mathbf{N}(\mathbf{p}-\mathbf{1})+\mathbf{p}}$; this amounts to taking $\lambda_r > 0$ for $r \geq l+1$. It is easy to see that $r \geq l+1$ implies $r \geq pl$. Hence estimate

$$\iint_{Q_n} (u^l - k_{n+1}^l)_+^{\frac{l+1}{l}} dxd\tau \leq \gamma \frac{2^{n\frac{r-(l+1)}{l}}}{k^{r-(l+1)}} \iint_{Q_n} (u^l - k_n^l)_+^{\frac{r}{l}} dxd\tau,$$

Enforcing $C\rho \leq 1$, by means of the previous inequalities, estimate (2.46) becomes

$$\sup_{t_n\leq\tau\leq t}\int_{K_\rho} [(u^l - k_{n+1}^l)_+\zeta]^{\frac{l+1}{l}}(x,t)dx + \varpi \iint_{Q_n} |D[(u^l - k_{n+1}^l)_+\zeta]|^p dxd\tau \leq \gamma\frac{2^{nr}}{(1-\sigma)^p t}\,\frac{1}{k^{r-(l+1)}}\iint_{Q_n}(u^l - k_n^l)_+^{\frac{r}{l}} dxd\tau.$$

Set

$$Y_n = \frac{1}{|Q_n|}\iint_{Q_n} (u^l - k_n^l)_+^{\frac{r}{l}} dxd\tau,$$

and denote $q = \frac{p(Ln+l+1)}{NL}$; then $\frac{r}{l} - q > 0$ and we can estimate

$$Y_{n+1} \leq \frac{1}{|Q_{n+1}|}\iint_{Q_n} (u^l - k_n^l)_+^{\frac{r}{l}} dxd\tau \leq \frac{1}{|Q_{n+1}|}\iint_{Q_n} (u^l - k_n^l)_+^{\frac{r}{l}-q+q} dxd\tau \leq \frac{||u||_{\infty,Q_0}}{|Q_{n+1}|}\iint_{Q_n}(u^l - k_n^l)_+^q dxd\tau.$$

Applying the embedding Proposition B.3.1, the previous inequality can be rewritten as

$$Y_{n+1} \leq \gamma\frac{2^{rn(\frac{p+N}{N})}}{(1-\sigma)^{\frac{p}{N}(N+p)}}\left(\frac{\rho^p}{t}\right)||u||_{\infty,Q_0}^{r-lq}\,\frac{1}{k^{(r-(l+1))(\frac{p+N}{N})}}Y_n^{\frac{p+N}{N}}. \tag{2.48}$$

By means of Lemma B.4.1, $Y_n \to 0$, provided k is chosen to satisfy

$$Y_0 = \iint_{Q_0} u^r dxd\tau = \gamma\frac{(1-\sigma)^(N+p)}{2^{Nr\frac{(N+p)}{p^2}}}\left(\frac{t}{\rho^p}\right)^{\frac{N}{p}}||u||_{\infty,Q_0}^{-(r-lq)\frac{N}{p}}k^{(r-(l+1))\frac{(p+N)}{p}}, \tag{2.49}$$

which yields

$$M_\sigma \le \gamma \frac{M^{\frac{(r-lq)N}{(r-(l+1))(p+N)}}}{(1-\sigma)^{\frac{p}{r-(l+1)}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{(r-(l+1))(p+N)}} \left(\iint_{Q_0} u^r dx d\tau\right)^{\frac{p}{(r-(l+1))(p+N)}}.$$

By Lemma B.4.2 we conclude that

$$\sup_{K_{\sigma\rho}\times(-\sigma t, t]} u \le \frac{\gamma}{(1-\sigma)^{\frac{(N+p)p}{\lambda_r}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda_r}} \left(\iint_{Q_0} u^r dx d\tau\right)^{\frac{p}{\lambda_r}}. \qquad \square$$

2.7 L^r_{loc} Estimates backward in time in the range $2 < m+p < 3$

Proposition 2.7.1 *Let u be a locally bounded, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T, and assume that $u \in L^r_{loc}(E_T)$ for some $r > 1$, satisfying $\lambda_r = N(p+m-3)+rp > 0$. There exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$ and r, such that either*

$$C\rho > \min\{1, M_r^{\pm}, (M_r^{\pm})^{\frac{p+m-2}{p-1}}\},$$

where

$$M_r^{\pm} := \left(\sup_{s\le\tau\le t} ⨍_{K_\rho(y)} u_\pm^r(x,\tau)dx\right)^{\frac{1}{r}},$$

or

$$\sup_{s\le\tau\le t} \int_{K_\rho(y)} u_\pm^r(x,\tau)dx \le \gamma \left[\int_{K_{2\rho}(y)} u_\pm^r(x,s)dx + \left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{3-m-p}}\right]$$

for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$ The constant $\gamma = \gamma(\text{data}, r) \to +\infty$ as $r \to 1$.

Proof of Proposition 2.7.1, case $\mathbf{m > 1}$ The proof will be given for non-negative solutions. Assume $(y,s) = (0,0)$, fix $\sigma \in (0,1]$ and choose $\zeta \in C_0^\infty(K_{(1+\sigma)\rho})$ satisfying

$$0 \le \zeta \le 1 \text{ in } K_{(1+\sigma)\rho}, \qquad \zeta = 1 \text{ in } K_\rho,$$

$$|D\zeta| \le \frac{\gamma}{\sigma\rho} \text{ in } K_{(1+\sigma)\rho},$$

for a constant γ depending only upon N. Let M be a positive constant to be chosen, and let q be a parameter in the range $\max\{r-1, 1\} < q < r$. In the weak formulation take $f(u)\zeta^p$, with

$$f(u) := u^{r-1}\left(\frac{(u-M)_+}{u}\right)^q,$$

as a testing function, modulo a standard Steklov averaging process. One verifies that

$$(r-1)u^{r-2}\left(\frac{(u-M)_+}{u}\right)^q \le f'(u) \le q u^{r-2}\left(\frac{(u-M)_+}{u}\right)^{q-1}.$$

Set

$$F(u) = \int_M^u f(v)dv,$$

and integrate over $Q_\tau = K_{(1+\sigma)\rho} \times (0,\tau]$, with $\tau \in (0,t]$. The weak formulation gives

$$\begin{aligned}0 = &\iint_{Q_\tau} u_s f(u)\zeta^p dxds + \iint_{Q_\tau} A(x,s,u,Du)\cdot Du\, f'(u)\zeta^p dxds\\ &+p\iint_{Q_\tau} A(x,s,u,Du)\cdot D\zeta f(u)\zeta^{p-1}dxds\\ &-\iint_{Q_\tau} B(x,s,u,Du)f(u)\zeta^p dxd\tau\\ &= T_1+T_2+T_3+T_4.\end{aligned}$$

By means of the structure conditions (2.2) estimate

$$\begin{aligned}T_1 &= \iint_{Q_\tau}\frac{\partial}{\partial s}\int_M^u f(v)dv\zeta^p dxds = \iint_{Q_\tau} F(u)_s\zeta^p dxds\\ &= \int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx - \int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx;\end{aligned}$$

$$\begin{aligned}T_2 &\ge C_0(r-1)\iint_{Q_\tau} u^{r-2}\Big(\frac{(u-M)_+}{u}\Big)^q (u^{m-1}|Du|^p)\zeta^p dxds\\ &\quad -qC^p\iint_{Q_\tau} u^{r-2}\Big(\frac{(u-M)_+}{u}\Big)^{q-1}\zeta^p dxds\\ &= C_0(r-1)\iint_{Q_\tau} f(u)u^{m-2}|Du|^p\zeta^p dxds\\ &\quad -qC^p\iint_{Q_\tau} u^{r-2}\Big(\frac{(u-M)_+}{u}\Big)^{q-1}\zeta^p dxds;\end{aligned}$$

$$|T_3| \le p\iint_{Q_\tau} f(u)(C_1u^{m-1}|Du|^{p-1} + C^{p-1}u^{\frac{m-1}{p}})|D\zeta|\zeta^{p-1}dxds;$$

$$|T_4| \le C\iint_{Q_\tau} u^{m-1}|Du|^{p-1}f(u)\zeta^p dxds + C^p\iint_{Q_\tau} u^{\frac{m-1}{p}}f(u)\zeta^p dxds.$$

Combining these remarks, since $T_1 \le -T_2 + |T_3| + |T_4|$,

$$\begin{aligned}&\int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx + C_0(r-1)\iint_{Q_\tau} f(u)u^{m-2}|Du|^p\zeta^p dxds\\ &\le \gamma\frac{pC_1}{\sigma\rho}(1+C\rho)\iint_{Q_\tau} f(u)u^{m-1}|Du|^{p-1}\zeta^{p-1}dxds\\ &+\gamma\Big(\frac{C^{p-1}}{\sigma\rho}+C^p\Big)\iint_{Q_\tau} f(u)u^{\frac{m-1}{p}}\zeta^{p-1}dxds\\ &+qC^p\iint_{Q_\tau} u^{r-2}\Big(\frac{(u-M)^+}{u}\Big)^{q-1}dxd\tau\\ &+\int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx.\end{aligned}$$

Applying Young's inequality, estimate

$$\begin{aligned}\iint_{Q_\tau} f(u)u^{\frac{m-1}{p}}\zeta^{p-1}dxds &\le \iint_{Q_\tau} u^{r-1+\frac{m-1}{p}}\zeta^{p-1}dxds\\ &\le \gamma\left(\iint_{Q_\tau} u^{r-1}dxds + \iint_{Q_\tau} u^{r+m-2}dxds\right);\end{aligned}$$

$$\begin{aligned}&\frac{pC_1}{\sigma\rho}\iint_{Q_\tau} f(u)u^{m-1}|Du|^{p-1}\zeta^{p-1}dxds\\ &\le \frac{r-1}{2}C_0\iint_{Q_\tau} f(u)u^{m-2}|Du|^p\zeta^p dxds + \frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} f(u)u^{p+m-2}dxds\\ &\le \frac{r-1}{2}C_0\iint_{Q_\tau} f(u)u^{m-2}|Du|^p\zeta^p dxds + \frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} u^{p+m-3+t}dxds,\end{aligned}$$

where $\gamma(r)$ depends only on r and the data. Also

$$\iint_{Q_\tau} u^{r-2}\Big(\frac{(u-M)^+}{u}\Big)^{q-1}dxds \le \iint_{Q_\tau}\frac{u^{r-1}}{M}dxds;$$

Enforcing $C\rho \le \min\{1, M_r^{\pm}, (M_r^{\pm})^{\frac{p+m-2}{p-1}}\}$, these remarks imply

$$\begin{aligned}&\int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx\\ &\le \int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx + \frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} u^{p+m-3+r}dxds\\ &+\gamma\frac{(\rho C)^{p-1}}{\sigma^p\rho^p}\iint_{Q_\tau}(u^{r-1}+u^{r+m-2})dxds + qC^p\iint_{Q_\tau}\frac{u^{r-1}}{M}dxds\\ &\le \int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx + \frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} u^{p+m-3+r}dxds\\ &+\frac{\gamma(\rho C)^{p-1}}{\sigma^p\rho^p}\Big(1+\frac{\rho C}{M}\Big)\iint_{Q_\tau} u^{r-1}dxds\\ &+\frac{\gamma(\rho C)^{p-1}}{\sigma^p\rho^p}\iint_{Q_\tau} u^{r+m-2}dxds\\ &\le \int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx + \frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} u^{p+m-3+r}dxds\\ &+\frac{\gamma(M_r)^{p+m-2}}{\sigma^p\rho^p}\Big(1+\frac{\rho C}{M}\Big)\iint_{Q_\tau} u^{r-1}dxds\\ &+\frac{\gamma(M_r)^{p-1}}{\sigma^p\rho^p}\iint_{Q_\tau} u^{r+m-2}dxds.\end{aligned} \tag{2.50}$$

By means of the Hölder inequality estimate

$$\frac{\gamma(r)}{\sigma^p\rho^p}\iint_{Q_\tau} u^{p+m-3+r}dxds \\ \leq \frac{\gamma(r)}{\sigma^p}\left(\sup_{0\leq\tau\leq t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}};$$

$$\frac{\gamma(M_r)^{p+m-2}}{\sigma^p\rho^p}\Big(1+\frac{\rho C}{M}\Big)\iint_{Q_\tau} u^{r-1}dxds \\ \leq \frac{\gamma(r)}{\sigma^p}\Big(1+\frac{\rho C}{M}\Big)\left(\sup_{0\leq\tau\leq t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}};$$

$$\frac{\gamma(M_r)^{p-1}}{\sigma^p\rho^p}\iint_{Q_\tau} u^{r+m-2}dxds \\ \leq \frac{\gamma(r)}{\sigma^p}\left(\sup_{0\leq\tau\leq t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}};$$

then inequality (2.50) becomes

$$\int_{K_{(1+\sigma)\rho}} F(u)(x,\tau)\zeta^p(x)dx \leq \int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx \\ +\frac{\gamma(r)}{\sigma^p}\Big(1+\frac{\rho C}{M}\Big)\left(\sup_{0\leq\tau\leq t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}}.$$

By elementary calculations and the Young inequality,

$$\int_{K_\rho\cap[u>M]} u^r(x,\tau)dx \leq 2r\sup_{0\leq\tau\leq t}\int_{K_\rho} F(u)(x,\tau)dx + \bar\gamma M^r|K_\rho|,$$

for a constant $\bar\gamma=\bar\gamma(r,p,m,q,C_0,C_1)$. From this

$$\sup_{0\leq\tau\leq t}\fint_{K_\rho} u^r(x,\tau)dx \leq 2r\Big(\sup_{0\leq\tau\leq t}\fint_{K_\rho} F(u)(x,\tau)dx + (1+\bar\gamma)M^r\Big).$$

Choosing

$$M=\frac{1}{[4r(1+\bar\gamma)]^{\frac{1}{r}}}M_r,$$

these inequalities yield

$$\sup_{0\le\tau\le t} \fint_{K_\rho} u^r(x,\tau)dx \le 2r \int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx$$
$$+\frac{\gamma(r)}{\sigma^p}\Big(1+\frac{\rho C}{M}\Big)\left(\sup_{0\le\tau\le t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}$$
$$+2r|K_\rho|(1+\bar\gamma)M^r$$
$$\le 2\int_{K_{(1+\sigma)\rho}} u^r(x,0)dx$$
$$+\frac{\gamma(r,\bar\gamma)}{\sigma^p}\left(\sup_{0\le\tau\le t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}$$
$$+\frac{1}{2}\sup_{0\le\tau\le t}\fint_{K_\rho} u^r(x,\tau)dx.$$

From this

$$\sup_{0\le\tau\le t} \fint_{K_\rho} u^r(x,\tau)dx \le 2r \int_{K_{(1+\sigma)\rho}} F(u)(x,0)\zeta^p(x)dx$$
$$\le \gamma\int_{K_{(1+\sigma)\rho}} u^r(x,0)dx$$
$$+\frac{\gamma(r,\bar\gamma)}{\sigma^p}\left(\sup_{0\le\tau\le t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.$$

Fix $R > 0$ and consider the sequence of radii

$$\rho_n = R\sum_{i=1}^{n} 2^{-i},$$

so that

$$\rho_{n+1} = (1+\sigma_n)\rho_n \qquad \text{for} \qquad \sigma_n = \frac{\rho_{n+1}-\rho_n}{\rho_n} \ge 2^{-n-2}.$$

Setting

$$Y_n = \sup_{0\le\tau\le t}\int_{K_n} u^r(x,\tau)dx$$

the previous inequality yields

$$Y_n \le \gamma\int_{K_{2R}} u^r(x,0)dx + \gamma(r,\bar\gamma)2^n\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} Y_{n+1}^{\frac{p+m-3+r}{r}}.$$

The proposition now follows from the interpolation Lemma B.4.2. □

Proof of Proposition 2.7.1, case $\mathbf{m < 1}$ Once more the proof will be given for non-negative solutions. Assume $(y,s) = (0,0)$, fix $\sigma \in (0,1]$ and choose $\zeta \in C_0^\infty(K_{(1+\sigma)\rho})$ satisfying

$$0 \le \zeta \le 1 \text{ in } K_{(1+\sigma)\rho}, \qquad \zeta = 1 \text{ in } K_\rho,$$

$$|D\zeta| \le \frac{\gamma}{\sigma\rho} \text{ in } K_{(1+\sigma)\rho},$$

for a constant γ depending only upon N. In the weak formulation, take $u^{r-1}\zeta^p$ as a test function, modulo a standard Steklov averaging process. Integrating over $Q_\tau = K_{(1+\sigma)\rho}\times(0,\tau]$, with $\tau \in (0,t]$, gives

$$\begin{aligned}
0 &= \frac{1}{r}\iint_{Q_\tau} (u^r)_s \zeta^p dxds \\
&+ (r-1)\iint_{Q_\tau} A(x,s,u,Du)\cdot Duu^{r-2}\zeta^p dxds \\
&+ p\iint_{Q_\tau} A(x,s,u,Du)\cdot D\zeta u^{r-1}\zeta^{p-1}dxds \\
&- \iint_{Q_\tau} B(x,x,u,Du)u^{r-1}\zeta^p dxds \\
&= \frac{1}{r}T_1 + (r-1)T_2 + T_3 + T_4.
\end{aligned}$$

Since ζ is independent of time

$$T_1 = \int_{K_{(1+\sigma)\rho}} u^r(x,\tau)\zeta^p(x)dx - \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^p(x)dx.$$

Next, by means of the structure conditions (2.3),

$$T_2 \ge C_0 \iint_{Q_\tau} u^{m+r-3}|Du|^p\zeta^p dxds - C^p \iint_{Q_\tau} u^{p+m-3+r}\zeta^p dxds.$$

$$\begin{aligned}
|T_3| &\le p\iint_{Q_\tau} u^{r-1}[C_1u^{m-1}|Du|^{p-1}|D\zeta| + C^{p-1}u^{p+m-2}|D\zeta|]\zeta^{p-1}dxds \\
&= pC_1\iint_{Q_\tau} u^{m+r-2}|Du|^{p-1}|D\zeta|\zeta^{p-1}dxds \\
&+ pC^{p-1}\iint_{Q_\tau} u^{p+m-3+r}\zeta^{p-1}dxds.
\end{aligned}$$

$$\begin{aligned}
|T_4| &\le \iint_{Q_\tau} Cu^{r-1}[u^{m-1}|Du|^{p-1} + C^{p-1}u^{p+m-2}]\zeta^p dxds \\
&= C\iint_{Q_\tau} u^{m+r-2}|Du|^{p-1}\zeta^p dxd\tau + C^p\iint_{Q_\tau} u^{p+m-3+r}\zeta^p dxds.
\end{aligned}$$

Combining the previous estimates

$$\begin{aligned}
&\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)\zeta^p(x)dx + (r-1)C_0 \iint_{Q_\tau} u^{m+r-3}|Du|^p\zeta^p dxds \\
&\quad \le \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^p(x)dx \\
&\quad +\gamma \iint_{Q_\tau} (C^p\zeta^p + C^{p-1}\zeta^{p-1}|D\zeta|)u^{p+m-3+r}dxds \\
&\quad +\gamma \iint_{Q_\tau} (C\zeta^p + |D\zeta|\zeta^{p-1})u^{m+r-2}|Du|^{p-1}dxds.
\end{aligned}$$

By Young's inequality, enforcing $|D\zeta| \le (\sigma\rho)^{-1}$ and $C\rho \le 1$,

$$\begin{aligned}
&\iint_{Q_\tau} (C\zeta^p + |D\zeta|\zeta^{p-1})u^{m+r-2}|Du|^{p-1}dxds. \\
&\quad \le \frac{\gamma}{\sigma\rho}(1+C\rho) \iint_{Q_\tau} u^{m+r-3}|Du|^{p-1}\zeta^{p-1}dxds \\
&\quad \le (r-1)C_0 \iint_{Q\tau} u^{m+r-3}|Du|^p\zeta^p dxds \\
&\quad +\frac{\gamma(r)}{(\sigma\rho)^p} \iint_{Q_\tau} u^{p+m-3+r}dxds,
\end{aligned}$$

where $\gamma(r)$ is a constant depending on r and the data $\{p,m,N,C_0,C_1\}$. Hence, enforcing again $C\rho \le 1$,

$$\begin{aligned}
&\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)\zeta^p(x)dx \\
&\quad \le \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^p(x)dx \\
&\quad +\gamma(r)\Big(\frac{C^{p-1}}{\sigma\rho} + \frac{1}{(\sigma\rho)^p}\Big) \iint_{Q_\tau} u^{p+m-3+r}dxds \\
&\quad \le \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^p(x)dx + \frac{\gamma(r)}{(\sigma\rho)^p} \iint_{Q_\tau} u^{p+m-3+r}dxds.
\end{aligned}$$

By means of Hölder's inequality estimate

$$\iint_{Q_\tau} u^{p+m-3+r}dxds \le \gamma t \left(\sup_{0\le\tau\le t} \int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}} \rho^{\frac{N}{r}(3-m-p)}.$$

Therefore

$$\begin{aligned}\int_{K_{(1+\sigma)\rho}} & u^r(x,\tau)\zeta^p(x)dx \\ & \le \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^p(x)dx \\ & + \frac{\gamma(r)}{\sigma^p}\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}}\left(\sup_{0\le\tau\le t}\int_{K_{(1+\sigma)\rho}} u^r(x,\tau)dx\right)^{\frac{p+m-3+r}{r}}.\end{aligned}$$

Fix $R > 0$ and consider the sequence of radii

$$\rho_n = R\sum_{i=1}^{n} 2^{-i},$$

so that

$$\rho_{n+1} = (1+\sigma_n)\rho_n \qquad \text{for} \qquad \sigma_n = \frac{\rho_{n+1}-\rho_n}{\rho_n} \ge 2^{-n-2}.$$

Setting

$$Y_n = \sup_{0\le\tau\le t}\int_{K_n} u^r(x,\tau)dx$$

the previous estimate yields the recursive inequality

$$Y_n \le \gamma\int_{K_{2R}} u^r(x,0)dx + \gamma(r,\bar{\gamma})2^{pn}\left(\frac{t^r}{R^{\lambda_r}}\right)^{\frac{1}{r}} Y_{n+1}^{\frac{p+m-3+r}{r}}.$$

The proposition now follows from the interpolation Lemma B.4.2. □

Chapter 3

Intrinsic Harnack estimates for some doubly nonlinear degenerate parabolic equations

3.1 Introduction

The aim of this chapter is to prove an intrinsic Harnack estimate for non-negative weak solutions to the parabolic degenerate equations (2.1). To this purpose, the most crucial property of such solutions is the "expansion of positivity". It asserts that, if one of these solutions is positive over a cube $K_\rho(y)$ at some time level, then the positivity expands in space at some further time, driven by the intrinsic geometry of these equations. The first step to prove this consists in propagating the positivity information to further times, within the same cube ("expansion of positivity in time"). Finally one expands the positivity set in the space of variables from $K_\rho(y)$ to $K_{2\rho}(y)$. By means of a proper changing of variables, this allows to prove an intrinsic Harnack estimate.

Moreover, in Section 4.2.5 we will show that the expansion of positivity is stable as $m + p \to 3$, hence all the results of this chapter continue to hold when $m + p = 3$.

In Section 3.1 we introduce the "expansion of positivity in time", whose proof is common to both the degenerate and the singular case. Section 3.2 and Section 3.3 are devoted to the proof, respectively, of the "expansion of positivity" and an intrinsic Harnack inequality for non-negative solutions to the parabolic degenerate equations (2.1). Finally, in Section 3.4, we show how the intrinsic Harnack inequality implies a Hölder continuity condition.

3.2 Expansion of positivity in time

Lemma 3.2.1 (Expansion of positivity in time) *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$ there holds*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)| \tag{3.1}$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist δ and ϵ in $(0,1)$, depending only upon the data $\{p, m, N, C_0, C_1\}$, and α, and independent of M, such that either

$$(C\rho)^p > \min\{1, M^{p+m-1}\}$$

or

$$|[u(\cdot,t) > \epsilon M] \cap K_\rho(y)| \geq \frac{1}{2}\alpha|K_\rho|$$

for all $t \in \left(s, s + \frac{\delta\rho^p}{M^{p+m-3}}\right]$.

Remark 3.2.2 *The proof is based on the energy estimates of Section 2.2, whose constants ϖ and γ are stable as $m + p \to 3$. Hence the constants $\delta = \delta(m,p)$ and $\epsilon = \epsilon(m,p)$ are stable as $m + p \to 3$.*

***Proof of Lemma 3.2.1 when* $\mathbf{m > 1}$** Assume $(y, s) = (0, 0)$, and for $k, t > 0$ set

$$A_{k,\rho}(t) = [u(\cdot,t) < k] \cap K_\rho.$$

Assumption (3.1) implies

$$|A_{M,\rho}(0)| \leq (1-\alpha)|K_\rho|. \tag{3.2}$$

Write down the energy estimates (2.8) for the truncated functions $(u - M)_-$ over the cylinder $K_\rho \times (0, \theta\rho^p]$, where $\theta > 0$ is to be chosen. The cutoff function ζ is taken independent of t, non-negative, and such that

$$\zeta = 1 \text{ on } K_{(1-\sigma)\rho}, \qquad |D\zeta| \leq \frac{1}{\sigma\rho},$$

where $\sigma \in (0,1)$ is to be chosen. Discarding the non-negative term containing $D(u - M)_-$ on the left-hand side, these estimates yield

$$\begin{aligned}
\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx &\leq \int_{K_\rho} (u-M)_-^2(x,0)dx \\
&+ \frac{\gamma}{(\sigma\rho)^p}\int_0^{\theta\rho^p}\int_{K_\rho} u^{m-1}(u-M)_-^p dxd\tau \\
&+ \gamma C^p \int_0^{\theta\rho^p}\int_{K_\rho} u^{m-1}(u-M)_-^p \zeta^p dxd\tau \\
&+ \gamma C^p \int_0^{\theta\rho^p}\int_{K_\rho} \chi_{[(u-M)_- > 0]}\zeta^p dxd\tau.
\end{aligned}$$

The first term on the right-hand side can be estimated by means of (3.2)

$$\int_{K_\rho} (u-M)_-^2(x,0)dx \leq M^2(1-\alpha)|K_\rho|$$

Assuming $(C\rho)^p \leq \min\{1, M^{p+m-1}\}$, we get

$$\begin{aligned}
&\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx \\
&\leq M^2\left[(1-\alpha) + \gamma\left(\frac{1}{(\sigma\rho)^p} + C^p\right)\theta\rho^p M^{p+m-3} + \gamma\frac{\theta}{M^2}(C\rho)^p\right]|K_\rho| \\
&\leq M^2\left[(1-\alpha) + \gamma\frac{\theta}{\sigma^p}M^{p+m-3}\right]|K_\rho|
\end{aligned}$$

for all $t \in (0, \theta\rho^p]$. The left-hand side is estimated below by

$$\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx \geq \int_{K_{(1-\sigma)\rho}\cap[u<\epsilon M]} (u-M)_-^2(x,t)dx$$
$$\geq M^2(1-\epsilon)^2|A_{\epsilon M,(1-\sigma)\rho}(t)|,$$

where $\epsilon \in (0,1)$ is to be chosen. Next we estimate

$$|A_{\epsilon M,\rho}(t)| = |A_{\epsilon M,(1-\sigma)\rho}(t) \cup (A_{\epsilon M,\rho}(t) \setminus A_{\epsilon M,(1-\sigma)\rho}(t))|$$
$$\leq |A_{\epsilon M,(1-\sigma)\rho}(t)| + |K_\rho \setminus K_{(1-\sigma)\rho}|$$
$$\leq |A_{\epsilon M,(1-\sigma)\rho}(t)| + N\sigma|K_\rho|.$$

Combining all the previous estimates we obtain

$$|A_{\epsilon M,\rho}(t)| \leq \frac{1}{(1-\epsilon)^2}[(1-\alpha) + \gamma\frac{\theta}{\sigma^p}M^{p+m-3} + N\sigma]|K_\rho|$$

for every $t \in (0, \theta\rho^p]$. Choose $\theta = \delta M^{3-m-p}$, and then choose σ, ϵ, and δ so close to zero, depending on α and the data, as to insure the conclusion of the lemma. □

***Proof of Lemma 3.2.1 when* $\mathbf{m < 1}$** Again assume $(y,s) = (0,0)$ and for $k, t > 0$ set

$$A_{k,\rho}(t) = [u(\cdot,t) < k] \cap K_\rho.$$

In the weak formulation (2.6) take the test function

$$\varphi = -(u^l - M^l)_-\zeta^p, \qquad l = \frac{m+p-2}{p-1} \in (0,1),$$

where $x \to \zeta(x)$ is a non-negative, piecewise smooth, cutoff function in K_ρ which equals one on $K_{(1-\sigma)\rho}$ and such that $|D\zeta| \leq (\sigma\rho)^p$($\sigma$ to be chosen). Proceeding as for the energy estimates on the cylinder $K_\rho \times (0, \theta\rho^p]$($\theta$ to be chosen), we estimate

$$\int_{K_\rho}\int_{u(x,t)}^{M} (M^l - s^l)_+ ds\zeta^p dx \leq \int_{K_\rho}\int_{u(x,0)}^{M} (M^l - s^l)_+ ds\zeta^p dx$$
$$+\gamma M^{lp} \iint_{K_\rho\times(0,\theta\rho^p]} \left(\frac{1}{(\sigma\rho)^p} + C^p\zeta^p\right)\chi_{[u<M]}dxd\tau,$$

for all times $0 < t < \theta\rho^p$. Either $(C\rho)^p > 1$ or

$$\int_{K_\rho}\int_{u(x,t)}^{M} (M^l - s^l)_+ ds\zeta^p dx \leq \int_{K_\rho}\int_{u(x,0)}^{M} (M^l - s^l)_+ ds\zeta^p dx$$
$$+\gamma M^{lp}\frac{\theta}{\sigma^p}|K_\rho|,,$$

for all times $0 < t < \theta\rho^p$. Estimate

$$\int_u^M (M^l - s^l)_+ ds \leq \frac{l}{l+1}(M^{l+1} - u^{l+1})\chi_{[u<M]};$$

then

$$\int_{K_\rho}\int_{u(x,0)}^{M}(M^l-s^l)_+ds\zeta^p dx \le \frac{l}{l+1}(1-\alpha)M^{l+1}|K_\rho|.$$

On the other hand

$$\begin{aligned}
&\int_{K_\rho}\int_{u(x,t)}^{M}(M^l-s^l)_+ds\zeta^p dx\\
&\quad\ge \int_{K_{(1-\sigma)\rho}\cap[u(x,t)<\epsilon M]}\int_{u(x,t)}^{M}(M^l-s^l)_+dsdx\\
&\quad\ge \int_{K_{(1-\sigma)\rho}\cap[u(x,t)<\epsilon M]}\int_{\epsilon M}^{M}(M^l-s^l)_+dsdx\\
&\quad= \left((1-\epsilon)M^{l+1}-\frac{M^{l+1}}{l+1}+\epsilon^{l+1}\frac{M^{l+1}}{l+1}\right)|A_{\epsilon M,(1-\sigma)\rho}(t)|\\
&\quad\ge \frac{l}{l+1}M^{l+1}\left(1-\frac{l+1}{l}\epsilon\right)|A_{\epsilon M,(1-\sigma)\rho}(t)|.
\end{aligned}$$

Set $\theta=\delta M^{3-m-p}$, where δ is to be chosen. Recalling that

$$|A_{\epsilon M,\rho}(t)|\le |A_{\epsilon M,(1-\sigma)\rho}(t)|+N\sigma|K_\rho|,$$

and combining all the previous estimates we obtain

$$|A_{\epsilon M,\rho}(t)|\le \frac{1}{(1-\frac{l+1}{l}\epsilon)}\left[(1-\alpha)+\gamma\frac{\delta}{\sigma^p}+N\sigma\right]|K_\rho|,$$

for every $t\in(0,\theta\rho^p]$. Finally choose σ,ϵ, and δ so close to zero, depending on α and the data, as to insure the conclusion of the lemma. □

3.3 Expansion of positivity

Proposition 3.3.1 (Expansion of positivity, $\mathbf{p+m-3>0}$) *Assume that*

$$u(x,s)\ge \xi M,\qquad x\in K_{2\rho}(y), \tag{3.3}$$

for some $(y,s)\in E_T, M>0, \xi\in(0,1]$. Then there exist positive constants γ, b and η, with $\eta\in(0,1)$, depending only on the data $\{m,p,N,C_0,C_1\}$, such that either

$$(C\rho)^p>\min\{1,\gamma(\xi M)^{p+m-1}\}$$

or

$$u(x,t)\ge \eta(\xi M) \tag{3.4}$$

for $x\in K_{4\rho}(y)$ and every t such that

$$s+\left(\frac{b}{\eta\xi M}\right)^{p+m-3}(16^p-4^p)\rho^p\le t\le s+\left(\frac{b}{\eta\xi M}\right)^{p+m-3}(16)^p\rho^p. \tag{3.5}$$

From now on we assume that $(C\rho)^p \leq \min\{1, (\xi M)^{p+m-1}\}$ and $p+m-3>0$. As a consequence of Lemma 2.4.1, we observe that choosing $\theta = \nu_0(\xi M)^{3-p-m}$, hypothesis (2.30) is automatically satisfied and therefore (2.31) gives, in particular,

$$u\left(x, \frac{\nu_0\rho^p}{(\xi M)^{p+m-3}}\right) \geqslant a\xi M \quad \text{in } K_\rho. \tag{3.6}$$

For every $\tau \geqslant 0$ we set

$$\xi_\tau = \frac{\xi}{f(\tau)}, \qquad \text{where} \qquad f(\tau) = e^{\frac{\tau}{p+m-3}}.$$

Since $\xi_\tau \leqslant \xi$, one still has $u(x,0) \geqslant \xi_\tau M$ in $K_{2\rho}$, by (3.3), and hence, replacing ξ by ξ_τ in (3.6) we obtain

$$u\left(x, \left(\frac{f(\tau)}{\xi M}\right)^{p+m-3} \nu_0\rho^p\right) \geqslant a\frac{\xi M}{f(\tau)},$$

for all $x \in K_\rho$ and every $\tau \geqslant 0$. Defining

$$w(x,\tau) = \frac{f(\tau)}{\xi M}(\nu_0\rho^p)^{\frac{1}{p+m-3}}\, u\left(x, \left(\frac{f(\tau)}{\xi M}\right)^{p+m-3} \nu_0\rho^p\right), \tag{3.7}$$

and fixing $a = 1/2$, we have

$$w(x,\tau) \geqslant \frac{1}{2}(\nu_0\rho^p)^{\frac{1}{p+m-3}} \stackrel{\text{def}}{=} k_0 \tag{3.8}$$

for every $\tau \geqslant 0$ and all $x \in K_\rho$. Let us first suppose $m > 1$. Recalling that $u \geqslant 0$, by formal computations it is easily seen that

$$w_\tau \geqslant \operatorname{div}\widetilde{A}(x,\tau,w,Dw) + \widetilde{B}(x,\tau,w,Dw),$$

where

$$\widetilde{A}(x,\tau,w,Dw) = \psi(\tau)^{p+m-2}A(x,\psi(\tau)^{p+m-3},\psi^{-1}w,\psi^{-1}Dw),$$
$$\widetilde{B}(x,\tau,w,Dw) = \psi(\tau)^{p+m-2}B(x,\psi(\tau)^{p+m-3},\psi^{-1}w,\psi^{-1}Dw),$$

with

$$\psi(\tau) = \frac{f(\tau)}{\xi M}(\nu_0\rho^p)^{\frac{1}{p+m-3}},$$

and A, B satisfying (2.2). Such a formal differential inequality can be made rigorous starting from the weak formulation (2.6), performing the corresponding change of variables from t into τ and taking positive test functions. The new functions $\widetilde{A}$, $\widetilde{B}$ preserve the structure conditions (2.2). Indeed, it is easily checked that

$$\begin{cases} \widetilde{A}(x,\tau,w,\eta)\cdot\eta \geqslant C_0 w^{m-1}|\eta|^p - \widetilde{C}(\tau)^p, \\ |\widetilde{A}(x,\tau,w,\eta)| \leqslant C_1 w^{m-1}|\eta|^{p-1} + \widetilde{C}(\tau)^{p-1}w^{\frac{m-1}{p}} \\ |\widetilde{B}(x,\tau,w,\eta)| \leqslant C w^{m-1}|\eta|^{p-1} + C\widetilde{C}^p(\tau)^{p-1}w^{\frac{m-1}{p}} \end{cases}$$

with

$$\widetilde{C}(\tau) = C\psi(\tau)^{1+\frac{m-1}{p}}.$$

At this point, the energy estimates that we need for w are the following

$$
\begin{aligned}
&\sup_{0<\tau\leqslant\theta(16\rho)^p}\int_{K_{16\rho}}(w-k)_-^2\zeta^p(x,\tau)dx\\
&\quad+\varpi\iint_{Q^+_{16\rho}(\theta)}\widetilde{w}^{m-1}|D[(w-k)_-\zeta]|^p dxds\\
&\leqslant\gamma\iint_{Q^+_{16\rho}(\theta)}(w-k)_-^2\zeta_\tau dxds+\gamma\iint_{Q^+_{16\rho}(\theta)}\widetilde{w}^{m-1}(w-k)_-^p|D\zeta|^p dxds\\
&+\gamma\iint_{Q^+_{16\rho}(\theta)}\Big(C^p\widetilde{w}^{m-1}(w-k)_-^p+\widetilde{C}^p(s)\chi_{\{(w-k)_->0\}}\Big)\zeta^p dxds,
\end{aligned}
\tag{3.9}
$$

where ζ is a piecewise smooth cutoff function in the cylinder $Q^+_{16\rho}(\theta)$ vanishing on the parabolic boundary of $Q^+_{16\rho}(\theta)$ and such that $0\leqslant\zeta\leqslant 1$, $\zeta_\tau\geqslant 0$.

Now our aim consists in proving the "expansion of positivity" for w. Namely we are going to extend (3.8) to $K_{2\rho}$ when τ is sufficiently large.

Proposition 3.3.2 *Set*

$$\mathcal{Q}_{8\rho}(\theta)=K_{8\rho}\times\big((16\rho)^p\theta-(8\rho)^p\theta,(16\rho)^p\theta\big].$$

Then, for every $\nu>0$ there exist $\sigma\in(0,1)$, depending upon the data and ν, $\gamma>1$ depending on the data and σ, such that either $(C\rho)^p>\min\{1,\gamma(\xi M)^{p+m-1}\}$ or

$$|\{w<\sigma k_0\}\cap\mathcal{Q}_{8\rho}(\theta_*)|\leqslant\nu|\mathcal{Q}_{8\rho}(\theta_*)|,$$

with $\theta_*=(\sigma k_0)^{p+m-3}$, and k_0 given in (3.8).

Proof Introduce the levels

$$k_j=\frac{k_0}{2^j}\qquad j=0,1,\dots,j_*,$$

with $j_*\in\mathbf{N}$, $j_*>1$ to be determined. Fix $j\in\{0,\dots,j_*-2\}$ and set

$$v_*=\max\{k_{j+2},w\}.$$

By writing the energy estimates (3.9) for $(w-k_j)_-$ and choosing a test function ζ such that

$$\zeta=1\quad\text{in}\quad\mathcal{Q}_{8\rho}(\theta),\qquad|D\zeta|\leqslant\frac{1}{8\rho},\qquad 0\leqslant\zeta_\tau\leqslant\frac{1}{\theta(8\rho)^p},$$

we obtain

$$
\begin{aligned}
&\iint_{\mathcal{Q}_{8\rho}(\theta)}w^{m-1}|D(w-k_j)_-|^p dxds\\
&\quad\leqslant\gamma\bigg(\frac{k_j^2}{\theta(8\rho)^p}+\frac{k_j^{p+m-1}}{(8\rho)^p}+C^pk_j^{p+m-1}+\big[\widetilde{C}\big((16\rho)^p\theta\big)\big]^p\bigg)|\mathcal{Q}_{8\rho}(\theta)|.
\end{aligned}
$$

It is immediate to see that

$$\begin{aligned}\iint_{\mathcal{Q}_{8\rho}(\theta)} & w^{m-1}|D(w-k_j)_-|^p dxds \\ & \geqslant \iint_{\mathcal{Q}_{8\rho}(\theta)\cap\{v_*=w\}} (v_*)^{m-1}|D(v_*-k_j)_-|^p dxds \\ & \geqslant \gamma_1 k_{j+2}^{m-1} \iint_{\mathcal{Q}_{8\rho}(\theta)} |D(v_*-k_j)_-|^p dxds.\end{aligned}$$

Setting $\theta=\theta_*=k_{j_*}^{3-p-m}$, by means of the last two inequalities, it turns out that

$$\begin{aligned}\iint_{\mathcal{Q}_{8\rho}(\theta_*)} |D(v_*-k_j)_-|^p dxds \leqslant \gamma \frac{k_j^p}{(8\rho)^p}\Bigg(& \frac{k_j^{2-p}k_{j+2}^{1-m}}{\theta_*} + \frac{k_j^{m-1}}{k_{j+2}^{m-1}} + \frac{C^p k_j^{m-1}(8\rho)^p}{k_{j+2}^{m-1}} \\ & + \frac{\big[\widetilde{C}\big((16\rho)^p\theta_*\big)\big]^p(8\rho)^p}{k_j^p k_{j+2}^{m-1}}\Bigg)|\mathcal{Q}_{8\rho}(\theta_*)|.\end{aligned}$$

It is easily seen that

$$\frac{k_j^{2-p}k_{j+2}^{1-m}}{\theta_*} \leqslant \gamma(\text{data}), \qquad \frac{k_j^{m-1}}{k_{j+2}^{m-1}} \leqslant \gamma(\text{data}).$$

Moreover, recalling the definitions of $\widetilde{C}$ and k_0, we obtain

$$\begin{aligned}\frac{\big[\widetilde{C}\big((16\rho)^p\theta_*\big)\big]^p(8\rho)^p}{k_j^p k_{j+2}^{m-1}} &= C^p\psi\big((16\rho)^p\theta_*\big)^{p+m-1}\frac{(8\rho)^p\, 2^{jp+(j+2)(m-1)}}{k_0^{p+m-1}} \\ &= \left(\frac{f\big((16\rho)^p\theta_*\big)}{\xi M}\right)^{p+m-1}\frac{C^p(8\rho)^p\, 2^{jp+(j+2)(m-1)}}{2^{-p-m+1}} \\ &\leqslant \gamma(\text{data};j_*)\frac{C^p\rho^p}{(\xi M)^{p+m-1}},\end{aligned}$$

as $\rho^p\theta_*=\gamma(\text{data};j_*)$. Thus, assuming that

$$(C\rho)^p \leqslant \gamma^{-1}(\text{data};j_*)(\xi M)^{p+m-1}$$

we have

$$\iint_{\mathcal{Q}_{8\rho}(\theta_*)} |D(v_*-k_j)_-|^p dxds \leqslant \gamma\frac{k_j^p}{(8\rho)^p}|\mathcal{Q}_{8\rho}(\theta_*)|, \tag{3.10}$$

with γ depending only on the data. Now, set

$$A_j(\tau)=\{v_*(\cdot,\tau)<k_j\}\cap K_{8\rho}, \qquad A_j=\{v_*<k_j\}\cap\mathcal{Q}_{8\rho}(\theta_*),$$

and notice that $A_j(\tau)=\{w(\cdot,\tau)<k_j\}\cap K_{8\rho}$, $A_j=\{w<k_j\}\cap\mathcal{Q}_{8\rho}(\theta_*)$, and the same holds true with $j+1$ replacing j, due to the choice of v_*. Moreover

$$|A_j|=\int_{\theta_*(16\rho)^p-\theta_*(8\rho)^p}^{\theta_*(16\rho)^p}|A_j(\tau)|d\tau.$$

From Lemma 2.2 of Chapter I in [16] it follows that

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \leqslant \frac{\gamma\rho^{N+1}}{|K_{8\rho} \setminus A_j(\tau)|} \int_{A_j(\tau)\setminus A_{j+1}(\tau)} |Dv_*|dx, \tag{3.11}$$

for every $\tau \in \big(\theta_*(16\rho)^p - \theta_*(8\rho)^p, \theta_*(16\rho)^p\big]$. On the other hand, by (3.8), we have

$$|K_{8\rho} \setminus A_j(\tau)| \geqslant |K_\rho| = \rho^N$$

and, consequently, (3.11) yields

$$\frac{1}{2}k_j|A_{j+1}(\tau)| \leqslant \gamma\rho \int_{A_j(\tau)\setminus A_{j+1}(\tau)} |Dv_*|dx.$$

Integrating both sides of the above inequality with respect to τ in the interval $\big(\theta_*\rho^p(16^p - 8^p), \theta_*(16\rho)^p\big]$, applying Hölder's inequality and using (3.10), we get

$$\begin{aligned} \frac{1}{2}k_j|A_{j+1}| &\leqslant \gamma\rho \left(\iint_{A_j\setminus A_{j+1}} |Dv_*|^p dx d\tau \right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \leqslant \\ &\leqslant \gamma\rho \left(\frac{k_j^p}{(8\rho)^p} |\mathcal{Q}_{8\rho}(\theta_*)| \right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\ &= \gamma k_j \, |\mathcal{Q}_{8\rho}(\theta_*)|^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}}. \end{aligned}$$

Raising both sides to the power $\frac{p}{p-1}$, and summing over j from 0 to $j_* - 2$ leads to

$$\sum_{j=0}^{j_*-2} |A_{j+1}|^{\frac{p}{p-1}} \leqslant \gamma \, |\mathcal{Q}_{8\rho}(\theta_*)|^{\frac{1}{p-1}} \sum_{j=0}^{j_*-2} |A_j \setminus A_{j+1}|.$$

Finally, since $A_{j+1} \subset A_j \subset A_0 \subset \mathcal{Q}_{8\rho}(\theta_*)$ for every j, we easily deduce that

$$(j_* - 1)|A_{j_*-1}|^{\frac{p}{p-1}} \leqslant \gamma \, |\mathcal{Q}_{8\rho}(\theta_*)|^{\frac{1}{p-1}} \sum_{j=0}^{j_*-2} (|A_j| - |A_{j+1}|) \leqslant \gamma \, |\mathcal{Q}_{8\rho}(\theta_*)|^{\frac{p}{p-1}}.$$

Thus, we have established that

$$|A_{j_*-1}| \leqslant \left(\frac{\gamma}{j_* - 1} \right)^{\frac{p-1}{p}} |\mathcal{Q}_{8\rho}(\theta_*)|.$$

At this point the statement follows immediately. Indeed, for any $\nu > 0$, we can choose j_* large enough to have $[\gamma/(j_* - 1)]^{\frac{p-1}{p}} \leqslant \nu$. Setting $\sigma = 1/2^{j_*-1} \in (0, 1)$ we conclude that

$$|\{w < \sigma k_0\} \cap \mathcal{Q}_{8\rho}(\theta_*)| = |A_{j_*-1}| \leqslant \nu |\mathcal{Q}_{8\rho}(\theta_*)|. \qquad \square$$

Proposition 3.3.3 (Expansion of positivity for w) *There exist $\sigma \in (0,1)$ and $\gamma > 1$, depending only upon the data, such that either $(C\rho)^p > \min\{1, \gamma(\xi M)^{p+m-1}\}$ or*

$$w(x,\tau) \geqslant \frac{1}{2}\sigma k_0 \qquad \textit{in} \ \ K_{4\rho} \times \left(\frac{(16^p - 4^p)\rho^p}{(\sigma k_0)^{p+m-3}}, \frac{(16\rho)^p}{(\sigma k_0)^{p+m-3}} \right].$$

Proof We first observe that $\mathcal{Q}_{8\rho}(\theta_*) = (0, \tau_*) + Q^-_{8\rho}(\theta_*)$, where $\tau_* = \theta_*(16\rho)^p$. Then, applying Lemma 2.3.1 (i) to the function w over the cylinder $(0,\tau_*) + Q^-_{8\rho}(\theta_*)$ with the choice $a = \frac{1}{2}$ and $\xi\omega$ replaced by σk_0, we find that if

$$\frac{|\{w < \sigma k_0\} \cap (0,\tau_*) + Q^-_{8\rho}(\theta_*)|}{|Q^-_{8\rho}(\theta_*)|} \leqslant \gamma \frac{[\theta_*(\sigma k_0)^{p+m-3}]^{\frac{N}{p}}}{[1 + \theta_*(\sigma k_0)^{p+m-3}]^{\frac{N+p}{p}}} \stackrel{\text{def}}{=} \delta_*, \tag{3.12}$$

with γ depending only on the data, then either $C^p \rho^p > \gamma(\xi M)^{p+m-1}$ or

$$w(x,\tau) \geqslant \frac{1}{2}\sigma k_0 \qquad \text{in} \ \ (0,\tau_*) + Q^-_{4\rho}(\theta_*).$$

Note that δ_* depends only on the data since we have $\theta_*(\sigma k_0)^{p+m-3} = 1$, by definition of θ_*. Applying Proposition 3.3.2 with $\nu = \delta_*$, we ensure condition (3.12) and hence the assertion is proved. □

End of the proof of Proposition 3.3.1 when $\mathbf{m > 1}$ To prove the claim, it now suffices to translate Proposition 3.3.3 into the original variables. As τ ranges over the interval

$$\left(\frac{(16^p - 4^p)\rho^p}{(\sigma k_0)^{p+m-3}}, \frac{(16\rho)^p}{(\sigma k_0)^{p+m-3}} \right],$$

recalling the definition of k_0, we find that

$$\begin{aligned} b_1 \ &\stackrel{\text{def}}{=} \ \exp\left\{ \frac{2^{p+m-3}(16^p - 4^p)}{(p+m-3)\sigma^{p+m-3}\nu_0} \right\} < f(\tau) \\ &\leqslant \ \exp\left\{ \frac{2^{p+m-3}16^p}{(p+m-3)\sigma^{p+m-3}\nu_0} \right\} \ \stackrel{\text{def}}{=} \ b_2, \end{aligned}$$

where σ, ν_0 are given by Proposition 3.3.3 and Lemma 2.4.1, respectively. It is worth observing that b_1 and b_2 depend only upon the data and are independent of ρ, M and u. Concerning u we obtain

$$u(x,t) \geqslant \frac{\sigma \xi M}{4 b_2} \stackrel{\text{def}}{=} \eta \xi M \tag{3.13}$$

for all $x \in K_{4\rho}$ and every t such that

$$\left(\frac{b_1}{\xi M} \right)^{p+m-3} \nu_0 \rho^p < t \leqslant \left(\frac{b_2}{\xi M} \right)^{p+m-3} \nu_0 \rho^p,$$

or, equivalently,

$$\left(\frac{b_1 \sigma}{4 b_2 \eta\, \xi M} \right)^{p+m-3} \nu_0 \rho^p < t \leqslant \left(\frac{\sigma}{4\eta\, \xi M} \right)^{p+m-3} \nu_0 \rho^p.$$

Choosing b with the following property

$$\frac{b_1\sigma}{4b_2}\left(\frac{\nu_0}{16^p-4^p}\right)^{\frac{1}{p+m-3}} < b \leqslant \frac{\sigma}{4}\left(\frac{\nu_0}{16^p}\right)^{\frac{1}{p+m-3}},$$

we infer that (3.13) holds true in $K_{4\rho}$ for every t such that

$$\left(\frac{b}{\eta\xi M}\right)^{p+m-3}(16^p-4^p)\rho^p \leqslant t \leqslant \left(\frac{b}{\eta\xi M}\right)^{p+m-3}(16\rho)^p. \qquad \square$$

***Proof of Proposition 3.3.1 when* $\mathbf{m<1}$** We maintain the definitions (3.7) of w, and (3.8) of k_0. By formal computations we obtain

$$w_\tau \geqslant \operatorname{div}\widetilde{A}(x,\tau,w,Dw)+\widetilde{B}(x,\tau,w,Dw),$$

where now

$$\widetilde{A}(x,\tau,w,Dw)=\psi(\tau)^{p+m-2}A(x,\psi(\tau)^{p+m-3},\psi^{-1}w,\psi^{-1}Dw),$$
$$\widetilde{B}(x,\tau,w,Dw)=\psi(\tau)^{p+m-2}B(x,\psi(\tau)^{p+m-3},\psi^{-1}w,\psi^{-1}Dw),$$

with ψ as before, and A, B satisfying (2.3). The new functions $\widetilde{A}$, $\widetilde{B}$ preserve the structure conditions (2.3). Indeed, it is easily checked that

$$\begin{cases} \widetilde{A}(x,\tau,w,\eta)\cdot\eta \geqslant C_0 w^{m-1}|\eta|^p - C^p w^{p+m-1}, \\ |\widetilde{A}(x,\tau,w,\eta)| \leqslant C_1 w^{m-1}|\eta|^{p-1} + C^{p-1}w^{p+m-2}, \\ |\widetilde{B}(x,\tau,w,\eta)| \leqslant C w^{m-1}|\eta|^{p-1} + C^p w^{p+m-2}, \end{cases}$$

where C_0, C_1, C are the same constants of (2.3); w then satisfies energy estimates like (2.9)

$$\begin{aligned} &\sup_{0(16\rho)^p}\int_{K_{16\rho}(y)}(w-k)_-^2\zeta^p(x,t)dx \\ &\quad + \varpi\, k^{m-1}\iint_{Q^+_{16\rho}(\theta)}|D[(w-k)_-\zeta]|^p dxdt \\ &\quad \leqslant \bar{\gamma}\bigg(k^2\iint_{Q^+_{16\rho}(\theta)}\chi_{[w<k]}\zeta\zeta_t dxdt \\ &\quad + k^{m+p-1}\iint_{Q^+_{16\rho}(\theta)}\chi_{[w<k]}(\phi+\zeta^p+|D\zeta|^p)dxdt\bigg), \end{aligned}$$

where

$$l=\frac{m+p-2}{p-1}, \qquad \phi=\phi(C,\zeta,D\zeta)=C^p\zeta^p+|D\zeta|^p,$$

ζ is a piecewise smooth cutoff function in the cylinder $Q^+_{16\rho}(\theta)$, vanishing on the parabolic boundary of $Q^+_{16\rho}(\theta)$, and such that $0\le\zeta\le 1, \zeta_\tau \ge 0$. In particular we set

$$\mathcal{Q}_{8\rho}(\theta)=K_{8\rho}\times\big((16\rho)^p\theta-(8\rho)^p\theta,(16\rho)^p\theta\big],$$

and we require

$$\zeta = 1 \text{ in } \mathcal{Q}_{8\rho}(\theta), \qquad |D\zeta| \leq \frac{1}{8\rho}, \qquad \zeta_\tau \leq \frac{1}{\theta(8\rho)^p}.$$

With these choices, the previous energy estimates become

$$\iint_{\mathcal{Q}_{8\rho}(\theta)} |D(w-k)_-|^p dx \leq \gamma k^{1-m} \left[\frac{k^2}{\theta(8\rho)^p} + \frac{k^{p+m-1}}{(8\rho)^p} \right].$$

Starting from these estimates, we can prove the following two statements, whose proofs are analogous to those of Proposition 3.3.2 and Proposition 3.3.3, respectively.

Proposition 3.3.4 *For every $\nu > 0$ there exist $\sigma \in (0,1)$, depending upon the data, and ν, $\gamma > 1$ depending on the data and σ, such that either $(C\rho)^p > 1$ or*

$$|\{w < \sigma k_0\} \cap \mathcal{Q}_{8\rho}(\theta_*)| \leqslant \nu |\mathcal{Q}_{8\rho}(\theta_*)|,$$

with $\theta_* = (\sigma k_0)^{p+m-3}$ and k_0 given in (3.8).

Proposition 3.3.5 *There exist $\sigma \in (0,1)$ and $\gamma > 1$, depending only upon the data, such that either $(C\rho)^p > 1$ or*

$$w(x,\tau) \geqslant \frac{1}{2}\sigma k_0 \qquad \text{in } \; K_{4\rho} \times \left(\frac{(16^p - 4^p)\rho^p}{(\sigma k_0)^{p+m-3}}, \frac{(16\rho)^p}{(\sigma k_0)^{p+m-3}} \right].$$

Translating Proposition 3.3.5 into the original variables we get the claim of Proposition 3.3.1 also in the case $m < 1$. $\square$

The stability of the constants in the expansion of positivity will be discussed in the next chapter, together with the analogous result for the singular case.

3.4 Intrinsic Harnack inequality

Theorem 3.4.1 *Let u be a continuous, non-negative, local weak solution to (2.1) in E_T. Let $(x_0,t_0) \in E_T$ be such that $u(x_0,t_0) > 0$. Then there exist constants $c, \gamma > 0$ and $\kappa > 1$, depending only upon the data, such that for all cylinders*

$$(x_0,t_0) + Q^{\pm}_{4\rho}(\theta) \subset E_T, \qquad \theta = \left(\frac{c}{u(x_0,t_0)} \right)^{p+m-3},$$

either $C\rho > \min\{1, \gamma u(x_0,t_0)^{\frac{p+m-1}{p}}\}$ or

$$\kappa^{-1} \sup_{K_\rho(x_0)} u(x, t_0 - \theta\rho^p) \leq u(x_0,t_0) \leqslant \kappa \inf_{K_\rho(x_0)} u(x, t_0 + \theta\rho^p). \tag{3.14}$$

Let us fix a point $(x_0,t_0) \in E_T$ with $u(x_0,t_0) > 0$. Let us consider the intrinsic cylinders

$$(x_0,t_0) + Q^{\pm}_{4\rho}(\theta), \qquad \theta = \left(\frac{c}{u(x_0,t_0)} \right)^{p+m-3},$$

where c is to be determined. By the following change of variables

$$x' = \frac{x - x_0}{\rho}, \qquad t' = u(x_0, t_0)^{p+m-3} \frac{t - t_0}{\rho^p},$$

these cylinders become

$$Q^+ = K_4 \times (0, 4^p c^{p+m-3}], \qquad Q^- = K_4 \times (-4^p c^{p+m-3}, 0].$$

Moreover, the rescaled function

$$v(x', t') = \frac{1}{u(x_0, t_0)} u \left(x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+m-3}} \right) \tag{3.15}$$

satisfies $v(0, 0) = 1$ and is a non-negative, local weak solution to

$$v_{t'} - \operatorname{div}_{x'} \mathcal{A}(x', t', v, D_{x'} v) = \mathcal{B}(x', t', v, D_{x'} v)$$

with

$$\begin{aligned} \mathcal{A}(x', t', v, D_{x'} v) = {} & \frac{\rho^{p-1}}{u(x_0, t_0)^{p+m-2}} \times \\ & A\Big(x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+m-1}}, u(x_0, t_0) v, \frac{u(x_0, t_0)}{\rho} D_{x'} v \Big), \end{aligned}$$

$$\begin{aligned} \mathcal{B}(x', t', v, D_{x'} v) = {} & \frac{\rho^{p}}{u(x_0, t_0)^{p+m-2}} \times \\ & B\Big(x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+m-3}}, u(x_0, t_0) v, \frac{u(x_0, t_0)}{\rho} D_{x'} v \Big). \end{aligned}$$

One can check the following structure conditions

$$m > 1 \quad \begin{cases} \mathcal{A}(x', t', v, \eta) \cdot \eta \geqslant C_0 |v|^{m-1} |\eta|^p - \check{C}^p, \\ |\mathcal{A}(x', t', v, \eta)| \leqslant C_1 |v|^{m-1} |\eta|^{p-1} + \check{C}^{p-1} |v|^{\frac{m-1}{p}}, \\ |\mathcal{B}(x', t', v, \eta)| \leqslant \bar{C} |v|^{m-1} |\eta|^{p-1} + \bar{C} \check{C}^{p-1} |v|^{\frac{m-1}{p}}, \end{cases}$$

with

$$\bar{C} = C\rho, \qquad \check{C} = \frac{C\rho}{u(x_0, t_0)^{1 + \frac{m-1}{p}}},$$

or

$$m < 1 \quad \begin{cases} \mathcal{A}(x', t', v, \eta) \cdot \eta \geqslant C_0 |v|^{m-1} |\eta|^p - (\rho C)^p |v|^{p+m-1}, \\ |\mathcal{A}(x', t', v, \eta)| \leqslant C_1 |v|^{m-1} |\eta|^{p-1} + (\rho C)^{p-1} |v|^{p+m-2}, \\ |\mathcal{B}(x', t', v, \eta)| \leqslant \bar{\rho} C |v|^{m-1} |\eta|^{p-1} + (\rho C)^p |v|^{p+m-2}, \end{cases}$$

where C_0, C_1, and C are the same constants of (2.2)–(2.3).
In order to keep the notation simple, from now on we will write (x, t) instead of (x', t'). Establishing the right-hand inequality of (3.14) in Theorem 3.4.1 is equivalent to proving the following theorem.

Theorem 3.4.2 *There exist constants $\gamma, \gamma_0 > 0$ and $\gamma_1 > 1$ which depend only upon the data, such that either $C\rho > \min\{1, \gamma u(x_0, t_0)^{\frac{p+m-1}{p}}\}$ or*

$$v(x, \gamma_1) \geqslant \gamma_0 \qquad a.e. \ in \ K_1.$$

We split the proof of Theorem 3.4.2 into three simpler steps.

First Step. Let us introduce the family of nested cylinders $\{Q_\tau\}$, $\tau \in [0, 1)$, with the same vertex $(0, 0)$ defined by

$$Q_\tau = Q^-_\tau(1) = K_\tau \times (-\tau^p, 0],$$

and the families of non-negative numbers $\{m_\tau\}$ and $\{n_\tau\}$ given by

$$m_\tau = \sup_{Q_\tau} v, \qquad n_\tau = (1-\tau)^{-\beta},$$

where $\beta > 0$ is a parameter to be chosen. We point out that the choice of β will involve only the data. Therefore, all the subsequent quantities depending on β will depend on the data, as soon as β is fixed.

Let τ_0 be the largest root of the equation $m_\tau = n_\tau$. It exists because $m_0 = n_0 = 1$ and $n_\tau \to +\infty$ as $\tau \to 1^-$, while m_τ remains bounded. Since v is continuous, there exists $(\bar{x}, \bar{t}\,) \in \overline{Q}_{\tau_0}$ such that

$$v(\bar{x}, \bar{t}\,) = n_{\tau_0} = (1-\tau_0)^{-\beta}. \tag{3.16}$$

Moreover $(\bar{x}, \bar{t}\,) + Q_{\frac{1-\tau_0}{2}} \subset Q_{\frac{1+\tau_0}{2}} \subset Q_1$, so we have

$$\sup_{(\bar{x},\bar{t}\,)+Q_{\frac{1-\tau_0}{2}}} v \leqslant \sup_{Q_{\frac{1+\tau_0}{2}}} v < 2^\beta (1-\tau_0)^{-\beta}. \tag{3.17}$$

Let us consider the cylinder $(\bar{x}, \bar{t}\,) + Q^-_{R_0}(\theta_0)$, with

$$R_0 = \frac{1-\tau_0}{2}, \qquad \theta_0 = M_0^{3-m-p}, \qquad M_0 = 2^\beta (1-\tau_0)^{-\beta},$$

In order to employ the "expansion of positivity" (Proposition 3.3.1), we need to find a time level at which the function v is strictly positive over a whole cube. This is done in the next step, by using a measure-theoretical argument.

Second Step. We need the following technical lemma.

Lemma 3.4.3 *Assume that*

$$\iint_{Q_1} |Dw|^p dxdt \leqslant \alpha, \qquad \left| \left\{ w > \frac{1}{2} \right\} \cap Q_1 \right| > \mu.$$

Then, there exists $\bar{s} \in (-1, -\mu/4]$ such that

$$\int_{K_1} |Dw(\cdot, \bar{s})|^p dx \leqslant \frac{2\alpha}{\mu} \qquad and \qquad \left| \left\{ w(\cdot, \bar{s}) \geqslant \frac{1}{2} \right\} \cap K_1 \right| \geqslant \frac{\mu}{2}. \tag{3.18}$$

Proof See [18, Lemma 9.1]. □

Proposition 3.4.4 *One has either $\rho C > \min\{1, u(x_0,t_0)^{\frac{p+m-1}{p}}\}$ or*

$$|\{v \geqslant 2^{-(\beta+1)}M_0\} \cap \{(\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0)\}| > \nu |Q^-_{\frac{R_0}{2}}(\theta_0)|, \tag{3.19}$$

where ν is defined by (2.28) with the choices $\xi = 1-2^{-\beta-1}$, $a = \xi^{-1}(1-3/2^{\beta+2})$, and $\mu_+ = \omega = M_0$, $\theta = \theta_0$. Note that ν depends on the data and β.

Proof If $|\{v \geqslant 2^{-(\beta+1)}M_0\} \cap \{(\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0)\}| \leqslant \nu |Q^-_{\frac{R_0}{2}}(\theta_0)|$, and $\rho C \leq \min\{1, u(x_0,t_0)^{\frac{p+m-1}{p}}\}$ then, by Lemma 2.3.1 (ii), with the indicated choice of the involved parameters, one gets

$$v(\bar{x},\bar{t}) \leqslant \frac{3}{4}(1-\tau_0)^{-\beta},$$

which would contradict (3.16). □

From now on we assume that $\rho C \leq \min\{1, u(x_0,t_0)^{\frac{p+m-1}{p}}\}$. It follows that (3.19) is satisfied, hence the set where v is bounded away from a given quantity occupies a sizable portion of the cylinder $(\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0)$. The next proposition asserts that there exists at least one subcylinder such that v remains large in any arbitrarily prefixed large portion of the subcylinder.

Proposition 3.4.5 *For every $\lambda_0 \in (0,1)$ and for every $\nu_0 \in (0,1)$, there exist $(y,s) \in (\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0)$, and a constant $\eta_0 \in (0,1)$, depending only upon the data, ν_0, λ_0, β, such that*

$$(y,s) + Q^-_{2\eta_0 R_0}(\theta_0) \subset (\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0),$$

and

$$|\{v < \lambda_0 2^{-(\beta+1)}M_0\} \cap \{(y,s) + Q^-_{\eta_0 R_0}(\theta_0)\}| \leqslant \nu_0 |Q^-_{\eta_0 R_0}(\theta_0)|. \tag{3.20}$$

Proof Let us first assume $m > 1$. Set

$$k = \frac{1}{2}(1-\tau_0)^{-\beta} = 2^{-(\beta+1)}M_0,$$

and consider the cylinders $(\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0) \subset (\bar{x},\bar{t}) + Q^-_{R_0}(\theta_0) \subset Q_{\frac{1+\tau_0}{2}}$. We write the energy estimates (2.8) for $\left(v - \frac{k}{2}\right)_+$ on $(\bar{x},\bar{t}) + Q^-_{R_0}(\theta_0)$, with the choice of a cutoff function ζ such that

$$\begin{aligned}
&\zeta = 1 && \text{in } (\bar{x},\bar{t}) + Q^-_{\frac{R_0}{2}}(\theta_0)\\
0 \leqslant \zeta_t \leqslant \frac{4^p}{\theta_0 R_0^p},\ &|D\zeta| \leqslant \frac{4}{R_0} && \text{in } (\bar{x},\bar{t}) + Q^-_{R_0}(\theta_0)\\
&\zeta = 0 && \text{on the parabolic boundary of } (\bar{x},\bar{t}) + Q^-_{R_0}(\theta_0).
\end{aligned}$$

Since $(v-k/2)_+ \leqslant v \leqslant 2^{\beta+1}k$ in $(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)$, due to (3.17), discarding the term containing the essential supremum, and using the fact that $R_0 \leqslant 1$, $2k>1$, $\check{C},\bar{C}\leqslant 1$, we get

$$\begin{aligned}
&\gamma_1\left(\frac{k}{2}\right)^{m-1}\iint_{(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)}\left|D\left(v-k/2\right)_+\right|^p\zeta^p dxd\tau\\
&\leqslant \gamma\iint_{(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)}\Big(v^{m-1}\left(v-k/2\right)^p_+|D\zeta|^p+\left(v-k/2\right)^2_+\zeta_t\Big)dxd\tau\\
&+\gamma\Big(\iint_{(\bar{x},\bar{t}\,)+Q^-_{R_0}(\theta_0)}\Big(\bar{C}^p v^{m-1}(v-k/2)^p_+ + \check{C}^p\chi_{\{(v-k/2)_+>0\}}\Big)\zeta^p dxd\tau\\
&\leqslant \gamma\frac{k^{p+m-1}}{R_0^p}|Q^-_{R_0}(\theta_0)|,
\end{aligned}$$

where γ is a constant depending upon the data and β. It follows, in particular, that

$$\iint_{(\bar{x},\bar{t}\,)+Q^-_{\frac{R_0}{2}}(\theta_0)}\left|D\left(v-k/2\right)_+\right|^p dxd\tau \leqslant \gamma\frac{k^p}{R_0^p}|Q^-_{R_0}(\theta_0)|. \tag{3.21}$$

Now, with respect to the new coordinates

$$x'=\frac{2(x-\bar{x})}{R_0}, \qquad t'=\frac{2^p(t-\bar{t}\,)}{\theta_0 R_0^p},$$

the cylinder

$$(\bar{x},\bar{t}\,)+Q^-_{\frac{R_0}{2}}(\theta_0)$$

becomes

$$Q_1 = K_1\times(-1,0].$$

Moreover, by (3.19) and (3.21), the function

$$w(x',t')=\frac{\big(v(x,t)-\frac{k}{2}\big)_+}{k}$$

satisfies

$$\left|\left\{w\geqslant\frac{1}{2}\right\}\cap Q_1\right|>\nu \qquad\text{and}\qquad \iint_{Q_1}|Dw|^p\leqslant\gamma,$$

respectively. Then, Lemma 3.4.3 applies and we get the existence of $\bar{s}\in(-1,-\nu/4])$ such that (3.18) is satisfied. At this point, by the result of [17] we find that for every $\bar{\lambda}$, $\bar{\nu}\in(0,1)$ there exist $\bar{y}\in K_1$, and $\bar{\varepsilon}\in(0,1)$, which can be determined a priori only in terms of $N,p,\bar{\nu},\bar{\lambda},\gamma$ and ν, such that

$$K_{\bar{\varepsilon}}(\bar{y})\subset K_1 \qquad\text{and}\qquad \left|\left\{w(\cdot,\bar{s})>\frac{\bar{\lambda}}{2}\right\}\cap K_{\bar{\varepsilon}}(\bar{y})\right|>(1-\bar{\nu})|K_{\bar{\varepsilon}}|.$$

Returning to the original variables and the original function v, we find that there exist $\hat{s}\in\big(\bar{t}-\theta_0(R_0/2)^p,\bar{t}-\theta_0(\nu/4)(R_0/2)^p]$, $\hat{y}\in K_{\frac{R_0}{2}}(\bar{x})$ and $\bar{\varepsilon}\in(0,1)$ such that $K_{\frac{\bar{\varepsilon}R_0}{2}}(\hat{y})\subset K_{\frac{R_0}{2}}(\bar{x})$ and

$$\left|\left\{v(\cdot,\hat{s})<\frac{\bar{\lambda}+1}{2}k\right\}\cap K_{\frac{\bar{\varepsilon}R_0}{2}}(\hat{y})\right|<\bar{\nu}\left|K_{\frac{\bar{\varepsilon}R_0}{2}}\right|. \tag{3.22}$$

In order to extend the previous inequality to a cylinder, we consider

$$s = \hat{s} + \bar{\theta}\left(\frac{\bar{\varepsilon}R_0}{2}\right)^p, \qquad \text{with} \quad \bar{\theta} = \bar{\nu}^p \theta_0,$$

and we write the energy estimates (2.8) for $(v - \lambda k)_-$, where $\lambda = \frac{\bar{\lambda}+1}{2}$, over the cylinders

$$(\hat{y}, s) + Q^-_{\frac{\bar{\varepsilon}R_0}{4}}(\bar{\theta}) \subset (\hat{y}, s) + Q^-_{\frac{\bar{\varepsilon}R_0}{2}}(\bar{\theta}).$$

The cutoff function ζ is chosen independent of t with $\zeta = 1$ on $K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})$, $\zeta = 0$ on the boundary of $K_{\frac{\bar{\varepsilon}R_0}{2}}(\hat{y})$, and such that $0 \leqslant \zeta \leqslant 1$, $|D\zeta| \leqslant 4\,(\bar{\varepsilon}R_0)^{-1}$. Discarding the term containing $|Dv|$, we obtain

$$\int_{K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})} (v - \lambda k)^2_-(x,t)dx \leqslant \int_{K_{\frac{\bar{\varepsilon}R_0}{2}}(\hat{y})} (v - \lambda k)^2_-(x,\hat{s})dx + \frac{\gamma k^{p+m-1}}{(\bar{\varepsilon}R_0)^p}\left|Q^-_{\frac{\bar{\varepsilon}R_0}{2}}(\bar{\theta})\right| \tag{3.23}$$

for every t such that $s - \bar{\theta}\left(\frac{\bar{\varepsilon}R_0}{2}\right)^p < t \leqslant s$. Since $\bar{\lambda} < \lambda$, we can estimate the left-hand side from below as follows

$$\int_{K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})} (v - \lambda k)^2_-(x,t)dx > \frac{1}{4}(1-\bar{\lambda})^2 k^2 \left|\{v(\cdot,t) < \bar{\lambda}k\} \cap K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})\right|$$

for every t such that $s - \bar{\theta}\left(\frac{\bar{\varepsilon}R_0}{2}\right)^p < t \leqslant s$. Concerning the right-hand side of (3.23), by (3.22) we have

$$\begin{aligned}\int_{K_{\bar{\varepsilon}R_0/2}(\hat{y})} (v - \lambda k)^2_-(x,\hat{s})dx &\leqslant (\lambda k)^2 \left|\{v(\cdot,\hat{s}) < \lambda k\} \cap K_{\bar{\varepsilon}R_0/2}(\hat{y})\right| \\ &\leqslant \gamma k^2 \bar{\nu} \left|K_{\bar{\varepsilon}R_0/4}\right|;\end{aligned}$$

moreover, referring to the definitions of $\bar{\theta}, \theta_0, k$, we get

$$\frac{\gamma k^{p+m-1}}{(\bar{\varepsilon}R_0)^p}\left|Q^-_{\frac{\bar{\varepsilon}R_0}{2}}(\bar{\theta})\right| = \frac{\gamma k^{p+m-1}}{(\bar{\varepsilon}R_0)^p}\bar{\theta}\Big(\frac{\bar{\varepsilon}R_0}{2}\Big)^p \left|K_{\frac{\bar{\varepsilon}R_0}{2}}\right| \leqslant \gamma k^2 \bar{\nu}\left|K_{\frac{\bar{\varepsilon}R_0}{4}}\right|. \tag{3.24}$$

Combining (3.23)–(3.24) we obtain

$$\left|\{v(\cdot,t) < \bar{\lambda}k\} \cap K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})\right| < \frac{\gamma\bar{\nu}}{(1-\bar{\lambda})^2}\left|K_{\frac{\bar{\varepsilon}R_0}{4}}\right| \tag{3.25}$$

for every t such that $s - \theta_0\left(\frac{\bar{\nu}\bar{\varepsilon}R_0}{4}\right)^p < t \leqslant s$.

Finally, we are ready to prove the thesis. Let us fix $\lambda_0 \in (0,1)$ and $\nu_0 \in (0,1)$. Choose $\bar{\lambda} = \lambda_0$ and $\bar{\nu} \in (0,1)$ such that $\frac{\gamma\bar{\nu}}{(1-\lambda_0)^2} \leqslant \nu_0$. Without loss of generality, we may suppose that $\bar{\nu}^{-1}$ is an integer. Let $\hat{y}, \bar{\varepsilon}$ be determined as above. We consider a partition of the cube $K_{\frac{\bar{\varepsilon}R_0}{4}}(\hat{y})$, up to a set of measure zero, into $\bar{\nu}^{-N}$ pairwise disjoint cubes congruent to $K_{\frac{\bar{\nu}\bar{\varepsilon}R_0}{4}}(\hat{y})$. For $j = 1, \ldots, \bar{\nu}^{-N}$, let y_j be the centers of such cubes. Up to a set of measure zero, the collection of cylinders

$$(y_j, s) + Q^-_{\eta_0 R_0}(\theta_0), \quad j = 1, \ldots, \bar{\nu}^{-N}, \qquad \text{where } \eta_0 = \frac{\bar{\nu}\bar{\varepsilon}}{4},$$

is a partition of the cylinder $(\hat{y}, s) + Q^{-}_{\frac{\bar{\varepsilon}R_0}{4}}(\bar{\theta})$ into $\bar{\nu}^{-N}$ sub-cylinders, each congruent to $Q^{-}_{\eta_0 R_0}(\theta_0)$. Since we proved (3.25), (3.20) holds true for at least one of these cylinders, and we are finished.

If $m < 1$, the only change we need to apply in the previous proof is to write the energy estimates (2.9) instead of (2.8). Such estimates lead to inequalities (3.21) and (3.23) as in the case $m > 1$. □

Corollary 3.4.6 *There exist* $(y, s) \in (\bar{x}, \bar{t}) + Q^{-}_{\frac{R_0}{2}}(\theta_0)$, *and* $\eta_0 \in (0, 1)$, *such that either* $\max\{\check{C}, \bar{C}\}$ *or*

$$v(x, s) \geqslant \frac{1}{8}(1 - \tau_0)^{-\beta} \qquad \forall x \in K_r(y), \tag{3.26}$$

with

$$r = \frac{\eta_0 R_0}{2} = \frac{1}{4}\eta_0(1 - \tau_0).$$

The constant η_0 *depends only upon* β *and the data.*

Proof Let ν_0 be determined by (2.20) for the choices $\mu_- = 0$, $\omega = M_0$, $\xi = 2^{-(\beta+2)}$, $a = \frac{1}{2}$ and $\theta = \theta_0$. Note that ν_0 depends on the data and on β. Let us fix $\lambda_0 = \frac{1}{2}$. By Proposition 3.4.5 we obtain that the cylinder $(y, s) + Q^{-}_{2\eta_0 R_0}(\theta)$ satisfies (3.20). We conclude the proof by means of Lemma 2.3.1 (i). □

Third Step. Now, combining all the results, we can conclude the proof of Theorem 3.4.2. Assuming (3.26), we apply Proposition 3.3.1 to the weak solution v, defined by (3.15), for the choices $\xi M = \frac{1}{8}(1 - \tau_0)^{-\beta}$ and $2\rho = r$. We have either $\max\{\bar{C}^p, \check{C}\}r^p > \gamma\big(\frac{1}{8}(1 - \tau_0)^{-\beta}\big)^{p+m-1}$ or

$$v(x, t) \geqslant \eta\, \xi M$$

for all $x \in K_{2r}(y)$ and for every t in the interval

$$s_1 \overset{\text{def}}{=} s + \left(\frac{b}{\eta\, \xi M}\right)^{p+m-3} (8^p - 2^p) r^p \leqslant t \leqslant s + \left(\frac{b}{\eta\, \xi M}\right)^{p+m-3} 8^p r^p \overset{\text{def}}{=} t_1.$$

In the second case we infer, in particular, that $v(x, s_1) \geqslant \eta\, \xi M$ and $v(x, t_1) \geqslant \eta\, \xi M$ for $x \in K_{2r}(y)$. By applying again the same Proposition, we get that either $\bar{C}^p(2r)^p > \gamma\big(\frac{1}{8}\eta(1 - \tau_0)^{-\beta}\big)^{p+m-1}$ or

$$v(x, t) \geqslant \eta^2\, \xi M$$

for all $x \in K_{4r}(y)$ and for every t in the interval

$$s_1 + \left(\frac{b}{\eta^2\, \xi M}\right)^{p+m-3} (8^p - 2^p)(2r)^p \leqslant t \leqslant t_1 + \left(\frac{b}{\eta^2\, \xi M}\right)^{p+m-3} 8^p (2r)^p.$$

By iteration, we get either $\check{C}^p(2^{k-1}r)^p > \gamma\big(\frac{1}{8}\eta^{k-1}(1 - \tau_0)^{-\beta}\big)^{p+m-1}$ or

$$v(x, t) \geqslant \eta^k \xi M$$

for all $x \in K_{2^k r}(y)$ and for every t in the interval $[s_k, t_k]$ with

$$s_k = s + \left(\frac{b}{\xi M}\right)^{p+m-3} (8^p - 2^p) r^p \sum_{j=1}^{k} \frac{2^{p(j-1)}}{\eta^{(p+m-3)j}}$$

$$t_k = s + \left(\frac{b}{\xi M}\right)^{p+m-3} 8^p r^p \sum_{j=1}^{k} \frac{2^{p(j-1)}}{\eta^{(p+m-3)j}},$$

for all $k = 1, 2, \ldots$.

Now, fix $n \in \mathbf{N}$ such that

$$2^{-n} \leqslant \frac{1}{8}\,\eta_0(1-\tau_0) < 2^{-n+1}.$$

It follows that

$$2 \leqslant 2^n r < 4 \tag{3.27}$$

and

$$2^{-3\beta-3}\eta_0^\beta(2^\beta\eta)^n \geqslant \eta^n\xi M > 2^{-4\beta-3}\eta_0^\beta(2^\beta\eta)^n.$$

We choose β such that $2^\beta\eta = 1$. Once β is fixed (depending only on the data), also η_0 turns out to depend only on the data. Then, in particular,

$$\eta^n\xi M > 2^{-4\beta-3}\eta_0^\beta \stackrel{\text{def}}{=} \gamma_0 \in (0,1). \tag{3.28}$$

Now, we have to distinguish two cases.

First Case: there exists $k \leqslant n$ such that $\check{C}^p(2^{k-1}r)^p > \gamma\big(\frac{1}{8}\eta^{k-1}(1-\tau_0)^{-\beta}\big)^{p+m-1}$. Then we have also $\check{C}^p(2^n r)^p > \gamma\big(\frac{1}{8}\eta^n(1-\tau_0)^{-\beta}\big)^{p+m-1}$. From (3.27) and (3.28) it follows that

$$\check{C}^p 4^p \geqslant \check{C}^p(2^n r)^p > \gamma\Big(\frac{1}{8}\eta^n(1-\tau_0)^{-\beta}\Big)^{p+m-1} \geqslant \gamma(\text{data}).$$

Recalling the definition of $\check{C}$, this is equivalent to saying that

$$C\rho \geqslant \gamma\, u(x_0,t_0)^{1+\frac{m-1}{p}},$$

with $\gamma = \gamma(\text{data})$.

Second Case: We have $\check{C}^p(2^{n-1}r)^p \leqslant \gamma\big(\frac{1}{8}\eta^{n-1}(1-\tau_0)^{-\beta}\big)^{p+m-1}$. Then

$$v(x,t) \geqslant \eta^n\xi M$$

for all $x \in K_{2^n r}(y)$ and for every t in the interval $[s_n, t_n]$. Taking (3.28) and (3.27) into account we infer that

$$v(x,t) \geqslant \gamma_0,$$

for every $x \in K_1 \subset K_2(y) \subset K_{2^n r}(y)$ and $t \in [s_n, t_n]$. It remains to estimate the time interval. Using (3.27) and (3.28) we have

$$t_n \geqslant s + \left(\frac{b}{\xi M}\right)^{p+m-3} r^p\,\frac{2^{p(n+2)}}{\eta^{(p+m-3)n}} \geqslant -1 + 8^p\left(\frac{8^{\beta+1}\,\bar{b}}{\eta_0^\beta}\right)^{p+m-3}.$$

If the right-hand side is larger than 1, we are done. Otherwise, we iterate the procedure k times more, until $t_{n+k} > 1$. Note that

$$t_{n+k} \geqslant -1 + 8^p\left(\frac{8^{\beta+1}\,\bar{b}}{\eta_0^\beta}\right)^{p+m-3}\frac{2^{pk}}{\eta^{k(p+m-3)}},$$

so that the choice of k is independent of u and depends only on the data. It follows that there exists $t = \gamma_1 > 1$ such that

$$v(x,\gamma_1) \geqslant \gamma_0 \qquad \text{for all } x \in K_1.$$

Thus,Theorem 3.4.2 is proved. Recalling (3.15), we can write the previous inequality in terms of u, and we obtain

$$u(x_0,t_0) \leqslant \frac{1}{\gamma_0} \inf_{K_\rho(x_0)} u(x, t_0+\theta\rho^p)$$

with

$$c = \gamma_1^{\frac{1}{p+m-3}} \qquad \text{and} \qquad \theta = \Big(\frac{c}{u(x_0,t_0)}\Big)^{p+m-3},$$

which is the right-hand side of (3.14). We now proceed to prove the left-hand side.

Once more fix $(x_0,t_0) \in E_T$, assume $u(x_0,t_0) > 0$, and let $(x_0,t_0) + Q^{\pm}_{4\rho}(\theta)$, with θ as above. Seek those values of $t < t_0$, if any, for which

$$u(x_0,t) = 2\kappa u(x_0,t_0), \tag{3.29}$$

where κ is the constant in the right-hand side of (3.14), which holds for all such cylinders. If such a t does not exist

$$u(x_0,t) < 2\kappa u(x_0,t_0) \tag{3.30}$$

for all $t \in (t_0 - \theta(4\rho)^p, t_0)$. We establish by contradiction that this in turn implies

$$\sup_{K_\rho(x_0)} u(\cdot, t_0-\theta\rho^p) \leq 2\kappa^2 u(x_0,t_0). \tag{3.31}$$

If not, by continuity there exists $x_* \in K_\rho(x_0)$ such that

$$u(x_*, t_0-\theta\rho^p) = 2\kappa^2 u(x_0,t_0).$$

Apply the intrinsic, forward inequality in (3.14) with (x_0,t_0) replaced by $(x_*, t_0-\theta\rho^p)$, to get

$$u(x_*, t_0-\theta\rho^p) \leq \kappa \inf_{K_\rho(x_*)} u(\cdot, t_0-\theta\rho^p+\theta_*\rho^p) \tag{3.32}$$

where

$$\theta_* = \Big(\frac{c}{u(x_*, t_0-\theta\rho^p)}\Big)^{m+p-3}.$$

Now $x_0 \in K_\rho(x_*)$ and, since $\kappa > 1$ and $m+p > 3$,

$$\begin{aligned} t_0-\theta\rho^p+\theta_*\rho^p &= t_0 - \Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p + \Big(\frac{c}{u(x_*, t_0-\theta\rho^p)}\Big)^{m+p-3}\rho^p \\ &= t_0 - \Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p + \frac{1}{(2\kappa^2)^{m+p-3}}\Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p \\ &= t_0 - [1-(2\kappa^2)^{3-m-p}]\Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p < t_0. \end{aligned}$$

Therefore from (3.30)-(3.32)

$$\begin{aligned} 2\kappa^2 u(x_0,t_0) = u(x_*, t_0-\theta\rho^p) &\leq \kappa u(x_0, t_0-\theta\rho^p+\theta_*\rho^p) \\ &< 2\kappa^2 u(x_0,t_0). \end{aligned} \tag{3.33}$$

The contradiction establishes (3.31). We now prove that there exists $t < t_0$ satisfying (3.29). Let $\tau < t_0$ be the first time for which (3.29) holds. For such a time

$$t_0 - \tau > \Big(\frac{c}{u(x_0,\tau)}\Big)^{m+p-3}\rho^p = \frac{1}{(2\kappa)^{m+p-3}}\Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p. \tag{3.34}$$

Indeed if such inequality were violated, applying the intrinsic, forward Harnack inequality in (3.14) with (x_0, t_0) replaced by (x_0, τ) would give

$$2\kappa u(x_0,t_0) = u(x_0,\tau) \le \kappa u(x_0,t_0).$$

Set

$$s = t_0 - \frac{1}{(2\kappa)^{m+p-3}}\Big(\frac{c}{u(x_0,t_0)}\Big)^{m+p-3}\rho^p.$$

From the definitions, the continuity of u and (3.34)

$$\tau < s < t_0 \qquad \text{and} \qquad u(x_0,s) \le 2\kappa u(x_0,t_o).$$

We claim that

$$u(y,s) < 2\kappa u(x_0,t_0) \qquad \text{for all } y \in K_\rho(x_0). \tag{3.35}$$

Proceeding by contradiction, let $y \in K_\rho(x_0)$ be such that

$$u(y,s) = 2\kappa u(x_0,t_0).$$

Apply the intrinsic, forward inequality in (3.14) with (x_0, t_0) replaced by (y, s) to obtain

$$u(y,s) \le \kappa \inf_{K_\rho(y)} u(\cdot, s+\theta_s\rho^p), \qquad \text{where } \theta_s = \Big(\frac{c}{u(y,s)}\Big)^{m+p-3}.$$

Using the definition of s and θ_s one computes

$$s + \theta_s\rho^p = t_0.$$

Therefore, since $y \in K_\rho(x_0)$

$$2\kappa u(x_0,t_0) = u(y,s) \le \kappa \inf_{K_\rho(y)} u(\cdot,t_0) \le \kappa u(x_0,t_0).$$

The contradiction implies that (3.35) holds true. Summarizing the results of these alternatives, either (3.31) holds or (3.35) is in force. The proof is now concluded by using the arbitrariness of ρ and by properly redefining κ. □

The stability of the constants in the intrinsic Harnack inequality will be discussed in the next chapter, together with the analogous result for the singular (super-critical) case.

3.5 Hölder continuity for non-negative solutions

The aim of the present section is to show that the intrinsic Harnack inequality implies a local Hölder continuity condition. Up to a translation, assume that the initial cylinder

$$Q_{R_0} = K_{R_0} \times (-R_0^{p-\xi}, 0],$$

with $0 \leqslant \xi < \min\{p, p+m-3\}$, is contained in the domain of u, which is a non-negative, local weak solution to (2.1). Set

$$\omega_0 = \operatorname*{osc}_{Q_{R_0}} u = \sup_{Q_{R_0}} u.$$

Let us define the intrinsic cylinder

$$Q_0 = K_{R_0} \times (-\theta_0 R_0^p, 0], \qquad \theta_0 = \left(\frac{c}{\omega_0}\right)^{p+m-3}$$

where $c > 0$ is to be determined only in dependence of the data. If $\omega_0 \leqslant cR_0^{\frac{\xi}{p+m-3}}$ for every cylinder as Q_{R_0} (keeping the same constant c), then u turns out to be locally Hölder continuous (see [39]). Thus, assume that there exists R_0 such that $\omega_0 > cR_0^{\frac{\xi}{p+m-3}}$. In this case, we have that $Q_0 \subset Q_{R_0}$ and, consequently,

$$\operatorname*{osc}_{Q_0} u \leqslant \omega_0.$$

The aim of the next theorem is to show that we can construct a sequence of nested and shrinking intrinsic cylinders $\{Q_n\}$ with the same vertex, such that the oscillation of u in Q_n tends to zero, as $n \to \infty$, in a way that can be quantitatively determined by means of the structure conditions (2.2)-(2.3). We point out that the proof of such a result is a little bit more involved than the one given in [18] since, in general, $\omega_0 - u$ is not a solution to (2.1). This fact is clear in the case of the model equation (2.4). The Hölder continuity will then follow from [39].

Theorem 3.5.1 *There exist positive constants c, γ and $\delta, \varepsilon \in (0,1)$, that can be quantitatively determined only in terms of the data such that, setting*

$$R_n = \varepsilon R_{n-1}, \quad \omega_n = \max\left\{\delta\omega_{n-1}, \gamma\big(CR_{n-1}\big)^{\frac{p}{p+m-1}}\right\},$$

$$\theta_n = \left(\frac{c}{\omega_n}\right)^{p+m-3}, \quad Q_n = Q^-_{R_n}(\theta_n),$$

for $n \in \mathbf{N}$, there hold $Q_{n+1} \subset Q_n$ and

$$\operatorname*{osc}_{Q_n} u \leq \omega_n.$$

Theorem 3.5.1 can be proved by using an iterative argument. For the sake of simplicity, we limit ourselves to the first iteration.

Let $P_0 = (0, -\theta_0 R_0^p/2)$ be the mid point of Q_0. We distinguish two cases.

3.5.1 First case

Assume first that $u(P_0) \geqslant \frac{1}{8}\omega_0$.
By Theorem 3.4.1 there exist $c, \kappa, \gamma > 0$, depending only upon the data, such that either

$$\gamma\, u(P_0)^{p+m-1} \leqslant C^p R_0^p \tag{3.36}$$

or

$$\frac{1}{8\kappa}\omega_0 \leqslant \inf_{Q^-_{\frac{1}{4}R_0}(\theta_0)} u(x,t). \tag{3.37}$$

Note that we have used an equivalent formulation of (3.14). Setting

$$\delta = 1 - \frac{1}{8\kappa}, \qquad \varepsilon = \frac{\delta^{\frac{p+m-3}{p}}}{4}, \qquad R_1 = \varepsilon R_0,$$

$$\omega_1 = \max\left\{\delta\omega_0, \gamma_1 (C R_0)^{\frac{p}{p+m-1}}\right\}, \qquad \theta_1 = \left(\frac{c}{\omega_1}\right)^{p+m-3},$$

with $\gamma_1 = 8\,\gamma^{-\frac{1}{p+m-1}}$, it is easily seen that the cylinder $Q_1 = K_{R_1} \times (-\theta_1 R_1^p, 0]$ is contained in $Q^-_{\frac{1}{4}R_0}(\theta_0)$. If (3.36) holds, then

$$\operatorname*{osc}_{Q_1} u \leqslant \omega_0 \leqslant 8u(P_0) \leqslant \gamma_1 (C R_0)^{\frac{p}{p+m-1}} \leqslant \omega_1.$$

If (3.37) is true then

$$\operatorname*{osc}_{Q_1} u \leqslant \delta\omega_0 \leqslant \omega_1.$$

In any case

$$\operatorname*{osc}_{Q_1} u \leqslant \omega_1. \tag{3.38}$$

3.5.2 Second case

Assume now that $u(P_0) < \frac{1}{8}\omega_0$. We are going to show that, also in this case, we can fix (possibly different) values of the constants $\varepsilon, \delta, c, \gamma$, dependent only on the data, such that (3.38) continues to hold.

Let us consider $Q_0^- = P_0 + Q^-_{R_0}\left(\frac{\theta_0}{2}\right)$. If $C^p R_0^p > \left(\frac{1}{2}\omega_0\right)^{p+m-1}$ then we can restart as from (3.36). From now on assume that $C^p R_0^p \leqslant \left(\frac{1}{2}\omega_0\right)^{p+m-1}$. Then

$$\left|\left\{u \leqslant \frac{1}{2}\,\omega_0\right\} \cap Q_0^-\right| \geqslant \nu |Q_0^-|, \tag{3.39}$$

where ν is determined by (2.20), for the choices $a = \xi = 1/2$, $\omega = \mu_+ = \omega_0$, $\theta = \theta_0/2$. Indeed, if (3.39) were not true, then Lemma 2.3.1 would imply that

$$u(x,t) \geqslant \frac{1}{4}\,\omega_0, \qquad \text{in } P_0 + Q^-_{\frac{R_0}{2}}\left(\frac{\theta_0}{2}\right).$$

In particular, $u(P_0) \geqslant \frac{1}{4}\,\omega_0$, which is impossible. Thus (3.39) is established. Note that ν depends only on the data (once c will be fixed only in dependence of the data). As for Proposition 3.4.5, one can see that the following lemma holds.

Lemma 3.5.2 *For every $\lambda > 1$ and $\eta \in (0,1)$, there exist $(y,s) \in Q_0^-$, and $\delta \in (0,1)$ such that $(y,s) + Q^-_{2\delta R_0}\left(\frac{\theta_0}{2}\right) \subset Q_0^-$, and*

$$\left|\left\{u \geqslant \lambda\,\frac{\omega_0}{2}\right\} \cap \left\{(y,s) + Q^-_{\delta R_0}\left(\frac{\theta_0}{2}\right)\right\}\right| \leqslant \eta \left|Q^-_{\delta R_0}\left(\frac{\theta_0}{2}\right)\right|.$$

Now, fix $\lambda = 3/2$ and $\eta = \nu$, where ν is obtained by (2.28) when $\mu_+ = \omega = \omega_0$, $a = 1/2$, $\xi = 1/4$ and $\theta = \theta_0/2$. It follows that there are $(\bar{y}, \bar{s}) \in Q_0^-$, and $\bar{\delta} \in (0,1)$ such that $(\bar{y}, \bar{s}) + Q^-_{2\bar{\delta}R_0}\left(\frac{\theta_0}{2}\right) \subset Q_0^-$, and

$$\left|\left\{u \geqslant \frac{3}{4}\,\omega_0\right\} \cap \left\{(\bar{y}, \bar{s}) + Q^-_{\bar{\delta}R_0}\left(\frac{\theta_0}{2}\right)\right\}\right| \leqslant \nu \left|Q^-_{\bar{\delta}R_0}\left(\frac{\theta_0}{2}\right)\right|.$$

Hence, Lemma 2.3.1 (ii) yields either $(C\bar{\delta}R_0)^p > 4^{-p}\omega_0^{p+m-1}$ (and in this case we finish the proof, as before), or

$$u \leqslant \frac{7}{8}\,\omega_0 \qquad \text{in } (\bar{y}, \bar{s}) + Q^-_{\bar{\rho}}\left(\frac{\theta_0}{2}\right), \tag{3.40}$$

where we have set

$$\bar{\rho} = \frac{\bar{\delta}R_0}{2}.$$

At this point, we change the time variable by

$$t' = \omega_0^{m-1} t,$$

and set

$$w(x, t') = u(x, t), \qquad v(x, t') = \omega_0 - w(x, t').$$

It turns out that v is a local weak solution to

$$v_{t'} = \mathrm{div} A'(x, t', w, Dv) + B(x, t', w, Dv)$$

with

$$\begin{aligned} A'(x, t', w, Dv) &= -\omega_0^{1-m} A(x, \omega_0^{1-m} t', w, -Dv) \\ B'(x, t', w, Dv) &= -\omega_0^{1-m} B(x, \omega_0^{1-m} t', w, -Dv). \end{aligned} \tag{3.41}$$

Let us momentarily assume $m > 1$. The structure conditions for the new coefficients are the following

$$\begin{cases} A'(x, t', w, \eta) \cdot \eta \geqslant \omega_0^{1-m} C_0 w^{m-1} |\eta|^p - \omega_0^{1-m} C^p, \\ |A'(x, t', w, \eta)| \leqslant \omega_0^{1-m} C_1 w^{m-1} |\eta|^{p-1} + \omega_0^{1-m} C^{p-1} w^{\frac{m-1}{p}} \\ |B'(x, t', w, \eta)| \leqslant \omega_0^{1-m} C w^{m-1} |\eta|^{p-1} + \omega_0^{1-m} C^p w^{\frac{m-1}{p}}. \end{cases}$$

To simplify the notation, from now on we write t instead of t'. The corresponding energy estimates are

$$
\begin{aligned}
&\sup_{s-\theta R^p<t\leqslant s}\int_{K_R(y)}(v-k)_-^2\zeta^p(x,t)dx-\int_{K_R(y)}(v-k)_-^2\zeta^p(x,s-\theta R^p)dx\\
&\qquad+\varpi\omega_0^{1-m}\iint_{(y,s)+Q_R^-(\theta)}w^{m-1}|D(v-k)_-|^p\zeta^p dxdt\\
&\leqslant\gamma\iint_{(y,s)+Q_R^-(\theta)}(v-k)_-^2\zeta^{p-1}\zeta_t dxdt\\
&+\omega_0^{1-m}\iint_{(y,s)+Q_R^-(\theta)}w^{m-1}(v-k)_-^p|D\zeta|^p dxdt\\
&+\gamma\iint_{(y,s)+Q_R^-(\theta)}(C^p\omega_0^{1-m}w^{m-1}(v-k)_-^p+\omega_0^{1-m}C^p\chi_{\{(v-k)_->0\}})\zeta^p dxdt.
\end{aligned}
\tag{3.42}
$$

If we choose levels

$$k\leqslant\frac{1}{8}\,\omega_0,$$

then $w\geqslant\frac{7}{8}\,\omega_0$, whenever $(v-k)_->0$. On the other hand, $w\leqslant\omega_0$ in $K_{R_0}\times(-\theta_0\omega_0^{m-1}R_0^p,0]$, so that (3.42) gives

$$
\begin{aligned}
&\sup_{s-\theta R^p<t\leqslant s}\int_{K_R(y)}(v-k)_-^2\zeta^p(x,t)dx-\int_{K_R(y)}(v-k)_-^2\zeta^p(x,s-\theta R^p)dx\\
&\qquad+\varpi\iint_{(y,s)+Q_R^-(\theta)}|D(v-k)_-|^p\zeta^p dxdt\\
&\leqslant\gamma\iint_{(y,s)+Q_R^-(\theta)}(v-k)_-^2\zeta^{p-1}\zeta_t dxdt\\
&+\gamma\iint_{(y,s)+Q_R^-(\theta)}(v-k)_-^p|D\zeta|^p dxdt\\
&+\gamma\iint_{(y,s)+Q_R^-(\theta)}\Big(C^p(v-k)_-^p+\omega_0^{1-m}C^p\chi_{\{(v-k)_->0\}}\Big)\zeta^p dxdt
\end{aligned}
\tag{3.43}
$$

for every cylinder $(y,s)+Q_R^-(\theta)\subset K_{R_0}\times(-\theta_0\omega_0^{m-1}R_0^p,0]$. Moreover, condition (3.40) leads to

$$v(x,t)\geqslant\frac{1}{8}\,\omega_0,$$

for $(x,t)\in(\bar{y},\omega_0^{m-1}\bar{s})+Q_{\bar{\rho}}^-\left(\frac{\omega_0^{m-1}\theta_0}{2}\right)$. In particular, we have

$$v(\cdot,s_0)\geqslant\frac{1}{8}\,\omega_0,\qquad\text{in } K_{\bar{\rho}}(\bar{y}),\quad\text{with } s_0=\omega_0^{m-1}\bar{s}.$$

Now, arguing as in [18], (note that the energy estimate (3.43) are the same as those considered there), one can check the following proposition.

Proposition 3.5.3 *Let $n \in \mathbf{N}$ be fixed. Then, there exist γ, σ, b_1, b_2, depending only on the data and n, such that either $C^p R_0^p > \gamma\, \omega_0^{p+m-1}$ or*

$$v(x,t) \geqslant \frac{\sigma\omega_0}{16 b_2},$$

in $K_{2^{n-1}\bar\rho}(\bar y) \times (s_0 + t_1, s_0 + t_2)$, where

$$t_i = \left(\frac{8\, b_i}{\omega_0}\right)^{p-2} \delta_0 \bar\rho^p, \qquad i = 1, 2.$$

Going now back to the function u, we find

$$u(x,t) \leqslant (1-\eta)\omega_0, \tag{3.44}$$

for every $x \in K_{2^{n-1}\bar\rho}(\bar y)$ and $t \in \left(\bar s + \omega_0^{1-m} t_1, \bar s + \omega_0^{1-m} t_2\right)$, where

$$\eta = \frac{\sigma}{16\, b_2}.$$

Recalling that $\bar s \in (-\theta_0 R_0^p, -\theta_0 R_0^p/2]$ and choosing n large enough, one can see that it is possible to find a positive constant c satisfying $\bar s + \omega_0^{1-m} t_2 > 0$ and $\bar s + \omega_0^{1-m} t_1 < 0$ and depending only on the data. Now,

$$\begin{aligned} \bar s + \omega_0^{1-m} t_1 &\leqslant -\frac{\theta_0 R_0^p}{2} + \omega_0^{1-m} t_1 \\ &= R_0^p \frac{c^{p+m-3}}{\omega_0^{p+m-3}} \left(-\frac{1}{2} + \frac{(8\, b_1)^{p-2}}{c^{p+m-3}} \frac{\delta_0 \bar\delta^p}{2^p}\right) \\ &= -\theta R_1^p \frac{(1-\eta)^{p+m-3}}{\varepsilon^p} \left(\frac{1}{2} - \frac{(8\, b_1)^{p-2}}{c^{p+m-3}} \frac{\delta_0 \bar\delta^p}{2^p}\right), \end{aligned}$$

where

$$\theta = \left(\frac{c}{(1-\eta)\omega_0}\right)^{p+m-3}, \qquad R_1 = \varepsilon R_0,$$

with $\varepsilon > 0$ to be determined. We require that

$$\frac{(1-\eta)^{p+m-3}}{\varepsilon^p} \left(\frac{1}{2} - \frac{(8\, b_1)^{p-2}}{c^{p+m-3}} \frac{\delta_0 \bar\delta^p}{2^p}\right) \geqslant 1$$

in order to conclude, by the previous estimate, that $\bar s + \omega_0^{1-m} t_1 \leqslant -\theta R_1^p$. This means that (3.44) is true in $K_{2^{n-1}\bar\rho}(\bar y) \times (-\theta R_1^p, 0]$ and, *a fortiori*, in $K_{2^{n-1}\bar\rho}(\bar y) \times (-\theta_1 R_1^p, 0]$, being

$$\theta_1 = \left(\frac{c}{\omega_1}\right)^{p+m-3}, \quad \omega_1 = \max\{(1-\eta)\omega_0, \gamma (C R_0)^{\frac{p}{p+m-1}}\}.$$

Finally, by choosing a possibly larger value of n (or a possibly smaller value of ε), we can ensure that $K_{2^{n-1}\bar\rho}(\bar y) \supset K_{R_1}$. We have then established that

$$\operatorname*{osc}_{Q_1} u \leq \omega_1,$$

where $Q_1 = Q^-_{R_1}(\theta_1)$, and the second case is concluded as before.

If $m < 1$, then the structure conditions for the new coefficients (3.41) are

$$\begin{cases} A'(x,t',w,\eta)\cdot\eta \geqslant \omega_0^{1-m} C_0 w^{m-1}|\eta|^p - \omega_0^{1-m} C^p w^{p+m-1}, \\ |A'(x,t',w,\eta)| \leqslant \omega_0^{1-m} C_1 w^{m-1}|\eta|^{p-1} + \omega_0^{1-m} C^{p-1} w^{p+m-2} \\ |B'(x,t',w,\eta)| \leqslant \omega_0^{1-m} C w^{m-1}|\eta|^{p-1} + \omega_0^{1-m} C^p w^{p+m-2}. \end{cases}$$

As before, to simplify the notation, from now on we write t instead of t'. The corresponding energy estimates are

$$\begin{aligned} &\sup_{s-\theta R^p<t\leqslant s}\int_{K_R(y)} (v-k)_-^2\zeta^p(x,t)dx - \frac{k}{l}\int_{K_R(y)}(v-k)_-\zeta^p(x,s-\theta R^p)dx \\ &\quad + \varpi\, \omega_0^{1-m}k^{m-1}\iint_{(y,s)+Q_R^-(\theta)} |D[(v-k)_-\zeta]|^p dxdt \\ &\quad \leqslant \bar\gamma\Big(k^2\iint_{(y,s)+Q_R^-(\theta)} \chi_{[v<k]}\zeta\zeta_t dxdt \\ &\quad +\omega_0^{1-m}k^{m+p-1}\iint_{(y,s)+Q_R^-(\theta)} \chi_{[v<k]}(C^p\zeta^p+\zeta^p+|D\zeta|^p)dxdt\Big), \end{aligned} \tag{3.45}$$

where

$$l = \frac{m+p-2}{p-1}.$$

If we choose levels

$$k \leqslant \frac{1}{8}\,\omega_0,$$

then $w \geqslant \frac{7}{8}\,\omega_0$, whenever $(v-k)_- > 0$. On the other hand, $w \leqslant \omega_0$ in $K_{R_0}\times(-\theta_0\omega_0^{m-1}R_0^p, 0]$, so that

$$\begin{aligned} &\sup_{s-\theta R^p<t\leqslant s}\int_{K_R(y)} (v-k)_-^2\zeta^p(x,t)dx - \frac{k}{l}\int_{K_R(y)}(v-k)_-\zeta^p(x,s-\theta R^p)dx \\ &\quad + \varpi\iint_{(y,s)+Q_R^-(\theta)} |D[(v-k)_-\zeta]|^p dxdt \\ &\quad \leqslant \bar\gamma\Big(k^2\iint_{(y,s)+Q_R^-(\theta)} \chi_{[v<k]}\zeta\zeta_t dxdt \\ &\quad +k^p\iint_{(y,s)+Q_R^-(\theta)} \chi_{[v<k]}(C^p\zeta^p+\zeta^p+|D\zeta|^p)dxdt\Big), \end{aligned} \tag{3.46}$$

for every cylinder $(y,s)+Q_R^-(\theta)\subset K_{R_0}\times(-\theta_0\omega_0^{m-1}R_0^p,0]$. Moreover, condition (3.40) leads to

$$v(x,t) \geqslant \frac{1}{8}\,\omega_0,$$

for $(x,t)\in(\bar y,\omega_0^{m-1}\bar s)+Q_{\bar\rho}^-\left(\frac{\omega_0^{m-1}\theta_0}{2}\right)$. In particular, we have

$$v(\cdot,s_0) \geqslant \frac{1}{8}\,\omega_0, \qquad \text{in } K_{\bar\rho}(\bar y), \quad \text{with } s_0 = \omega_0^{m-1}\bar s.$$

Now we can conclude the proof as before, applying Proposition 3.5.3. One can check that the previous estimates work as well as (3.43).

Chapter 4

Intrinsic Harnack estimates for some doubly nonlinear singular parabolic equations

4.1 Introduction

In this chapter we analyze the parabolic equations (2.1) to the purpose of extending, where possible, the intrinsic Harnack estimate (3.14) to the singular case ($m+p<3$). Indeed, the weak solutions to (2.1) behave differently depending on $m+p\,(<3)$ being close to either 3 or 2. The critical threshold is

$$m+p+\frac{p}{N}=3.$$

In the super-critical range ($3-\frac{p}{N}<m+p<3$), a form of the Harnack inequality similar to (3.14) holds. For the same range, a "time insensitive" Harnack estimate can be proved. An analysis of the model equation (2.4) suggests that neither of the previous Harnack inequalities holds in the sub-critical range ($2<m+p<3-\frac{p}{N}$); as discussed by Vespri in [54], the solutions to the Cauchy problem

$$\begin{cases} u_t = \mathrm{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \mathbf{R}^N\times(0,\infty) \\ u(x,0)=u_0(x)\in L^1(\mathbf{R}^N)\cap L^{(3-m-p)(N/p)}(\mathbf{R}^N), & u_0(x)\geq 0 \end{cases}$$

become extinct after a finite time, and this contradicts the estimate (3.14). Nevertheless, recent results of DiBenedetto, Gianazza and Vespri [21], for the p-Laplacian and the porous medium equations, suggest that a different form of a Harnack estimate might hold.

As for the degenerate case, the expansion of positivity is a crucial property in the proof of any Harnack inequality; Section 4.2 is devoted to its proof. Section 4.3 treats the super-critical case, yielding to the two Harnack estimates mentioned above. Finally, in Section 4.4, we prove a different form of Harnack estimate, which holds in the sub-critical range, introducing as well its connection with Hölder continuity.

4.2 Expansion of positivity

Proposition 4.2.1 (Expansion of positivity, $\mathbf{p+m-3<0}$) *Assume that for some $(y,s) \in E_T$ and some $\rho > 0$ there holds*

$$|[u(\cdot,s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants $\varepsilon, \delta, \eta \in (0,1)$, depending only upon the data $\{p, m, N, C_0, C_1\}$ and α, and independent of $(y,s), \rho, M$, such that either

$$C\rho > \min\{1, M^{p+m-1}\}$$

or

$$u(\cdot,t) \geq \eta M \qquad \text{in } K_{2\rho}(y)$$

for all times

$$s + (1-\varepsilon)\delta M^{3-m-p}\rho^p \leq t \leq s + \delta M^{3-m-p}\rho^p.$$

4.2.1 Transforming the variables and the equation

Assume $(y,s) = (0,0)$, let δ and ϵ be as determined in Lemma 3.2.1, and let ρ be so that

$$Q_{16\rho}(\delta M^{3-m-p}) = K_{16\rho} \times (0, \delta M^{3-m-p}] \subset E_T.$$

Introduce the change of variables and the new unknown function

$$z = \frac{x}{\rho}, \qquad -e^{-\tau} = \frac{t - \delta M^{3-m-p}\rho^p}{\delta M^{3-m-p}\rho^p}, \qquad v(z,\tau) = \frac{1}{M} u(x,t) e^{\frac{\tau}{3-m-p}}. \tag{4.1}$$

This maps the cylinder $Q_{16\rho}(\delta M^{3-m-p})$ into $K_{16} \times (0,\infty)$ and transforms the equation into

$$v_\tau - \operatorname{div}_z \bar{A}(z,\tau,v,D_z v) = \bar{B}(z,\tau,v,D_z v) + \frac{v}{3-m-p} \tag{4.2}$$

weakly in $K_{16} \times (0,\infty)$, where $\bar{A}$ and $\bar{B}$ are measurable functions of their arguments given by

$$\begin{aligned}
\bar{A}(x,\tau,v,\eta) = \delta M^{3-m-p}\rho^{p-1} e^{-\tau}\psi(\tau) \\
\times A\left(\rho z, \delta M^{3-m-p}\rho^p(1-e^{-\tau}), \frac{v}{\psi(\tau)}, \frac{\eta}{\rho\psi(\tau)}\right), \\
\bar{B}(x,\tau,v,\eta) = \delta M^{3-m-p}\rho^{p} e^{-\tau}\psi(\tau) \\
\times B\left(\rho z, \delta M^{3-m-p}\rho^p(1-e^{-\tau}), \frac{v}{\psi(\tau)}, \frac{\eta}{\rho\psi(\tau)}\right),
\end{aligned}$$

where $\psi(\tau) := \frac{e^{\frac{\tau}{3-m-p}}}{M}$. $\bar{A}, \bar{B}$ satisfy the structure conditions

$$m > 1 : \quad \begin{cases}
\bar{A}(z,\tau,v,\eta)\cdot\eta \geq \delta C_0 |v|^{m-1}|\eta|^p - \delta\tilde{C}^p, \\
|\bar{A}(z,\tau,v,\eta)| \leq \delta C_1 |v|^{m-1}|\eta|^{p-1} + \delta\tilde{C}^{p-1}|v|^{\frac{m-1}{p}}, \\
|\bar{B}(z,\tau,v,\eta)| \leq \delta\bar{C}|v|^{m-1}|\eta|^{p-1} + \delta\bar{C}\tilde{C}^{p-1}|v|^{\frac{m-1}{p}},
\end{cases}$$

$$m<1:\quad\begin{cases}\bar{A}(z,\tau,v,\eta)\cdot\eta\geq\delta[C_0|v|^{m-1}|\eta|^p-(C\rho)^p|v|^{m+p-1}],\\|\bar{A}(z,\tau,v,\eta)|\leq\delta[C_1|v|^{m-1}|\eta|^{p-1}+(C\rho)^{p-1}|v|^{m+p-2}],\\|\bar{B}(z,\tau,v,\eta)|\leq\delta[C|v|^{m-1}|\eta|^{p-1}+(C\rho)^p|v|^{m+p-2}],\end{cases}$$

a.e. in $K_{16}\times(0,\infty)$, with $\bar{C}=\rho C$ and $\tilde{C}=\rho\psi^{\frac{p+m-1}{p}}C$, where C_0,C_1, and C are the original constants in the structure conditions (2.2)-(2.3). In this setting, the information of Lemma 3.2.1 reads

$$|[v(\cdot,\tau)>\epsilon e^{\frac{\tau}{3-m-p}}]\cap K_1|\geq\frac{1}{2}\alpha|K_1|\qquad\forall\tau\in(0,+\infty).$$

Let τ_0 to be chosen and set

$$k_0:=\epsilon e^{\frac{\tau_0}{3-m-p}},\qquad k_j:=\frac{k_0}{2^j}\qquad\text{for } j=0,1,\dots,j_*,$$

where j_* is to be chosen. With this symbolism

$$|[v(\cdot,\tau)>k_j]\cap K_8|\geq\frac{1}{2}\alpha 8^{-N}|K_8|\qquad\forall\tau\in(\tau_0,+\infty),$$

for all $j\in\mathbb{N}$. Introduce the cylinders

$$\begin{aligned}Q_{\tau_0}&=K_8\times(\tau_0+k_0^{3-m-p},\tau_0+2k_0^{3-m-p}),\\Q'_{\tau_0}&=K_{16}\times(\tau_0,\tau_0+2k_0^{3-m-p}),\end{aligned}$$

and a non-negative, piecewise smooth, cutoff function in Q'_{τ_0} of the form $\zeta(z,\tau)=\zeta_1(z)\zeta_2(\tau)$, where

$$\begin{aligned}\zeta_1&=\begin{cases}1 & \text{in } K_8\\0 & \text{in } \mathbb{R}^N\setminus K_{16}\end{cases},\qquad |D\zeta_1|\leq\tfrac{1}{8},\\\zeta_2&=\begin{cases}1 & \text{for } \tau\geq\tau_0+k_0^{3-m-p}\\0 & \text{for } \tau<\tau_0\end{cases},\quad 0\leq\zeta_{2,\tau}\leq\tfrac{1}{k_0^{3-m-p}}.\end{aligned}$$

First assume $m>1$. In the weak formulation take as test function $-(v-k_j)_-\zeta^p$, for the indicated choice of ζ. Performing calculations analogous to those in Section 2.2, we get

$$\begin{aligned}&\iint_{Q_{\tau_0}}v^{m-1}|D(v-k_j)_-|^p dxd\tau\\&\quad\leq\gamma\iint_{Q'_{\tau_0}}[(v-k_j)_-^2\zeta_\tau+\delta v^{m-1}(v-k_j)_-^p|D\zeta|^p]dxd\tau\\&\quad+\gamma\delta\iint_{Q'_{\tau_0}}[(C\rho)^p v^{m-1}(v-k_j)_-^p+(C\rho)^p\chi_{(v-k_j)_->0}]dxd\tau.\end{aligned}$$

Define $\tilde{v}=\max\{\frac{k_j}{2},v\}$. Then, assuming $(C\rho)\leq 1$,

$$\begin{aligned}&\Big(\frac{k_j}{2}\Big)^{m-1}\iint_{Q_{\tau_0}}|D(\tilde{v}-k_j)_-|^p dxd\tau\\&\quad\leq\gamma\Big[\frac{k_j^2}{k_0^{3-m-p}}+\delta\frac{k_j^{p+m-1}}{8^p}+\delta[k_j^{p+m-1}+(C\rho)^p]\Big]|Q'_{\tau_0}|.\end{aligned}$$

Either $(C\rho)^p > (\epsilon M)^{p+m-1}$ for all $\tau \in (\tau_0, +\infty)$, or

$$\iint_{Q_{\tau_0}} |D(\tilde{v} - k_j)_-|^p dx d\tau \leq \gamma(\delta, \text{data}) k_j^p |Q_{\tau_0}|. \tag{4.3}$$

Now assume $m < 1$. In the weak formulation take as test function

$$-(v^l - k_j^l)_- \zeta^p, \qquad l = \frac{p+m-2}{p-1} \in (0,1),$$

for the indicated choice of ζ. Performing calculations analogous to those in Section 2.2, we get

$$\begin{aligned}\iint_{Q_{\tau_0}} &|D(v - k_j) - |^p dz d\tau \\ &\leq \gamma k_j^{1-m} \Big[\frac{k_j^2}{k_0^{3-m-p}} + (1 + (C\rho)^p) k_j^{p+m-1} \Big] |Q'_{\tau_0}| \\ &\leq \gamma(\delta, \text{data}) k_j^p |Q_{\tau_0}|,\end{aligned} \tag{4.4}$$

where we enforced $C\rho \leq 1$.

4.2.2 Estimating the measure of the set $[v < k_j]$ within Q_{τ_0}.

Notice that $[v(\cdot,\tau) < k_j] = [\tilde{v}(\cdot,\tau) < k_j]$ for all j. Set

$$A_j(\tau) = [v(\cdot,\tau) < k_j] \cap K_8 \qquad \text{and} \qquad A_j = [v < k_j] \cap Q_{\tau_0}.$$

By Lemma B.2.3 and (4.3)

$$\begin{aligned}(k_j - k_{j+1})|A_{j+1}(\tau)| &\leq \frac{\gamma(N)}{|K_8 \setminus A_j(\tau)|} \int_{K_8 \cup [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz \\ &\leq \frac{\gamma(N)}{\alpha} \int_{K_8 \cup [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz\end{aligned}$$

for all $\tau \geq \tau_0$. Integrate this in $d\tau$ over $(\tau_0 + k_0^{3-m-p}, \tau_0 + 2k_0^{3-m-p})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use either (4.4) or (4.3) to get

$$\begin{aligned}\frac{k_j}{2}|A_{j+1}| &\leq \gamma(\text{data}, \alpha) \iint_{A_j \setminus A_{j+1}} |Dv| dz \\ &\leq \gamma(\text{data}, \alpha) \Big(\iint_{A_j \setminus A_{j+1}} |Dv|^p dz \Big)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\ &\leq \gamma(\text{data}, \alpha, \delta) \Big(\iint_{Q_{\tau_0}} |D(v - k_j)|^p dz \Big)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\ &\leq \gamma(\text{data}, \alpha, \delta) k_j |Q_{\tau_0}|^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}}.\end{aligned}$$

Hence

$$|A_{j+1}|^{\frac{p}{p-1}} \leq \gamma(\text{data}, \alpha, \delta) |Q_{\tau_0}|^{\frac{1}{p-1}} |A_j \setminus A_{j+1}|.$$

Add these inequalities for $j = 0, 1, \dots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$\begin{aligned} j_* |A_{j_*}|^{\frac{p}{p-1}} &\le \gamma(\text{data}, \alpha, \delta) |Q_{\tau_0}|^{\frac{1}{p-1}} \sum_{j=0}^{j_*-1} (|A_j| - |A_{j+1}|) \\ &\le \gamma(\text{data}, \alpha, \delta) |Q_{\tau_0}|^{\frac{p}{p-1}}. \end{aligned}$$

Equivalently

$$|[v < k_{j_*}] \cap Q_{\tau_0}| \le \nu |Q_{\tau_0}|, \qquad \text{where } \nu = \Big(\frac{\gamma(\text{data}, \alpha, \delta)}{j_*} \Big)^{\frac{p-1}{p}}.$$

4.2.3 Segmenting Q_{τ_0}

Assume momentarily that j_*, and hence ν, has been determined. By possibly increasing j_* to be not necessarily integer, without loss of generality we may assume that $(2^{j_*})^{3-m-p}$ is an integer. Then subdivide Q_{τ_0} into $(2^{j_*})^{3-m-p}$ cylinders, each of length $k_{j_*}^{3-m-p}$, by setting

$$Q_n = K_8 \times (\tau_0 + k_0^{3-m-p} + n k_{j_*}^{3-m-p}, \tau_0 + k_0^{3-m-p} + (n+1) k_{j_*}^{3-m-p})$$

for $n = 0, 1, \dots, (2^{j_*})^{3-m-p} - 1$. For at least one of these, say Q_n, there must hold

$$|[v < k_{j_*}] \cap Q_n| \le \nu |Q_n|.$$

Apply Lemma 2.3.1 to v over Q_n with $\xi\omega = k_{j_*}, a = \frac{1}{2}$, and $\theta = k_{j_*}^{3-m-p}$. It gives

$$v(z, \tau_0 + k_0^{3-m-p} + (n+1) k_{j_*}^{3-m-p}) \ge \frac{1}{2} k_{j_*} \qquad \text{a.e. in } K_4,$$

provided $C\rho \le 1$, and

$$\frac{|[v \le k_{j_*}] \cap Q_n|}{|Q_n|} \le \nu_- = \gamma(\text{data}).$$

Choose now j_*, and hence ν, from this and the definition of ν_-. Summarizing, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$\tau_0 + k_0^{3-m-p} < \tau_1 < \tau_0 + 2k_0^{3-m-p}$$

such that

$$v(z, \tau_1) \ge \sigma_0 e^{\frac{\tau_0}{3-m-p}}, \qquad \text{where } \sigma_0 = \epsilon 2^{-(j_*+1)}.$$

Remark 4.2.2 *Notice that j_*, and hence ν, are determined only in terms of the data and are independent of the parameter τ_0, which is still to be chosen.*

4.2.4 Returning to the original coordinates

In terms of the original coordinates and the original function $u(x,t)$, this implies

$$u(\cdot, t_1) \ge \sigma_0 M e^{\frac{\tau_0 - \tau_1}{3-m-p}} =: M_0 \qquad \text{in } K_{4\rho}$$

where $t_1 = \delta M^{3-m-p}(1-e^{-\tau_1})$. Apply now Lemma 2.4.1 with M replaced by M_0 and $\xi = 1$ over the cylinder

$$(t_1, 0) + Q^+_{4\rho}(\theta) \times (t_1, t_1 + \theta(4\rho)^p].$$

By choosing $\theta = \nu_0 M_0^{3-m-p}$, where $\nu_0 = \nu_0(\text{data})$, the assumption of Lemma 2.4.1 is satisfied, and it yields to

$$u(\cdot, t) \geq \frac{1}{2} M_0 = \frac{1}{2}\sigma_0 M e^{\frac{\tau_0-\tau_1}{3-m-p}} \geq \frac{1}{2}\sigma_0 M e^{\frac{-2e^{3-m-p}}{3-m-p}} e^{\tau_0} \tag{4.5}$$

in $K_{2\rho}$, for all times $t_1 < t \leq t_1 + \nu_0 M_0^{3-m-p}(4\rho)^p$. We require $\delta M^{3-m-p}\rho^p = t_1 + \nu_0 M_0^{3-m-p}(4\rho)^p$, which determines τ_0:

$$\delta M^{3-m-p}\rho^p e^{-\tau_1} = \delta M^{3-m-p}\rho^p - t_1 = \nu_0(\sigma_0 M)^{3-m-p} e^{-(\tau_1-\tau_0)}(4\rho)^p$$

$$\Rightarrow \qquad e^{\tau_0} = \frac{\delta}{4^p \nu_0 \sigma_0^{3-m-p}}.$$

This determines quantitatively $\tau_0 = \tau_0(\text{data})$; in particular, (4.5) holds for all times

$$t_1 = \delta M^{3-m-p}\rho^p - \nu_0 M^{3-m-p}(4\rho)^p \leq t \leq \delta M^{3-m-p}\rho^p.$$

From the previous definitions and transformations one estimates

$$\begin{aligned} t_1 &= M^{3-m-p}\rho^p(\delta - \nu_0 4^p) = M^{3-m-p}\rho^p \delta\Big(1 - \frac{1}{\sigma_0^{3-m-p} e^{\tau_0}}\Big) \\ &\leq (1-\varepsilon)\delta M^{3-m-p}\rho^p, \qquad \text{where } \varepsilon = e^{-\tau_0 - 2e^{\tau_0}}. \end{aligned}$$

□

4.2.5 Stability of the expansion of positivity

The proof of Proposition 3.3.1 for the degenerate case $p+m>3$, shows that the constants b and η in (3.4)-(3.5) depend on m and p as

$$b \approx \exp\Big(\gamma_b \frac{h^{p+m-3}}{p+m-3}\Big), \qquad \eta \approx \exp\Big(-\gamma_\eta \frac{k^{p+m-3}}{p+m-3}\Big)$$

for constants $\gamma_b, \gamma_\eta, h, k$ all larger than 1, depending only upon the data $\{N, C_0, C_1\}$, and independent of p, m. Thus the ratio $(b/\eta)^{p+m-3}$ that determines the "waiting time" needed to preserve and expand the positivity, deteriorates as $p+m \to \infty$. However it is stable as $p+m \to 3$ and (3.5) remains meaningful for $p+m$ near 3. On the other hand $\eta(p,m) \to 0$ as $p+m \to 3$ and (3.4) becomes vacuous. Likewise, in the proof of Proposition 4.2.1, for the singular case $2 < m+p < 3$, the change of variables (4.1) and the subsequent arguments yield constants that deteriorate as $m+p \to 3$. Nevertheless, the next proposition states that both Proposition 3.3.1, for $m+p>3$, and Proposition 4.2.1, for $2<m+p<3$, continue to hold with constants that are stable as $m+p \to 3$.

Proposition 4.2.3 *Let u be a non-negative, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *for $p+m>2$ in E_T. Let*

$$K_{8\rho}(y) \times \Big(s, s + \frac{\delta\rho^p}{M^{p+m-3}}\Big] \subset E_T$$

and assume that for some $(y,s) \in E_T$ *and some* $\rho > 0$ *there holds*

$$|[u(\cdot,s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ *and some* $\alpha \in (0,1)$. *There exist constants* $\delta, \sigma_*, \eta_* \in (0,1)$ *and* $\gamma_* > 1$, *depending only upon the data* $\{N, C_0, C_1\}$ *and* α, *independent of* $(y,s), M, p, m$ *and* ρ, *such that, for all* $|p+m-3| \leq \sigma_*$, *either*

$$\gamma_* C\rho > \min\{1, M\}$$

or

$$u(x,t) \geq \eta_* M \qquad \text{for all } x \in K_{2\rho}(y)$$

for all

$$s + \frac{\frac{1}{2}\delta\rho^p}{M^{p+m-3}} \leq t \leq s + \frac{\delta\rho^p}{M^{p+m-3}}.$$

Assume that $(y,s) = (0,0)$ and let $\epsilon(p,m)$ and $\delta(p,m)$ be the constants corresponding to α, claimed by Lemma 3.2.1. The lemma does not distinguish between $p+m > 3$ and $2 < p+m < 3$ and it implies

$$|[u(\cdot,t) > \epsilon M] \cap K_{4\rho}| > \frac{1}{2}\alpha 4^{-N} |K_{4\rho}|, \qquad \text{for all } t \in (0, \delta M^{3-m-p}\rho^p). \tag{4.6}$$

By Remark 3.2.2 the constants $\epsilon(p,m)$ and $\delta(p,m)$ are stable as $p+m \to 3$. The proof now proceeds for $p+m$ near 3 irrespective of the degeneracy ($p+m > 3$) or singularity ($p+m < 3$) of the partial differential equation. For this reason we denote by $|p+m-3|$ the proximity of $p+m$ to 3 from either side.

Lemma 4.2.4 *For every* $\nu_* \in (0,1)$, *there exist constants* $\sigma_*, \epsilon_{\nu_*} \in (0,1)$ *and* $\gamma_* > 1$, *depending only upon the data* $\{N, C_0, C_1\}$ *and* α, *independent of* u, M, p, m *and* ρ, *such that, for all* $|p+m-3| \leq \sigma_*$, *either*

$$\gamma_* C\rho > \min\{1, M\}$$

or

$$|[u < \epsilon_{\nu_*} M] \cap \mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})| \leq \nu_* |\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})|,$$

where $\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p}) = K_{4\rho} \times (0, \delta M^{3-m-p}\rho^p]$.

Proof Consider the levels $k_j = \frac{\epsilon M}{2^j}$ for $j = 0, 1, \ldots, j_*$ where $j_* \in \mathbb{N}$ is to be chosen, and ϵ is from Lemma 3.2.1, and a non-negative, piecewise smooth, cutoff function ζ that equals one on $\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})$, and such that

$$|D\zeta| \leq \frac{1}{4\rho}, \qquad |\zeta_t| \leq \frac{1}{\delta M^{3-m-p}\rho^p}.$$

First assume $m > 1$. Write down the energy estimates (2.7) for $(u-k_j)_-$ over the cylinder $\mathcal{Q}^+_{8\rho}(\delta M^{3-m-p})$

$$\begin{aligned}
&\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})} u^{m-1} |D(u-k_j)_-|^p \, dxd\tau \\
&\quad \leq \gamma \iint_{\mathcal{Q}^+_{8\rho}(\delta M^{3-m-p})} [(u-k_j)^2_- \zeta_\tau + u^{m-1}(u-k_j)^p_- |D\zeta|^p] \, dxd\tau \\
&\quad \leq \gamma \iint_{\mathcal{Q}^+_{8\rho}(\delta M^{3-m-p})} [C^p u^{m-1}(u-k_j)^p_- + C^p \chi_{[u<k_j]}] \, dxd\tau.
\end{aligned}$$

Define $\tilde{u} := \max\{\frac{k_j}{2}, u\}$, then

$$\begin{aligned}
&\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})} |D(\tilde{u}-k_j)_-|^p dx d\tau \\
&\quad \le \gamma\Big(\frac{2}{k_j}\Big)^{m-1}\Big[\frac{k_j^2}{\delta M^{3-m-p}\rho^p} + \frac{k_j^{p+m-1}}{(4\rho)^p} + C^p k_j^{p+m-1} + C^p\Big]|\mathcal{Q}^+_{4\rho}| \\
&\quad \le \gamma \frac{k_j^p}{\delta\rho^p}\Big[\frac{k_j^{3-m-p}}{M^{3-m-p}} + 1 + (C\rho)^p + \frac{(C\rho)^p}{k_j^p}\Big]|\mathcal{Q}^+_{4\rho}| \\
&\quad \le \gamma \frac{k_j^p}{\delta\rho^p}\Big[\frac{\epsilon^{3-m-p}}{2^{j(3-m-p)}} + 1 + (C\rho)^p + \frac{(C\rho)^p}{k_j^p}\Big]|\mathcal{Q}^+_{4\rho}| \\
&\quad \le \gamma \frac{k_j^p}{\delta\rho^p}\Big[\epsilon^{3-m-p}2^{j_*|3-m-p|} + 1 + (C\rho)^p + \frac{(C\rho)^p}{k_j^p}\Big]|\mathcal{Q}^+_{4\rho}|.
\end{aligned}$$

Since ϵ^{3-m-p} is limited, being $m+p$ close to 3, either $C\rho > \min\{1, \frac{M}{2^{j_*}}\}$ or

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})} |D(\tilde{u}-k_j)_-|^p dx d\tau \le \gamma \frac{k_j^p}{\delta\rho^p} 2^{j_*|3-m-p|}|\mathcal{Q}^+_{4\rho}|,$$

for a constant γ depending only upon the data $\{N, C_0, C_1\}$ and independent of u, M, ρ, p and m. Recall the definition

$$A_{k,\rho}(t) = [u(\cdot,t) < k] \cap K_\rho.$$

Apply the discrete isoperimetric inequality of Lemma B.2.3 to the levels

$$l = k_j = \frac{\epsilon M}{2^j} \qquad \text{and} \qquad k = k_{j+1} = \frac{\epsilon M}{2^{j+1}} \qquad \text{for } j = 0, 1\dots$$

and take into account (4.6) to obtain

$$k_{j+1}|A_{k_{j+1},4\rho}(t)| \le \frac{\gamma}{\alpha}\rho\int_{K_{4\rho}\cap[k_{j+1}<u<k_j]} |Du(\cdot,t)|dx.$$

Integrate this over $(0, \delta M^{3-m-p}\rho^p)$ and set

$$A_j = [u < k_j] \cap \mathcal{Q}^+_{4\rho}(\delta M^{3-m-p}) = \int_0^{\delta M^{3-m-p}\rho^p} |A_{k_j}(\tau)|d\tau.$$

Then the previous inequality yields

$$\begin{aligned}
k_{j+1}|A_{j+1}| &\le \gamma\rho \iint_{\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})\cap[k_{j+1}<u<k_j]} |Du| dx d\tau \\
&\le \gamma\rho\left(\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})} |D(u-k_j)_-|^p dx d\tau\right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\
&\le \gamma k_j \Big(|\mathcal{Q}^+_{4\rho}(\delta M^{3-m-p})|\Big)^{\frac{1}{p}} (|A_j| - |A_{j+1}|)^{\frac{p-1}{p}}.
\end{aligned}$$

where we have used the energy estimates. Next divide by $k_{j+1} = \frac{k_j}{2}$, and take the power $\frac{p}{p-1}$ on both sides to obtain the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \leq \gamma |\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|^{\frac{1}{p-1}}(|A_j| - |A_{j+1}|).$$

Add these inequalities for $j = 0, 1, \ldots, j_* - 1$. Minorize the terms on the left-hand side by their smallest value $|A_{j_*}|^{\frac{p}{p-1}}$ and majorize the right-hand side with the corresponding telescopic series. The indicated estimations yield

$$\begin{aligned} j_* |A_{j_*}|^{\frac{p}{p-1}} &\leq \gamma |\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|^{\frac{1}{p-1}} \sum_{j=0}^{\infty} (|A_j| - |A_{j+1} \\ &\leq \gamma |\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|^{\frac{p}{p-1}}. \end{aligned}$$

From this

$$|A_{j_*}| \leq \frac{\gamma}{j_*^{\frac{p-1}{p}}} |\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|. \tag{4.7}$$

Choosing

$$\epsilon_{\nu_*} = \frac{\epsilon}{2^{j_*}} \qquad \text{and} \qquad \nu_* = \frac{\gamma}{j_*^{\frac{p-1}{p}}} \tag{4.8}$$

proves the lemma for $m > 1$. When $m < 1$ we repeat the same argument starting from the energy estimates (2.9). $\square$

To conclude the proof of Proposition 4.2.3, apply Lemma 2.3.1 with $\mu_- = 0, \xi = \epsilon_{\nu_*}, a = \frac{1}{2}, \omega = M, \theta = \delta M^{3-m-p}$ and ρ replaced by 2ρ. The lemma yields

$$u > \frac{1}{2}\epsilon_{\nu_*} M \qquad \text{in } K_{2\rho} \times (\frac{1}{2}\delta\rho^p, \delta\rho^p),$$

provided

$$Y_0 = \frac{|[u < \epsilon_{\nu_*}] \cap \mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|}{|\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|} = \frac{|A_{j_*}|}{|\mathcal{Q}_{4\rho}^{+}(\delta M^{3-m-p})|} = \nu_-,$$

where the number ν_* is taken from the proof of Lemma 2.3.1. For $p + m > 3$ compute

$$\begin{aligned} Y_0 &= \nu_- = \gamma \frac{(\delta \epsilon_{\nu_*}^{p+m-3})^{\frac{N}{p}}}{(1 + \delta \epsilon_{\nu_*}^{p+m-3})^{\frac{N + p\chi_{[m>1]} + \max\{p,2\}\chi_{[m<1]}}{p}}} \\ &= \gamma \frac{(\delta \epsilon^{p+m-3} 2^{j_*(3-m-p)})^{\frac{N}{p}}}{(1 + \delta \epsilon^{p+m-3} 2^{j_*(3-m-p)})^{\frac{N + p\chi_{[m>1]} + \max\{p,2\}\chi_{[m<1]}}{p}}} \leq \nu_*. \end{aligned}$$

Stipulate to choose $|p + m - 3| \leq \sigma_*$ and then σ_* so small that $2^{j_*|p+m-3|} \in (1, 2)$. Then, from (4.7)-(4.8) choose j_* so large as to satisfy the requirement. $\square$

4.3 Intrinsic Harnack inequality for super-critical, singular equations

In addition to the structure conditions (2.2)-(2.3) we now assume

- $C = 0$, namely the operator is homogeneous;
- the Comparison Principle holds (see [53]).

Let u be a continuous, non-negative, local, weak solution to the singular equations (2.1)-(2.2)-(2.3) in E_T, for p, m in the super-critical range

$$3 - \frac{p}{N} < m + p < 3. \tag{4.9}$$

Fix $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$ and construct the cylinders

$$(x_0, t_0) + Q_\rho^\pm(\theta) \qquad \text{where} \qquad \theta = u(x_0, t_0)^{3-m-p}. \tag{4.10}$$

These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_0, t_0). The Harnack inequality holds in such an intrinsic geometry, as made precise in Theorems 4.3.1–4.3.2 below. The first is an intrinsic, mean vale Harnack inequality in a form similar to Theorem 3.4.1 of Chapter 2, for degenerate equations. This Harnack estimate is stable as $p+m \to 3$. The second is a "time insensitive" mean value Harnack inequality, valid for all times t ranging in the intrinsic geometry of (4.10), including t_0. This inequality is unstable as $p + m \to 3$.

Theorem 4.3.1 (The Intrinsic, Mean Value, Harnack Inequality) *Let u be a continuous, non-negative, local, weak solution to the singular equations (2.1)-(2.2)-(2.3) in E_T, for p, m in the super-critical range (4.9). There exist constants $\epsilon \in (0, 1)$ and $\gamma > 1$ depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and all the intrinsic cylinders $(x_0, t_0) + Q_{8\rho}^\pm(\theta)$ as in (4.10), contained in E_T,*

$$\begin{aligned} \gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot, t_0 - \epsilon u(x_0, t_0)^{3-m-p}\rho^p) &\le u(x_0, t_0) \\ &\le \gamma \inf_{K_\rho(x_0)} u(\cdot, t_0 + \epsilon u(x_0, t_0)^{3-m-p}\rho^p). \end{aligned} \tag{4.11}$$

The constant $\gamma \to \infty$ as $m + p + \frac{p}{N} \to 3$, but it is stable as $m + p \to 3$.

Theorem 4.3.2 (Time insensitive, Intrinsic, Mean Value, Harnack Inequalities) *Let u be a continuous, non-negative, local, weak solution to the singular equations (2.1)-(2.2)-(2.3) in E_T, for p, m in the super-critical range (4.9), and consider the intrinsic cylinders of the form (4.10), where c is the constant of Theorem 4.3.1. There exists constants $\bar{\epsilon} \in (0, 1)$ and $\bar{\gamma} > 1$, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, and all the intrinsic cylinders $(x_0, t_0) + Q_{8\rho}^\pm(\theta)$ as in (4.10), contained in E_T,*

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_0)} u(\cdot, \sigma) \le u(x_0, t_0) \le \bar{\gamma} \inf_{K_\rho(x_0)} u(\cdot, \tau) \tag{4.12}$$

for any pair of time levels σ, τ in the range

$$t_0 - \bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p \le \sigma, \tau \le t_0 + \bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p.$$

The constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as either $p + m + \frac{p}{N} \to 3$ or as $p + m \to 3$.

Remark 4.3.3 *The Theorems have been stated for continuous solutions, to give meaning to $u(x_0, t_0)$. It is known that locally bounded, local, weak solutions to* (2.1)*, for all $m + p > 2$ are locally Hölder continuous (see [53]). The intrinsic Harnack inequality* (4.11)*, in turn, can be used to prove that these local solutions, irrespective of their signum, are indeed locally Hölder continuous within their domain of definition.*

The proofs of Theorem 4.3.1 and Theorem 4.3.2 are intertwined. In either case the key inequalities to establish are the right-hand side estimates in (4.11) and (4.12). The left estimates will follow from these by geometrical arguments. In all the cases the proofs involve in an essential way the number

$$\lambda = N(p + m - 3) + p.$$

The requirement that p, m be in the super-critical range (4.9) is equivalent to requiring that $\lambda > 0$. The main components of the proof are the expansion of positivity for singular equations of section 4.2, a $L^1_{loc} - L^\infty_{loc}$ Harnack-type estimate valid for $\lambda > 0$, which we present next, and the Comparison Principle.

Theorem 4.3.4 *Let u be a non-negative, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *in E_T, for p, m in the super-critical range* (4.9)*. There exists a positive constant γ, depending only upon the data $\{p, m, N, C_0, C_1\}$, such that for all cylinders*

$$K_{2\rho}(y) \times [s - (t - s), s + (t - s)] \subset E_T,$$

$$\begin{aligned} \sup_{K_\rho(y)\times[s,t]} u \quad &\le \quad \frac{\gamma}{(t-s)^{\frac{N}{\lambda}}} \Big(\inf_{2s-t<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx \Big)^{\frac{p}{\lambda}} \\ &+ \quad \gamma \Big(\frac{t-s}{\rho^p} \Big)^{\frac{1}{3-m-p}}, \end{aligned} \tag{4.13}$$

where $\lambda = N(m + p - 3) + p$.

Proof Apply first Proposition 2.6.1, with $r = 1$, and then Proposition 2.5.1. □

4.3.1 An auxiliary proposition

We rephrase the right-hand side of (4.12) in this way

Proposition 4.3.5 *Let u be a continuous, locally bounded, non-negative, local, weak solution to the singular equation* (2.1) *in the super-critical range* (4.9)*. There exist positive constants $\bar\epsilon$ and $\bar\gamma$, that can be determined quantitatively, a priori only in terms of the data $\{p, m, N, C_0, C_1\}$, such that*

$$u(x_0, t_0) \le \bar\gamma \inf_{K_R(x_0)} u(\cdot, t)$$

for all times

$$t_0 - \bar\epsilon u(x_0, t_0)^{3-m-p} R^p \le t \le t_0 + \bar\epsilon u(x_0, t_0)^{3-m-p} R^p.$$

The constant $\bar\epsilon$ and $\bar\gamma$ tend to zero as either $m + p \to 3$ or $\lambda \to 0$.

The first step is to render the equation (2.1) "dimensionless" and to identify the largest value of the solution u within $Q_{8R}(x_0,t_0)$. Introduce the change of variables and unknown function

$$x \to \frac{x-x_0}{R}, \qquad t \to \frac{t-t_0}{u(x_0,t_0)^{3-m-p}R^p}, \qquad v = \frac{u}{u(x_0,t_0)}. \tag{4.14}$$

This maps $Q_{8R}(x_0,t_0)$ into

$$Q_8 = K_8 \times (-8^p, 8^p].$$

The function v is a weak solution to

$$v_\tau - \text{div}\bar{A}(z,\tau,v,Dv) = 0 \qquad \text{in } Q_8, \tag{4.15}$$

where the transformed function

$$\bar{A}(z,\tau,v,Dv) \quad = \quad R^{p-1}u(x_0,t_0)^{2-m-p}A(x,t,u,Du),$$

satisfies

$$m>1: \quad \begin{cases} \bar{A}(z,\tau,v,\eta)\cdot\eta \ge C_0|v|^{m-1}|\eta|^p, \\ |\bar{A}(z,\tau,v,\eta)| \le C_1|v|^{m-1}|\eta|^{p-1} + |v|^{\frac{m-1}{p}}, \end{cases} \tag{4.16}$$

$$m<1: \quad \begin{cases} \bar{A}(z,\tau,v,\eta)\cdot\eta \ge [C_0|v|^{m-1}|\eta|^p - |v|^{m+p-1}], \\ |\bar{A}(z,\tau,v,\eta)| \le [C_1|v|^{m-1}|\eta|^{p-1} + |v|^{m+p-2}], \end{cases} \tag{4.17}$$

where C_0, C_1, are the original constants in the structure conditions (2.2)-(2.3). Establishing the Proposition consists in finding positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, depending only upon the data, such that

$$v(\cdot,t) \ge \bar{\gamma}^{-1} \qquad \text{in } K_1 \text{ for all } t \in [-\bar{\epsilon},\bar{\epsilon}].$$

Hereafter we relabel by x, t the new coordinates z, τ.

4.3.2 Locating the supremum of v in K_1

For $\tau \in (0,1)$ introduce the family of nested expanding cubes $\{K_\tau\}$ centered at the origin, and the increasing family of positive numbers

$$M_\tau = \sup_{K_\tau} v, \qquad N_\tau = (1-\tau)^{\frac{p}{3-m-p}}.$$

By definition, $M_0 = N_0$ and $N_\tau \to +\infty$, as $\tau \to 1$, whereas M_τ remains finite. Therefore the equation $M_\tau = N_\tau$ has roots. Denoting by τ_* the largest root

$$M_{\tau_*} = (1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{and} \qquad M_\tau \le N_\tau \text{ for all } \tau \ge \tau_*.$$

Since v is continuous, the supremum M_{τ_*} is achieved at some $\bar{x} \in K_{\tau_*}$. Choose $\bar{\tau} \in (0,1)$ from

$$(1-\bar{\tau})^{-\frac{p}{3-m-p}} = 4(1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{i.e.} \qquad \bar{\tau} = 1 - 4^{-\frac{3-m-p}{p}}(1-\tau_*).$$

Set also

$$2r := \bar{\tau} - \tau_* = (1 - 4^{-\frac{3-m-p}{p}})(1-\tau_*).$$

For those choices, $K_{2r}(\bar{x}) \subset K_{\bar{\tau}}, M_{\bar{\tau}} \le N_{\bar{\tau}}$, and

$$\begin{aligned} \sup_{K_{\tau_*}} v(\cdot,0) &= (1-\tau_*)^{-\frac{p}{3-m-p}} = v(\bar{x},0) \le \sup_{K_{2r}(\bar{x})} v(\cdot,0) \\ &\le \sup_{K_{\bar{\tau}}} v(\cdot,0) \le 4(1-\tau_*)^{-\frac{p}{3-m-p}}. \end{aligned}$$

4.3.3 Estimating the supremum of v in some intrinsic neighbourhood about $(\bar{x}, 0)$

Consider the cylinder centered at $(\bar{x}, 0)$

$$\begin{aligned} Q_{2r} &= [(\bar{x},0) + Q^-_{2r}(\theta_*)] \cup [(\bar{x},0) + Q^+_{2r}(\theta_*)] \\ &= K_{2r}(\bar{x}) \times (-\theta_*(2r)^p, \theta_*(2r)^p], \end{aligned}$$

where $\theta_* = (1-\tau_*)^{-p}$. Such a cylinder is included in the box Q_8 since

$$\theta_*(2r)^p = (1-\tau_*)^{-p}(1-4^{-\frac{3-m-p}{p}})^p(1-\tau_*)^p \le 8.$$

Lemma 4.3.6 *There exists a positive constant γ_1, depending only on the data $\{p, m, N, C_0, C_1\}$, and independent of R, such that*

$$\sup_{Q_r} v \le \gamma_1(1-\tau^*)^{-\frac{p}{3-m-p}}.$$

The constant $\gamma_1 \to +\infty$ as $p+m \to 3$ or $\lambda \to 0$.

Proof Apply Theorem 4.3.4 to the function v over the pair of cylinders $Q_r \subset Q_{2r}$. Apply it first for the choice

$$s = 0, \quad t = \theta_*(2r)^p,$$

and then apply it again, for the choice

$$s = -\theta_*(2r)^p, \quad t = 0.$$

We obtain

$$\begin{aligned} \sup_{Q_r} v &\le \frac{\gamma}{(\theta_*(2r)^p)^{\frac{N}{\lambda}}} \Big(\inf_{-2\theta_*(2r)^p < \tau < \theta_*(2r)^p} \int_{K_{2r}(\bar{x})} v(x,\tau)dx \Big)^{\frac{p}{\lambda}} \\ &\quad + \gamma(2^p\theta_*)^{\frac{1}{3-m-p}} \\ &\le \gamma(1-\tau_*)^{-p\frac{N}{\lambda}} \Big(\fint_{K_{2r}(\bar{x})} u(x,0)dx \Big)^{\frac{p}{\lambda}} + \gamma 2^{\frac{p}{3-m-p}}(1-\tau_*)^{-\frac{p}{3-m-p}} \\ &\le \gamma(1-\tau_*)^{-\frac{p}{3-m-p}}[4^{\frac{p}{\lambda}} + 2^{\frac{p}{3-m-p}}] = \gamma_1(1-\tau_*)^{-\frac{p}{3-m-p}}. \qquad \square \end{aligned}$$

Introduce next the cylinder

$$Q_r(\bar{\delta}\theta_*) = K_r(\bar{x}) \times (-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p] \subset Q_{2r},$$

where $\bar{\delta} \in (0,1)$ is to be chosen.

Lemma 4.3.7 *There exist numbers* $\bar{\delta}, \bar{c}$, *and* $\alpha \in (0,1)$, *depending only upon the data* $\{p, m, N, C_0, C_1\}$, *and independent of* R, *such that*

$$|[v(\cdot,t) \geq \bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}} \cap K_r| > \alpha |K_r| \qquad \text{for all } t \in [-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p],$$

where $\theta_* = (1-\tau_*)^{-p}$. *The constants* $\bar{c}$, *and* α *tend to zero as either* $p+m \to 3$ *or as* $\lambda \to 0$. *The constant* $\bar{\delta}$ *tend to zero as* $p+m \to 3$.

Proof Apply Theorem 4.3.4 to the function v over the pair of cylinders

$$Q_{\frac{r}{2}}(\bar{\delta}\theta_*) \subset Q_r(\bar{\delta}\theta_*),$$

For all $t \in [-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$\begin{aligned}
(1-\tau_*)^{-\frac{p}{3-m-p}} &= v(\bar{x},0) \leq \sup_{K_{\frac{r}{2}}(\bar{x})} v(\cdot,0) \\
&\leq \frac{\gamma}{(\bar{\delta}\theta_* r^p)^{\frac{N}{\lambda}}} \Big(\int_{K_\rho} v(x,t)dx \Big)^{\frac{p}{\lambda}} + \gamma \Big(\frac{2^p \bar{\delta}\theta_* r^p}{r^p} \Big)^{\frac{1}{3-m-p}} \\
&\leq \gamma \frac{(1-\tau_*)^{-p\frac{N}{\lambda}}}{\bar{\delta}^{-\frac{N}{\lambda}}} \Big(\fint_{K_r} v(x,t)dx \Big)^{\frac{p}{\lambda}} + \gamma (2^p \bar{\delta})^{\frac{1}{3-m-p}} (1-\tau_*)^{-\frac{p}{3-m-p}}.
\end{aligned}$$

Choose $\bar{\delta}$ from

$$\gamma (2^p \bar{\delta})^{\frac{1}{3-m-p}} \leq \frac{1}{2},$$

and set

$$\gamma_2 = 2\gamma, \qquad \gamma_3 = \frac{2^{\frac{N}{\lambda}(3-m-p)}\gamma_2}{\bar{\delta}^{\frac{N}{\lambda}}}.$$

For such choices, the constants $\bar{\delta}$, γ_2, and γ_3 depend only upon the data $\{p, m, N, C_0, C_1\}$. Then, for all $t \in [-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$\begin{aligned}
\frac{1}{\gamma_2}(1-\tau_*)^{-\frac{p}{3-m-p}} &\leq \frac{1}{\gamma}(1-\tau_*)^{-\frac{p}{3-m-p}} (1-\gamma(2^p\bar{\delta})^{\frac{1}{3-m-p}}) \\
&\leq \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{\bar{\delta}^{\frac{N}{\lambda}}} \Big(\fint_{K_r} v(x,t)dx \Big)^{\frac{p}{\lambda}}.
\end{aligned}$$

From this, for $\bar c \in (0,1)$,

$$\begin{aligned}
\frac{1}{\gamma_3}(1-\tau_*)^{-\frac{p}{3-m-p}} &\le \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{2^{\frac{N}{\lambda}(3-m-p)}}\Big(\fint_{K_r} v(x,t)dx\Big)^{\frac{p}{\lambda}}\\
&\le \frac{(1-\tau_*)^{p\frac{N}{\lambda}}}{2^{\frac{N}{\lambda}(3-m-p)}}\\
&\times\Big[\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)<\bar c(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx+\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)\ge\bar c(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Big]^{\frac{p}{\lambda}}\\
&\le (1-\tau_*)^{p\frac{N}{\lambda}}\Big(\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)<\bar c(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Big)^{\frac{p}{\lambda}}\\
&+(1-\tau_*)^{p\frac{N}{\lambda}}\Big(\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)\ge\bar c(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Big)^{\frac{p}{\lambda}}\\
&\le \bar c^{\frac{p}{\lambda}}(1-\tau_*)^{-\frac{p}{3-m-p}}+\gamma_1^{\frac{p}{\lambda}}(1-\tau_*)^{-\frac{p}{3-m-p}}\Big(\fint_{K_r}\chi_{[v(\cdot,t)\ge\bar c(1-\tau_*)^{-\frac{p}{3-m-p}}]}dx\Big)^{\frac{p}{\lambda}}.
\end{aligned}$$

To prove the thesis choose

$$\bar c^{\frac{p}{\lambda}}=\frac{1}{2\gamma_3} \qquad \text{and set} \qquad \alpha=\frac{1}{\gamma_1}\Big(\frac{1}{2\gamma_3}\Big)^{\frac{\lambda}{p}}. \qquad \square$$

4.3.4 Expanding the positivity of v

The information provided by Lemma 4.3.7 is the assumption required by the expansion of positivity for all

$$\bar\delta\theta_* r^p \le s \le \bar\delta\theta_* r^p.$$

Apply then the expansion of positivity (Proposition 4.2) to v with $\rho = r$, $M = \bar c(1-\tau_*)^{-\frac{p}{3-m-p}}$ and for s ranging in the indicated interval. It gives

$$v(\cdot,t) > \eta\bar c(1-\tau_*)^{-\frac{p}{3-m-p}} \qquad \text{in } K_{2r}(\bar x) \tag{4.18}$$

and for all times

$$-\bar\delta\theta_* r^p + (1-\varepsilon)\delta M^{3-m-p} r^p < t < \bar\delta\theta_* r^p, \tag{4.19}$$

for constants $\delta, \bar\delta, \varepsilon \in (0,1)$ depending only upon the data $\{p, m, N, C_0, C_1\}$ and the constant α, which itself is determined only in terms of the data.

4.3.5 Expanding the positivity of w and applying the Comparison Principle

Consider the boundary value problem

$$\begin{cases} w \in L^\infty(t_I,1;L^2(K_4(\bar x))), \qquad w^{\frac{m+p-1}{p-1}} \in L^p(t_I,1;W^{1,2}(K_4(\bar x))),\\ w_t - \operatorname{div}\bar A(x,t,w,Dw) = 0, \quad \text{in } K_4(\bar x)\times[t_I,1],\\ w\lfloor_{\partial K_4(\bar x)} = 0,\\ w(x,t_I) = \begin{cases} \eta\bar c(1-\tau_*)^{-N}, & x\in K_{2r}(\bar x),\\ 0, & x\in K_4(\bar x)\setminus K_{2r}(\bar x),\end{cases}\end{cases} \tag{4.20}$$

where $t_I = -\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{3-m-p} r^p$. The problem has a unique solution. Moreover

$$w\lfloor_{\partial K_4(\bar{x})} = 0 \leq v\lfloor_{\partial K_4(\bar{x})},$$

and

$$\begin{aligned} v(x,t_I) - w(x,t_I) &\geq \eta\bar{c}(1-\tau_*)^{-\frac{p}{3-m-p}} - \eta\bar{c}(1-\tau_*)^{-N} \\ &\geq \eta\bar{c}(1-\tau_*)^{-N}[(1-\tau_*)^{-\frac{\lambda}{3-m-p}} - 1] > 0. \end{aligned}$$

Therefore, by the Comparison Principle

$$u \geq w \text{ in } K_4(\bar{x}) \times [t_I, 1].$$

To prove the Proposition 4.3.5, it suffices to show that we can determine two constants $\bar{\gamma}$ and $\bar{\varepsilon}$, depending only upon the data, such that

$$w(x,t) \geq \bar{\gamma}^{-1} \qquad \text{in } K_1 \text{ for all } t \in [-\bar{\varepsilon}, \bar{\varepsilon}].$$

Assume $\bar{x} = 0$. Let $\bar{\theta} \in (0,1)$ to be chosen. From Theorem 4.3.4, applied with $y = 0$, $s = 0, t = \bar{\theta}$, and $\rho = 2$, we deduce

$$\sup_{K_2 \times [0,\bar{\theta}]} w(\cdot,t) \leq \gamma\bar{\theta}^{-\frac{N}{\lambda}}\eta\bar{c} + \gamma\bar{\theta}^{-\frac{3-m-p}{p}} = \gamma_*(N,m,p,C_0,C_1,\eta,\bar{c},\bar{\theta}).$$

On the other hand, by Proposition 2.5.1, for all $t \in [0,\bar{\theta}]$

$$\int_{K_1} w(x,0)dx \leq \gamma \int_{K_2} w(x,t)dx + \gamma\bar{\theta}^{\frac{1}{3-m-p}}.$$

By the definition of $w(\cdot,0)$

$$\int_{K_1} w(x,0)dx = \eta\bar{c}.$$

We choose $\bar{\theta}$ from

$$\gamma\bar{\theta}^{\frac{1}{3-m-p}} = \frac{1}{2}\eta\bar{c}.$$

It follows that

$$\iint_{K_2 \times [0,\bar{\theta}]} w(x,t)dxdt \geq \frac{1}{2}\eta\bar{c}.$$

Next, for all $t \in [0,\bar{\theta}]$

$$\begin{aligned} &\int_{K_2} w(x,t)dxdt \\ &\quad \leq \int_{K_2 \cap [w(\cdot,t) < c_0]} w(x,t)dx + \int_{K_2 \cap [w(\cdot,t) \geq c_0]} w(x,t)dx \\ &\quad \leq c_0|K_2| + \gamma_*|[w(\cdot,t) \geq c_0] \cap K_2|, \end{aligned}$$

where c_0 is any positive number. Choosing

$$c_0 = \frac{1}{4|K_2|}\eta\bar{c},$$

the previous inequality becomes

$$|[w(\cdot,t) \geq c_0] \cap K_2| \geq \alpha |K_2|, \qquad \alpha = \frac{1}{4\gamma_* |K_2|} \eta \bar{c},$$

for all $t \in [0, \bar{\theta}]$. By the expansion of positivity (Proposition 4.2.1)

$$w(x,t) \geq \eta c_0 \qquad \text{in } K_4(\bar{x}) \qquad \text{for all } t \in [-\bar{\varepsilon}, \bar{\varepsilon}],$$

for a sufficiently small $\bar{\varepsilon}$ depending only the data, α, and c_0. By the Comparison Principle the proof is now finished.

4.3.6 Proof of the right-hand side Harnack inequality of Theorem 4.3.1

The estimate in the proof of Theorem 4.3.2 deteriorate as $p+m \to 3$ and as $m+p+\frac{p}{N} \to 3$. Stable estimates for $p+m \to 3$ required in the proof of the right-hand side inequality of Theorem 4.3.1 can be derived as in Proposition 4.2.3 by almost identical arguments. As remarked in that contest, there exists $\sigma_* \in (0,1)$, that can be determined a priori only in terms of $\{N, C_0, C_1\}$ and independent of p, m, such that, for $|p+m-3| < \sigma_*$, the expansion of positivity for non-negative solutions to the class of equations (2.1)-(2.2)-(2.3) behaves as if these equations were neither degenerate nor singular. Henceforth we let σ_* be the number claimed by Proposition 4.2.3 and let $|p+m-3| < \sigma_*$. With such a restriction at hand, a "forward", intrinsic Harnack inequality can be derived for non-negative, local, solutions to these equations, by the same arguments as in Theorem 3.4.1, both for the degenerate case $p+m > 3$ and the singular case $p+m < 3$.

Having fixed $(x_0, t_0) \in E_T$ such that $u(x_0, t_0) > 0$, let

$$\theta = u(x_0, t_0)^{3-m-p},$$

and consider the cylinder

$$(x_0, t_0) + Q^{\pm}_{8\rho}(\theta) \subset E_T.$$

Introduce the change of variable (4.14) with R replaced by ρ. This maps

$$(x_0, t_0) + Q^{\pm}_{8\rho}(\theta) \subset E_T \qquad \text{into} \qquad Q^{\pm}_{8}(c^{3-m-p})$$

and v solves (4.15)-(4.16)-(4.17) in

$$Q^{-}_{8}(c^{p+m-3}) \cup Q^{+}_{8}(c^{p+m-3}).$$

For $\tau \in [0,1)$, introduce the family of nested cylinders $\{Q^-_\tau\}$ with the same "vertex" at $(0,0)$, and the families of non-negative numbers $\{M_\tau\}$, $\{N_\tau\}$ defined by

$$Q^-_\tau = K_\tau \times (-\tau, 0], \qquad M_\tau = \sup_{Q_\tau} v, \qquad N_\tau = (1-\tau)^{-\beta}$$

where $\beta > 1$ is to be chosen. Let τ_* be the largest root of the equation $M_\tau = N_\tau$, and let $(\bar{x}, \bar{t}) \in Q^-_{\tau_*}$ be a point where v achieves its maximum M_{τ_*}. Consider the cylinder

$$Q_0 = [|x - \bar{x}| < \frac{1}{2}(1-\tau_*)] \times (\bar{t} - \frac{1}{2}(1-\tau_*), \bar{t}] \subset Q^-_{\frac{1}{2}(1+\tau_*)}.$$

From the definitions

$$v(\bar{x},\bar{t}) = M_{\tau_*} = (1-\tau_*)^{-\beta} \le \sup_{Q_0} v \le \sup_{Q^-_{\frac{1}{2}(1+\tau_*)}} v \le N_{\frac{1}{2}(1+\tau_*)} = 2^{\beta}(1-\tau_*)^{-\beta}.$$

Set

$$r = \frac{1}{2}(1-\tau_*), \qquad \text{and} \qquad M_\beta = 2^\beta (1-\tau^*)^{-\beta}$$

and consider the cylinder with "vertex" at $(\bar{x},\bar{t})$

$$(\bar{x},\bar{t}) + Q^-_r(M_\beta^{3-m-p}) = K_r(\bar{x}) \times (\bar{t} - M_\beta^{3-m-p} r^p, \bar{t}].$$

This can be taken as the starting cylinder in the proof of the "forward" intrinsic Harnack inequality of Theorem 3.4.1, provided its geometry is "intrinsic", that is if

$$\sup_{(\bar{x},\bar{t})+Q^-_r(M_\beta^{3-m-p})} v \le M_\beta.$$

This occurs if $(\bar{x},\bar{t}) + Q^-_r(M_\beta^{3-m-p}) \subset Q_0$ or equivalently if

$$2^{\beta(3-m-p)}(1-\tau_*)^{-\beta(3-m-p)}(1-\tau_*)^{p-1} = 2^{p-1}. \tag{4.21}$$

Assuming this inclusion for the moment, proceed as in the proof of the "forward", intrinsic Harnack inequality of Theorem 3.4.1. The proof will determine quantitatively the constants ϵ, γ and c, by a quantitative determination of the parameter β depending only on the data $\{p, m, N, C_0, C_1\}$ and stable as $p + m \to 3$.

The condition (4.21) does not enter in the determination of β. It is needed only to ensure that $(\bar{x},\bar{t}) + Q^-_r(\theta_\beta)$ possesses the correct intrinsic geometry. Having determined β, the condition (4.21) is satisfied by choosing $\beta(3-m-p) = p-1$. The right-hand side of the Harnack inequality (4.10)-(4.11) then holds with the constants ϵ, γ and c stable for

$$|p+m-3| < \sigma_{**} = \min\{\sigma_*, (1-\sigma_*)\beta^{-1}\}.$$

To establish the right-hand side inequality of Theorem 4.3.1 assume first $3 - \frac{p}{N} < p + m < 3 - \sigma_{**}$ and proceed as in the proof of Theorem 4.3.2. This will produce constants $\bar{\gamma}(p,m), \bar{\epsilon}(p,m)$ that deteriorate as $p+m \to 3$. For $0 < 3 - m - p < \sigma_{**}$ proceed as above, to establish the inequality with constants that are stable as $p + m \to 3$. □

Remark 4.3.8 *If $3 - \frac{p}{N} < m + p < 3 - \sigma_{**}$ the proof of the right-hand side, intrinsic Harnack inequality* (4.11) *is a particular case of the right-hand side inequality* (4.12) *of Theorem 4.3.2. Having fixed m, p such that $m + p \in (3 - \frac{p}{N}, 3 - \sigma_{**}]$ and having determined $\bar{\epsilon}(m,p)$ and $\bar{\gamma}(m,p)$, the inequality continues to hold for any smaller $\bar{\epsilon}$ for the same constant $\bar{\gamma}$.*

If $3 - \sigma_{**} < m + p \le 3$ the proof of the Harnack inequality follows instead the proof of Proposition 3.3.1 for the degenerate case $m + p > 3$. In that case, the constants c and κ have a functional dependence, made quantitative in Section 3.4. Having determined c and κ, the parameter c can be taken to be smaller, provided κ is taken larger.

4.3.7 Proof of the left-hand side Harnack inequality of Theorem 4.3.1

Let ϵ, γ be the constants appearing on the right-hand side Harnack inequality of Theorem 4.3.1. Set

$$\bar{t} = t_0 - \epsilon u(x_0, t_0)^{3-m-p}\rho^p.$$

Let $\alpha \in (0,1)$ to be chosen, consider the cube $K_{\alpha\rho}(x_0)$, and introduce the set

$$\mathcal{U}_\alpha = K_{\alpha\rho}(x_0) \cap [u(\cdot, \bar{t}) \le \gamma u(x_0, t_0)].$$

Since u is continuous, $\mathcal{U}_\alpha$ is closed. The parameter α will be chosen, depending only upon γ, such that $\mathcal{U}_\alpha$ is also open. Then, if $\mathcal{U}_\alpha$ is not empty, it coincides with $K_{\alpha\rho}$, thereby establishing the left-hand side, intrinsic Harnack inequality in (4.10)-(4.11), modulo a suitable re-definition of ρ and ϵ.

Assume momentarily that $\mathcal{U}_\alpha$ is not empty, and fix $z \in \mathcal{U}_\alpha$. Since u is continuous, there exists a cube $K_\varepsilon(z) \subset K_{\alpha\rho}(x_0)$ such that

$$u(y, \bar{t}) \le 2\gamma u(x_0, t_0) \qquad \text{for all } y \in K_\varepsilon(z). \tag{4.22}$$

For each $y \in K_\varepsilon(z)$ construct the intrinsic p-paraboloid

$$\mathcal{P}(y, \bar{t}) = [|t - \bar{t}| \ge \epsilon u(y, \bar{t})^{3-m-p}|x - y|^p].$$

If $(x_0, t_0) \in \mathcal{P}(y, \bar{t})$, by the right-hand side Harnack inequality in (4.10)-(4.11)

$$u(y, \bar{t}) \le \gamma u(x_0, t_0)$$

and hence $y \in \mathcal{U}_\alpha$, proving U_α to be open. This occurs if

$$\epsilon u(y, \bar{t})^{3-m-p}|y - x_0|^p \quad \le \quad \epsilon u(x_0, t_0)^{3-m-p}\rho^p,$$

that is if

$$|y - x_0| < \alpha\rho \qquad \text{where} \qquad \alpha = (2\gamma)^{\frac{m+p-3}{p}}.$$

The right-hand side Harnack inequality can be applied since, in view of (4.22), the cylinder

$$(y, \bar{t}) + Q^{\pm}_{8\rho}(\bar{\theta}) \qquad \text{with} \qquad \bar{\theta} = u(y, \bar{t})^{3-m-p}$$

can be assumed to be contained in E_T.

It remains to show that $\mathcal{U}_\alpha \neq \emptyset$. Having determined α, consider the cylinder

$$K_{\alpha\rho}(x_0) \times (\bar{t}, \bar{t} + \nu_0(\gamma u(x_0, t_0))^{3-m-p}(\alpha\rho)^p],$$

where $\nu_0 \in (0,1)$ is to be chosen, depending only on the data $\{p, m, N, C_0, C_1\}$. Such a cylinder crosses the time level t_0 if

$$t_0 - \epsilon u(x_0, t_0)^{3-m-p}\rho^p + \nu_0(\gamma u(x_0, t_0))^{3-m-p}(\alpha\rho)^p > t_0.$$

Recalling the value of α, this occurs if

$$\nu_0\gamma^{3-m-p}\alpha^p > \epsilon \qquad \Rightarrow \qquad \epsilon < \nu_0 2^{p+m-3},$$

which, by reducing ϵ if necessary, we assume. Note that such a reduction of ϵ is possible by increasing γ accordingly to Remark 4.3.8. If $\mathcal{U}_\alpha = \emptyset$, then

$$u(\cdot, \bar{t}) > \gamma u(x_0, t_0) \qquad \text{in } K_{\alpha\rho}(x_0).$$

Apply Lemma 2.4.1, with 2ρ replaced by $\alpha\rho$, and with

$$a = \frac{1}{2}, \qquad \xi = 1, \qquad M = \gamma u(x_0, t_0), \qquad \theta = \nu_0(\gamma u(x_0, t_0))^{3-m-p},$$

where ν_0 is the number in the hypothesis (2.30) of Lemma 2.4.1. For such a choice of θ, (2.30) is satisfied and the lemma yields

$$u(x_0, t_0) > \frac{1}{2}\gamma u(x_0, t_0) \qquad \text{for all } x \in K_{\frac{1}{2}\alpha\rho}(x_0).$$

Computing this for $x = x_0$ gives a contradiction if $\gamma > 2$, which without loss of generality we may assume. □

4.3.8 Proof of the left-hand side Harnack inequality of Theorem 4.3.2

Let the assumptions of Theorem 4.3.2 be in force and consider first the left-hand side inequality (4.12) for the specific value of σ

$$\bar{\sigma} = t_0 - \bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p.$$

For such fixed value of σ, the left-hand side inequality in (4.12) can be derived exactly as in the case of the left-hand side inequality (4.11) of Theorem 4.3.1 as established in the previous section. Thus, by possibly redefining $\bar{\gamma}$ and $\bar{\epsilon}$,

$$\sup_{K_\rho(x_0)} u(\cdot, \bar{\sigma}) \leq \bar{\gamma} u(x_0, t_0).$$

Apply Theorem 4.3.4 over the cubes $K_{\frac{1}{2}\rho}(x_0) \subset K_\rho(x_0)$ for the time levels

$$s = \bar{\sigma} \qquad \text{and} \qquad t_0 \leq t \leq t_0 + \bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p \tag{4.23}$$

so that

$$\bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p \leq t - s \leq 2\bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p.$$

With these choices,

$$\begin{aligned}
\sup_{K_{\frac{1}{2}\rho}(x_0)} u(\cdot, t) \quad &\leq \quad \frac{\gamma}{\bar{\epsilon}^{\frac{N}{\lambda}} u(x_0, t_0)^{\frac{N(3-m-p)}{\lambda}}} \left(\fint_{K_\rho(x_0)} u(x, \bar{\sigma}) dx \right)^{\frac{p}{\lambda}} \\
&\qquad + \gamma (2\bar{\epsilon})^{\frac{m+p-1}{p(3-m-p)}} u(x_0, t_0)^{\frac{m+p-1}{p}} \\
&\leq \quad (\gamma \bar{\gamma}^{\frac{p}{\lambda}} \bar{\epsilon}^{-\frac{N}{\lambda}} + \gamma (2\bar{\epsilon})^{\frac{1}{3-m-p}}) u(x_0, t_0) \\
&= \quad \bar{\bar{\gamma}} u(x_0, t_0).
\end{aligned}$$

This establishes the left-hand side inequality (4.12) for all $\sigma = t$ in the range (4.23), by possibly redefining $\bar{\gamma}$ and $\bar{\epsilon}$. For σ in the range

$$t = t_0 - \bar{\epsilon} u(x_0, t_0)^{3-m-p}\rho^p \leq \sigma \leq t_0$$

the proof is the same, starting from (**??**)-(4.13) of Theorem 4.3.4 with the time levels

$$t = t_0 + \bar{\epsilon} u(x_0, t_0)^{3-m-p} \rho^p \qquad \text{and} \qquad s = \sigma$$

for σ in the indicated range. □

4.4 Harnack estimates for sub-critical singular equations

In this final section we switch back to the complete operator and we do not require the Comparison Principle anymore. Let u be a non-negative, local, weak solutions to the singular equation (2.1) in E_T, for p, m in the critical and sub-critical range

$$2 < p + m \leq 3 - \frac{p}{N}. \tag{4.24}$$

As we mentioned in Section 4.1, a Harnack estimate in any of the forms (4.11)-(4.12) fails to hold when p, m are in the critical and sub-critical range (4.24). Nevertheless a different form of Harnack estimate holds for p, m in such a range, with constants depending on the ratio of some integral norms of the solution u. Fix $(x_0, t_0) \in E_T$ and ρ such that $K_{4\rho}(x_0) \subset E$, and introduce the quantity

$$\theta = \Big[\varepsilon \Big(\fint_{K_\rho(x_0)} u^q(\cdot, t_0) dx \Big)^{\frac{1}{q}} \Big]^{3-m-p}, \tag{4.25}$$

where $\varepsilon \in (0, 1)$ is to be chosen, and $q \geq 1$ is arbitrary. If $\theta > 0$ assume that

$$(x_0, t_0) + Q^-_{8\rho}(\theta) = K_{8\rho}(\theta) \times (t_0 - \theta(8\rho)^p, t_0] \subset E_T, \tag{4.26}$$

and set

$$\sigma = \left[\frac{\Big(\fint_{K_\rho(x_0)} u^q(\cdot, t_0) dx \Big)^{\frac{1}{q}}}{\Big(\fint_{K_{4\rho}(x_0)} u^r(\cdot, t_0 - \theta\rho^p) dx \Big)^{\frac{1}{r}}} \right]^{\frac{rp}{\lambda_r}}, \tag{4.27}$$

$$M_q = \left(\sup_{t_0 - \theta\rho^p < s \leq t_0} \fint_{K_{2\rho}(x_0)} u^q(\cdot, s) dx \right)^{\frac{1}{q}}, \tag{4.28}$$

where $r \geq 1$ is any number such that

$$\lambda_r = N(p + m - 3) + rp > 0. \tag{4.29}$$

Theorem 4.4.1 *Let u be a non-negative, locally bounded, local, weak solution to the singular equation (2.1) in E_T, for $2 < m + p < 3$. Introduce θ as in (4.25) and assume that $\theta > 0$. There exist constants $\varepsilon \in (0, 1)$, and $\gamma, \beta > 1$, depending only on the data $\{p, m, N, C_0, C_1\}$ and the parameters q, r, such that either*

$$C\rho > \min\{1, M_q, M_r, M_q^{\frac{p+m-2}{p-1}}, M_r^{\frac{p+m-2}{p-1}}\} \tag{4.30}$$

or

$$\inf_{(x_0,t_0)+Q^-_\rho(\frac{1}{2}\theta)} u \geq \gamma \sigma^\beta \sup_{(x_0,t_0)+Q^-_\rho(\theta)} u, \tag{4.31}$$

where σ is defined in (4.27), $q \geq 1$ and $r \geq 1$ satisfies (4.29). The constant $\varepsilon \to 0$, and $\gamma, \beta \to \infty$ as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Remark 4.4.2 *Inequality* (4.31) *is not a Harnack inequality per se, since σ depends upon the solution itself. It would reduce to a Harnack inequality if $\sigma \geq \sigma_0$ for some absolute constant σ_0 depending only upon the data. This however cannot occur since a Harnack inequality for solutions to* (4.1) *does not hold.*

Inequality (4.31) *can be regarded as a "weak" form of a Harnack estimate valid for all $2 < m + p < 3$.*

4.4.1 Components of the proof of Theorem 4.4.1

The first is the expansion of positivity presented in 4.2.1; this property of non-negative, local solutions to the singular, quasi-linear parabolic equation (2.1) holds in the entire range $2 < p+m < 3$. The second is Proposition 2.7.1, which states some L^r_{loc} estimates backward in time. The last one is a consequence of Proposition 2.6.1 and Proposition 2.7.1, which we present next.

Theorem 4.4.3 *Let u be a non-negative, locally bounded, local, weak solutions to the singular equation* (2.1) *in E_T, for $2 < p + m < 3$, and let $r \geq 1$ satisfy* (4.29)*. There exists a positive constant γ_r, depending only upon the data $\{p, m, N, C_0, C_1\}$, and r, such that either*

$$C\rho > \min\Big\{1, M_r, M_r^{\frac{p+m-2}{p-1}}, \Big(\frac{t-s}{\rho^p}\Big)^{\frac{p+m-1}{p(3-m-p)}}\Big\}$$

or

$$\begin{aligned}\sup_{K_\rho(y)\times[s,t]} u \quad &\leq \quad \frac{\gamma_r}{(t-s)^{\frac{N}{\lambda_r}}}\Big(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx\Big)^{\frac{p}{\lambda_r}} \\ &+ \quad \gamma_r\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{3-m-p}}\end{aligned}$$

for all cylinders

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T.$$

The constant $\gamma_r \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Proof Apply first Proposition 2.6.1 and then Proposition 2.7.1. □

Remark 4.4.4 *Theorem 4.4.3 assumes that u is locally bounded, and turns such a qualitative information into a quantitative estimate in terms of the $L^r_{\rm loc}$ integrability of $u(\cdot, t)$.*

4.4.2 Estimating the positivity set of the solution

Having fixed $(x_0, t_0) \in E_T$, assume it coincides with the origin, write $K_\rho(0) = K_\rho$ and introduce the quantity θ as in (4.25), which is assumed to be positive. Assume moreover that (4.30) is always violated. Apply Theorem 2.7.1 for $r = q, y = 0$, and $s \in (-\theta\rho^p, 0]$. Using the definition (4.25) of θ gives

$$\begin{aligned}\int_{K_\rho} u^q(x,0)dx \quad &\leq \quad \gamma_q \int_{K_{2\rho}} u^q(x,\tau)dx + \gamma_q\Big(\frac{(\theta\rho^p)^q}{\rho^{\lambda_q}}\Big)^{\frac{1}{3-m-p}} \\ &= \quad \gamma_q \int_{K_{2\rho}} u^q(x,\tau)dx + \gamma_q\varepsilon^q \int_{K_\rho} u^q(x,0)dx,\end{aligned}$$

for all $q \ge 1$ and all $\tau \in (-\theta\rho^p, 0]$, for a constant γ_q depending only on the data $\{p, m, N, C_0, C_1\}$ and q. Choosing ε from

$$\gamma_q \varepsilon^q \le \frac{1}{2},$$

yields

$$\int_{K_{2\rho}} u^q(x,\tau)dx \ge \frac{1}{2\gamma_q} \int_{K_\rho} u^q(x,0)dx \tag{4.32}$$

for all $\tau \in (-\theta\rho^p, 0]$. Next apply Theorem 4.4.3 over the cylinder

$$K_{2\rho} \times (-\frac{1}{2}\theta\rho^p, 0]$$

with $r \ge 1$ such that $\lambda_r > 0$, to get

$$\begin{aligned} \sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u &\le \gamma_r \frac{(4\rho)^{\frac{Np}{\lambda_r}}}{(\theta\rho^p)^{\frac{N}{\lambda_r}}} \Big(\fint_{K_{4\rho}} u^r(x,-\theta\rho^p)dx \Big)^{\frac{1}{r}\frac{rp}{\lambda_r}} + \gamma_r \theta^{\frac{1}{3-m-p}} \\ &\le \frac{\gamma_r'}{\varepsilon^{\frac{N(3-m-p)}{\lambda_r}}} \frac{1}{\sigma} \Big(\fint_{K_\rho} u^q(x,0)dx \Big)^{\frac{1}{q}} + \gamma_r' \varepsilon \Big(\fint_{K_\rho} u^q(x,0)dx \Big)^{\frac{1}{q}} \\ &= \gamma_r' \varepsilon \Big(1 + \frac{1}{\sigma \varepsilon^{\frac{rp}{\lambda_r}}} \Big) \Big(\fint_{K_\rho} u^q(x,0)dx \Big)^{\frac{1}{q}} \end{aligned}$$

for a constant γ_r' depending only upon the data $\{p, m, N, C_0, C_1\}$ and r. One verifies that $\gamma_r' \to \infty$, as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Assume momentarily that $0 < \sigma < 1$ so that in the round brackets containing σ, the second term dominates the first one. In such a case

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u \le \frac{1}{\varepsilon'\sigma} \Big(\fint_{K_\rho} u^q(x,0)dx \Big)^{\frac{1}{q}} =: M, \tag{4.33}$$

where

$$\varepsilon' = \frac{\varepsilon^{\frac{N(3-m-p)}{\lambda_r}}}{2\gamma_r'}.$$

From this

$$\varepsilon'\sigma M = \Big(\fint_{K_\rho} u^q(x,0)dx \Big)^{\frac{1}{q}}.$$

Let $\nu \in (0,1)$ to be chosen. Using (4.32) and (4.34) estimate

$$\begin{aligned} (\varepsilon'\sigma M)^q &\le 2^{N+1}\gamma_q \fint_{K_{2\rho}} u^q(x,\tau)dx \\ &\le 2^{N+1}\gamma_q \Bigg(\fint_{K_{2\rho}\cap[u<\nu\sigma M]} u^q(x,\tau)dx + \fint_{K_{2\rho}\cap[u\ge\nu\sigma M]} u^q(x,\tau)dx \Bigg) \\ &\le 2^{N+1}\gamma_q \nu^q (\sigma M)^q + 2^{N+1}\gamma_q M^q \frac{|[u(\cdot,\tau) > \nu\sigma M] \cap K_{2\rho}|}{|K_{2\rho}|} \end{aligned}$$

for all $\tau \in (-\frac{1}{2}\theta\rho^p, 0]$. From this

$$|[u(\cdot,\tau) > \nu\sigma M] \cap K_{2\rho}| \geq \alpha\sigma^q |K_{2\rho}|, \tag{4.34}$$

where

$$\alpha = \frac{\varepsilon'^q - \nu^q 2^{N+1}\gamma_q}{2^{N+1}\gamma^q},$$

for all $\tau \in (-\frac{1}{2}\theta\rho^p, 0]$. By choosing $\nu \in (0,1)$ sufficiently small, only dependent on the data $\{p, m, N, C_0, C_1\}$ and γ_q, we can ensure that $\alpha \in (0,1)$ depends only upon the data $\{p, m, N, C_0, C_1\}$ and q, and is independent of σ. We summarize

Proposition 4.4.5 *Let u be a non-negative, locally bounded, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3)*, for $2 < p + m < 3$. Fix $(x_0, t_0) \in E_T$, let $K_{4\rho}(x_0) \subset E$ and let θ and σ be defined by*(4.25)-(4.27) *for some $\varepsilon \in (0,1)$. Suppose $0 < \sigma < 1$. For every $r \geq 1$ satisfying* (4.29) *and every $q \geq 1$, there exist constants $\varepsilon, \nu, \alpha \in (0,1)$, depending only upon the data $\{p, m, N, C_0, C_1\}$, q and r, such that*

$$|[u(\cdot,t) > \nu\sigma M] \cap K_{2\rho}(x_0)| \geq \alpha\sigma^q |K_{2\rho}|$$

for all $t \in (t_0 - \frac{1}{2}\theta\rho^p, t_0]$.

4.4.3 A first form of the Harnack inequality

The definitions (4.25) of θ and the parameters ε' and α imply that

$$\frac{1}{2}\theta = \epsilon(\nu\sigma M)^{3-m-p}, \qquad \text{where } \epsilon = \frac{1}{2}\Big(\frac{\varepsilon\varepsilon'}{\nu}\Big)^{3-m-p}.$$

By Proposition 4.2.1 with M replaced by $\nu\sigma M$ and α replaced by $\alpha\sigma^q$, there exist constants η and δ in $(0,1)$, depending upon the data $\{p, m, N, C_0, C_1\}$ and α, σ and ϵ such that

$$u(\cdot,t) > \eta(\alpha\sigma^q, \epsilon)\nu\sigma M \qquad \text{in } K_{4\rho}(x_0),$$

for all times

$$t \in (t_0 - \frac{1}{2}\theta\rho^p + \delta(\nu\sigma M)^{3-m-p}(2\rho)^p, t_0]$$

where δ includes the quantity $1-\varepsilon$ of Proposition 4.2.1. Without loss of generality we can assume that this time interval contains $(t_0 - \frac{1}{4}\theta\rho^p, t_0]$.

Proposition 4.4.6 (A first form of the Harnack inequality) *Let u be a non-negative, locally bounded, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3)*, for $2 < p + m < 3$. Fix $(x_0, t_0) \in E_T$, let $K_{4\rho}(x_0) \subset E$ and let θ and σ be defined by*(4.25)-(4.27) *for some $\varepsilon \in (0,1)$. Suppose $0 < \sigma < 1$. For every $r \geq 1$ satisfying* (4.29) *and every $q \geq 1$, there exist constants $\varepsilon, \delta \in (0,1)$, and a continuous, increasing function $\sigma \to f(\sigma)$ defined in $\mathbf{R}^+$ and vanishing at $\sigma = 0$, that can be quantitatively determined a priori only in terms of the data $\{p, m, N, C_0, C_1\}$, q, and r, such that*

$$\inf_{K_{4\rho}(x_0)} u(\cdot,t) \geq f(\sigma) \sup_{(x_0,t_0)+Q^-_{2\rho(\frac{1}{4}\theta)}} u, \tag{4.35}$$

for all

$$t \in (t_0 - \frac{1}{4}\theta\rho^p, t_0],$$

provided $(x_0, t_0) + Q^-_{8\rho(\theta)} \subset E_T$.

Remark 4.4.7 *The proof of Proposition 4.2.1 shows that the function* $f(\cdot)$ *can be taken of the form*

$$f(\sigma) \approx \sigma B^{-\frac{1}{\sigma^d}},$$

for constants $B, d > 1$ *depending only upon the data,* q *and* r.

Remark 4.4.8 *The function* $f(\cdot)$ *depends on* θ *only through the parameter* ε *in the definition* (4.25) *of* θ.

Remark 4.4.9 *The inequality* (4.35) *is a Harnack-type estimate of the same form as stated in section 4.3, where however the constant* $f(\sigma)$ *depends on the solution itself, through* σ *defined in* (4.27)*, as a proper quotient of the* L^q_{loc} *and* L^r_{loc} *averages of* u*, respectively at time* $t = t_0$ *on the cube* $K_\rho(x_0)$*, and at time* $t = t_0 - \theta\rho^p$ *on the cube* $K_{4\rho}(x_0)$.

Remark 4.4.10 *The inequality* (4.35) *has been derived by assuming that* $0 < \sigma < 1$. *If* $\sigma \geq 1$ *the same proof gives* (4.35) *where* $f(\sigma) \geq f(1)$*, thereby establishing a strong form of the Harnack estimate for these solutions. Such a strong form is false for* p, m *in the critical, and sub-critical range* $2 < p + m \leq 3 - \frac{p}{N}$.

In Vespri [53], local Hölder continuity for solutions to (2.1)-(2.2)-(2.3) has been proved for all $m + p > 2$. We recall this result in the next proposition.

Proposition 4.4.11 *Let* u *be a locally bounded, local, weak solution to the singular equations* (2.1)-(2.2)-(2.3) *for* $2 < p + m < 3$*, in* E_T. *There exist constants* $\bar{\gamma}, A > 1$ *and* $\epsilon_0 \in (0, 1)$*, depending only upon the data* $\{p, m, N, C_0, C_1\}, q$ *and* r*, such that for all* $(x_0, t_0) \in E_T$*, setting*

$$M = \operatorname*{ess\,sup}_{(x_0,t_0)+Q^-_R(1)} u \qquad \textit{for } (x_0, t_0) + Q^-_R(1) \subset E_T,$$

there holds

$$\operatorname*{ess\,osc}_{(x_0,t_0)+Q^-_\rho(\theta_M)} u \leq \bar{\gamma} M \Big(\frac{\rho}{R}\Big)^{\epsilon_0}, \qquad \textit{where } \theta_M = \Big(\frac{M}{A}\Big)^{3-m-p}$$

for all $0 < \rho \leq R$*, and all cylinders*

$$(x_0, t_0) + Q^-_\rho(\theta_M) \subset (x_0, t_0) + Q^-_R(1) \subset E_T.$$

4.4.4 Proof of Theorem 4.4.1 concluded

Assume (x_0, t_0) coincides with the origin of $\mathbf{R}^{N+1}$ and determine ν and α as in section 4.4.2. We may assume that

$$|[u(\cdot, 0) \leq \nu\sigma M] \cap K_\rho| > 0.$$

Indeed otherwise (4.34) would hold with $\alpha\sigma^q = 1$ and the proof could be repeated leading to (4.35) with f depending only on the data $\{p, m, N, C_0, C_1\}$ and independent of σ. Moreover, by (4.33)

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u \le M$$

with θ given by (4.25). Since u is locally Hölder continuous, there exists $x_1 \in K_\rho$ such that

$$u(x_1, 0) = \nu\sigma M.$$

Using the parameter A claimed by Proposition 4.4.11, construct the cylinder with "vertex" at $(x_1, 0)$

$$(x_1, 0) + Q^-_{2r}\Big[\Big(\frac{\nu\sigma M}{A}\Big)^{3-m-p} r^p\Big] \subset K_{2\rho} \times \Big(-\frac{1}{4}\theta\rho^p, 0\Big].$$

In the definition (4.25) of θ, such an inclusion can be realized by possibly increasing A by a fixed quantitative factor depending only on the data, and by choosing r sufficiently small. Assuming the choice of r has been made, by Proposition 4.4.11

$$|u(x,t) - u(x,0)| \le \bar\gamma M \Big(\frac{r}{\rho}\Big)^{\epsilon_0}$$

for all

$$(x,t) \in \tilde Q_1 =: (x_1, 0) + Q^-_r\Big[\Big(\frac{\nu\sigma M}{A}\Big)^{3-m-p} r^p\Big].$$

From this

$$u(x,t) \ge \frac{1}{2}\nu\sigma M \qquad \text{for all } (x,t) \in \tilde Q_1,$$

provided r is chosen to be so small that

$$\frac{\bar\gamma}{\nu\sigma}\Big(\frac{r}{\rho}\Big)^{\epsilon_0} = \frac{1}{2}, \qquad \text{that is } r = \varepsilon_1 \sigma^{\frac{1}{\epsilon_0}} \rho, \qquad \text{where } \varepsilon_1 = \Big(\frac{\nu}{2\bar\gamma}\Big)^{\frac{1}{\epsilon_0}}.$$

Therefore, by Proposition 4.2.1

$$u \ge \eta(\nu\sigma M) \qquad \text{in} \qquad (x_1, 0) + Q^-_{2r}\Big[\Big(\frac{\eta(\nu\sigma M)}{A}\Big)^{3-m-p} (2r)^p\Big]$$

for an absolute constant $\eta \in (0, 1)$. This process can now be iterated to give

$$u \ge \eta^n(\nu\sigma M) \qquad \text{in} \qquad (x_1, 0) + Q^-_{2^n r}\Big[\Big(\frac{\eta^n(\nu\sigma M)}{A}\Big)^{3-m-p} (2^n r)^p\Big]$$

for all $n \in \mathbf{N}$. Choose n as the smallest integer for which

$$2^n r \ge 4\rho \qquad \text{that is} \qquad n \ge \log_2\left(\frac{4}{\varepsilon_1 \sigma^{\frac{1}{\epsilon_0}}}\right).$$

For such a choice

$$u \ge \gamma\sigma^\beta M \qquad \text{in } Q^-_{2\rho}\Big[\Big(\frac{\gamma\sigma^\beta M}{A}\Big)^{3-m-p} \rho^p\Big]$$

for some $\beta = \beta(\text{data})$. $\square$

Conclusions and future prospects

In this work we dealt with two different subjects: an Optimal Transportation problem and a class of doubly nonlinear partial differential equations. For both the arguments we concentrated on regularity issues, proving some new results.

We analyzed the Optimal Transportation problem (1.2), which has been introduced by Gangbo and McCann in [27]. The peculiarity of this problem is that the masses are distributed on boundaries of convex domains, leading to multi-valued optimal mappings, instead of a single-valued optimal map. The novelty of this work is the quantification of the continuity of such multi-valued mappings, $t^{\pm}$. In particular, we proved Hölder continuity for the restrictions of $t^{\pm}$ to certain subsets of their domains, under the additional hypothesis that the masses are distributed on n-dimensional spheres. According to [27], the domain of the mapping t^{+} can be partitioned into three subsets $\mathbf{S}^N = S_0 \cup S_1 \cup S_2$, where S_2 represents the domain of t^{-}. A conjecture by Gangbo and McCann, expressed in Remark 4.5 of [27], claims Hölder continuity for t^{+} on $S_1 \cup S_2$, when the masses are supported on boundaries of more general convex domains. Since the Hölder constants of Theorem 1.2.4 blow up approaching S_0, the result cannot be extended to $S_1 \cup S_2$. On the other hand we believe that Theorem 1.2.4 continues to hold when the measures are supported on boundaries of convex domains more general than spheres. Recalling Remark 1.4.3, it would be sufficient to extend Theorem 1.6.1 to such domains, which seems doable at least for domains which can be obtained by small perturbations of a sphere.

As for the second subject of my thesis, we proved some Harnack estimates for nonnegative weak solutions of the class of doubly nonlinear parabolic equations (2.1), satisfying the structure conditions (2.2)-(2.3). We analyzed both the degenerate and the singular case by means of purely measure theoretical arguments. As suggested for the model equation (2.4) in [54], the singular case presents a critical threshold under which only a weak "Harnack-type" estimate can be proven. Indeed, for the degenerate and the singular super-critical range, we proved an intrinsic Harnack estimate whose constants depend only on the data; for the singular sub-critical range such inequalities cannot hold, and a weaker Harnack-type estimate has been proven, with coefficients depending on the solution itself. However, by means of these results, we showed that weak solutions of (2.1) are Hölder continuous in the degenerate case. We were not able to prove the Harnack inequality for the general operator in the singular super-critical range; we can conclude that the theory of nonlinear parabolic equations is still at its inception, and some of its aspects are largely open to a better understanding.

Appendix A

Some results related to Chapter 1

A.1 Some useful results

In the following we collect some well-known results, but we omit their seemingly well-known proofs.

Proposition A.1.0.1 *Subtracting from the cost function c a smooth function $x \to \phi(x)$ that depends only on x does not change the solution of the optimal transportation problem. The potential will be changed according to the rule $\psi \to \psi + \phi(x)$.*

Proposition A.1.0.2 (Strong convexity) *Let f belong to $C^2(\mathbf{R}, \mathbf{R})$.*

- *If $f'' \geq \alpha$, we have, for all $t_0, t_1 \in \mathbf{R}$ and $\theta \in [0,1]$,*

$$\theta f(t_0) + (1-\theta) f(t_1) \geq f(\theta t_0 + (1-\theta) t_1) + \frac{1}{2} \alpha \theta (1-\theta) |t_1 - t_0|^2.$$

- *In all cases, for all $t_0, t_1 \in \mathbf{R}$ and $\theta \in [0,1]$, we have*

$$|\theta f(t_0) + (1-\theta) f(t_1) - f(\theta t_0 + (1-\theta) t_1)| \leq \frac{1}{2} ||f''||_{L^\infty(t_0,t_1)} \theta (1-\theta) |t_1 - t_0|^2.$$

The following proposition can be found in [22].

Proposition A.1.0.3 *Let $E \subset \mathbf{R}^N$ and $f : E \to \mathbf{R}^N$ be a Lipschitz function with Lipschitz constant $Lipf$. Then*

$$\mathcal{H}^s(f(E)) \leq (Lipf)^s \mathcal{H}^s(E) \qquad \forall s \geq 0.$$

Theorem A.1.0.4 (Mean value theorem) *Suppose $h : \mathbf{R}^N \to \mathbf{R}$ is C^1. Let $y = (y^1, \ldots, y^N), z = (z^1, \ldots, z^N) \in \mathbf{R}^N$. Then*

$$h(y) - h(z) = \nabla h(u) \cdot (y - z),$$

for some u on the line segment between y and z.

A.2 Proofs from Section 1.7

Proof of Lemma 1.7.5

Using (1.31) and (1.32), we have

$$H = \psi(X_0) \leq -\nabla_X c(X_m, Y_0) \cdot (X_0 - X_m) + C_1|X_0 - X_m|^2, \tag{A.1}$$

$$H = \psi(X_1) \leq -\nabla_X c(X_m, Y_1) \cdot (X_1 - X_m) + C_1|X_1 - X_m|^2, \tag{A.2}$$

where C_1 depends only on $\|c\|_{C^2}$. By possibly rotating coordinates, we can certainly suppose that x_0 is parallel to e_{n+1}. Consider now the function $f : t \to \psi(X_0 + t(X_1 - X_0))$ on $[0,1]$. Lemma 1.6 of [27] states that, if ω is a Lipschitz domain, then, at dVol*-a.e. boundary point x, ψ is tangentially differentiable. Hence f is* dVol*-a.e. differentiable. At the points where f'' exists we have*

$$\begin{aligned} f''(t) = & \sum_{i,k=1}^{n} \frac{\partial\psi}{\partial x^i \partial x^k}(\pi_{x_0}^{-1}(X_t))(X_1 - X_0)(X_1 - X_0)^k - o(|X_1 - X_0|^2) \\ \geq & \quad -||D^2_{xx}c||_{L^\infty(\mathbf{S}^N \times \mathbf{S}^N)}|X_1 - X_0|^2 - o(|X_1 - X_0|^2), \end{aligned}$$

with $X_t = X_0 + t(X_1 - X_0)$. We approximate ψ with a C^2-function $\tilde{\psi}$ which satisfies

$$\begin{gathered} \tilde{\psi}(X_0) = \psi(X_0), \tilde{\psi}(X_1) = \tilde{\psi}(X_1), \\ \tilde{\psi}(X) = \psi(X) + o(|X_1 - X_0|^2) \quad \text{on } [X_0, X_1], \\ \tilde{f}''(t) \geq -||D^2_{xx}c||_{L^\infty(\mathbf{S}^N \times \mathbf{S}^N)}|X_1 - X_0|^2 - o(|X_1 - X_0|^2), \end{gathered}$$

where $\tilde{f}(t) = \tilde{(\psi)}(X_0 + t(X_1 - X_0))$ on $[0,1]$. Applying the first part of Proposition A.1.0.2 to $\tilde{f}$ we find

$$\tilde{\psi}(X) \leq H + \tilde{C}_2|X_1 - X_0|^2 \qquad \text{for all} \quad X \in [X_0, X_1],$$

where $\tilde{C}_2$ depends only on $\|c\|_{C^2}$. It follows

$$\psi(X) \leq H + C_2|X_1 - X_0|^2 \qquad \text{for all} \quad X \in [X_0, X_1],$$

where C_2 depends only on $\|c\|_{C^2}$. We now consider two cases. The first case is when $-\nabla_X c(X_m, Y_0)\cdot (X_0 - X_m)$ and $-\nabla_X c(X_m, Y_1) \cdot (X_1 - X_m)$ are not both positive: let us assume for example that $-\nabla_x c(X_m, Y_0) \cdot (X_0 - X_m)$ is negative. Then, using (A.1), we have $H \leq C_1|X_0 - X_m|^2$, and using (A.3) we conclude

$$\psi \leq C_3|X_1 - X_0|^2 \leq C_3|X_1 - X_0||Y_1 - Y_0| \qquad \text{for all} \quad X \in [X_0, X_1],$$

where $C_3 = C_1 + C_2$. The second case is when $-\nabla_X c(X_m, Y_0) \cdot (X_0 - X_m)$ and $-\nabla_X c(X_m, Y_1) \cdot (X_1 - X_m)$ are both positive. This implies that

$$\begin{gathered} -\nabla_X c(X_m, Y_0) \cdot (X_0 - X_m) \leq -\nabla_X c(X_m, Y_0) \cdot (X_0 - X_1), \\ -\nabla_X c(X_m, Y_1) \cdot (X_1 - X_m) \leq -\nabla_X c(X_m, Y_1) \cdot (X_1 - X_0). \end{gathered}$$

Combining with (A.1) and (A.2) we have

$$\begin{aligned} 2H \quad \leq & \quad -\nabla_X c(X_m, Y_0) \cdot (X_0 - X_1) - \nabla_X c(X_m, Y_1) \cdot (X_1 - X_0) \\ + & \quad 2C_1|X_1 - X_0|^2 \\ \leq & \quad |\nabla_X c(X_m, Y_0) - \nabla_X c(X_m, Y_1)||X_1 - X_0| + 2C_1|X_1 - X_0|^2 \\ \leq & \quad (\|c\|_{C^2} + 2C_1)(|X_1 - X_0||Y_1 - Y_0| + |X_1 - X_0|^2). \end{aligned}$$

Using $|X_1 - X_0| \le |Y_1 - Y_0|$*, and then (A.3) we conclude.* □

Proof of Lemma 1.7.6
For $y \in \bar{N}(x)$ *we have*

$$\begin{aligned} -c(X,Y) + c(X_m,Y) &= -c(X,Y_\theta) + c(X_m,Y_\theta) + \\ &+ \int_0^1 [\nabla_Y c(X_m, Y_\theta + s(Y - Y_\theta)) - \nabla_Y c(X, Y_\theta + s(Y - Y_\theta))] \cdot (Y - Y_\theta) ds \\ &\le -c(X,Y_\theta) + c(X_m,Y_\theta) + C_4|Y - Y_\theta||X - X_m|, \end{aligned}$$

where C_4 *depends only on* $\|c\|_{C^2}$*. Combining this with Lemma 1.7.5 to estimate* $\psi(X_m)$*, we get*

$$\begin{aligned} f_Y(X) &= -c(X,Y) + c(X_m,Y) + \psi(X_m) \\ &\le -c(X,Y_\theta) + c(X_m,Y_\theta) + C_4|Y - Y_\theta||X - X_m| + C_3|Y_1 - Y_0||X_1 - X_0| \\ &=: F_Y(X). \end{aligned}$$

Inequality $\psi - f_Y \ge 0$ *on* $\partial B_\eta(X_m)$ *will be satisfied if we have* $F_Y(X) \le \Psi(X)$ *on the set*$\{X : |X - X_m| = \eta\}$*, for some* $\eta > 0$ *and* Ψ *defined in (1.33). First we restrict* $\theta \in \left[\frac{1}{4}, \frac{3}{4}\right]$*, then* $F_Y(X) \le \Psi(X)$ *reads*

$$\frac{3}{16}|Y_1 - Y_0|^2\eta^2 - \upsilon\eta^3 \ge C_4|Y_1 - Y_\theta|\eta + C_3|Y_1 - Y_0||X_1 - X_0|.$$

The previous inequality will be satisfied if the three following inequalities are satisfied

$$\begin{aligned} C_3|Y_1 - Y_0||X_1 - X_0| &\le \frac{1}{16}|Y_1 - Y_0|^2\eta^2, \\ C_4|Y - Y_\theta|\eta &\le \frac{1}{16}|Y_1 - Y_0|^2\eta^2, \\ \upsilon\eta^3 &\le \frac{1}{16}|Y_1 - Y_0|^2\eta^2. \end{aligned}$$

To satisfy the first inequality we define

$$\eta^2 = 16C_3\frac{|X_1 - X_0|}{|Y_1 - Y_0|}.$$

To satisfy the second inequality we define

$$\rho = \frac{1}{16C_4}\eta|Y_1 - Y_0|^2,$$

and consider Y *such that* $|Y - Y_\theta| \le \mu$*. The third inequality will then be implied by*

$$\upsilon\eta \le \frac{1}{16}|Y_1 - Y_0|^2,$$

which follows from (1.34). □

Appendix B

Parabolic spaces, embeddings, and technical facts

This appendix contains some well known results of functional analysis whose proofs can be found in many textbooks (see for example [16]).

B.1 Poincaré and Sobolev inequalities

Let E be a bounded domain in $\mathbf{R}^N$ of boundary ∂E. If $f \in L^q(E)$ for some $1 \le q \le \infty$, denote by $\|f\|_{q,E}$ the $L^q(E)$-norm of f over E. We also write $\|f\|_q$ whenever the specification of the domain E is unambiguous from the context. The function $f \in L^q_{loc}(E)$ if $\|f\|_{q,K} < \infty$ for all compact subsets $K \subset E$. For $f \in C^1(E)$ denote by $Df = (f_{x_1}, \dots, f_{x_N})$ its gradient and set

$$\|f\|_{1,p;E} = \|f\|_{p,E} + \|Df\|_{p,E}.$$

The spaces $W^{1,p}(E)$ and $W^{1,p}_0(E)$ for $p \ge 1$ are defined as

$$\begin{array}{ll} W^{1,p}(E) & \text{the completion of } C^\infty(E) \text{ under } \|\cdot\|_{1,p;E}, \\ W^{1,p}_0(E) & \text{the completion of } C^\infty_0(E) \text{ under } \|\cdot\|_{1,p;E}. \end{array}$$

Equivalently $W^{1,p}(E)$ is the Banach space of the functions $f \in L^p(E)$ whose generalized derivatives f_{x_i} belong to $L^p(E)$ for all $i = 1, \dots, N$.

A function $f \in W^{1,p}_{loc}(E)$ if $\|f\|_{1,p;K} < \infty$ for every compact subset $K \subset E$.

Let $W^{1,\infty}(E)$ denote the Banach space of functions $f \in L^\infty(E)$ whose distributional derivatives $f_{x_i} \in L^\infty(E)$, for $i = 1, \dots, N$.

The space $W^{1,\infty}_{loc}(E)$ is defined analogously.

Theorem B.1.1 (Gagliardo-Nirenberg inequality) *Let $v \in W^{1,p}_0(E)$ for some $p \ge 1$. For every $s \ge 1$ there exists a constant C depending only upon N, p, q, and s, and independent of E, such that*

$$\|v\|_{q,E} \le C \|Dv\|^{\alpha}_{p,E} \|v\|^{1-\alpha}_{s,E}$$

where $\alpha \in [0,1]$ *and* $p, q \geq 1$, *are linked by*

$$\alpha = \Big(\frac{1}{s} - \frac{1}{q}\Big)\Big(\frac{1}{N} - \frac{1}{p} + \frac{1}{s}\Big)^{-1}$$

and their admissible range is

$$\begin{array}{llll} if & N = 1, & \alpha \in \Big[0, \frac{p}{p+s(p-1)}\Big], & q \in [s, \infty]; \\ if & 1 \leq p \leq N, & \alpha[0,1], & \begin{cases} q \in [s, \frac{Np}{N-p}] & if \quad s \leq \frac{Np}{N-p}, \\ q \in [\frac{Np}{N-p}] & if \quad s \geq \frac{Np}{N-p}; \end{cases} \\ if & 1 < N \leq p, & \alpha \in \Big[0, \frac{Np}{Np+s(p-N)}\Big), & q \in [s, \infty). \end{array}$$

Corollary B.1.2 *Let* $v \in W_0^{1,p}(E)$, *and assume* $p \in [1, N)$. *There exists a constant* $\gamma = \gamma(N, p)$ *such that*

$$\|v\|_{q,E} \leq \gamma \|Dv\|_{p,E}, \qquad where \qquad q = \frac{Np}{N-p}. \tag{B.1}$$

The boundary ∂E is *piecewise smooth* if it is the union of finitely many portions of $(N-1)$-dimensional hypersurfaces of class $C^{1,\lambda}$, for some $\lambda \in (0,1)$.

If ∂E is piecewise smooth, functions v in $W^{1,p}(E)$ are defined up to ∂E via their traces denoted by $v\lfloor_{\partial E}$.

Theorem B.1.3 *Let* ∂E *be piecewise smooth. There exists a constant* C *depending only upon* N, p *and the structure of* ∂E *such that*

$$\|v\|_{q,\partial E} \leq C \|v\|_{W^{1,p}(E)},$$

where

$$\begin{array}{ll} q \in [1, \frac{(N-1)p}{N-p}], & if \quad 1 < p < N \\ q \in [1, \infty), & if \quad p = N. \end{array}$$

If ∂E is piecewise smooth, the space $W_0^{1,p}(E)$ can be defined equivalently as the set of functions $v \in W^{1,p}(E)$ whose trace on ∂E is zero.

Remark B.1.4 *The embedding inequalities of Theorem B.1.1 and Corollary B.1.2 continue to hold for functions* v *in* $W^{1,p}(E)$ *not necessarily vanishing on* ∂E *in the sense of the traces, provided* ∂E *is piecewise smooth and*

$$\int_E v(x)dx = 0.$$

In such a case the constant C *depends upon* s, p, q, N *and the structure of* ∂E. *However it does not depend on the* size *of* E, *and in particular it does not change by dilations of* E.

B.2 Cuts and truncations of functions in $W^{1,p}(E)$ and their embeddings

Let k be any real number and for a function $v \in W^{1,p}(E)$ consider the truncations of v given by

$$\begin{aligned} (v-k)_+ &= \max\{(v-k); 0\}, \\ (v-k)_- &= \max\{-(v-k); 0\}. \end{aligned}$$

Lemma B.2.1 (Stampacchia) *Let $v \in W^{1,p}(E)$. Then $(v-k)_\pm \in W^{1,p}(E)$ for all $k \in \mathbf{R}$. If in addition the trace of v on ∂E is essentially bounded and*

$$\|v\|_{\infty,\partial E} \le M \qquad \textit{for some} \quad M > 0$$

then $(v-k)_\pm \in W_0^{1,p}(E)$ for all $k \ge M$.

Corollary B.2.2 *Let $v_i \in W^{1,p}(E)$ for $i = 1, \dots, n \in \mathbf{N}$. Then*

$$w = \min\{v_1, \dots, v_n\} \in W^{1,p}(E).$$

For a function v defined in E and real numbers $k < l$, set

$$\begin{aligned}
[v > l] &= \{x \in E \mid v(x) > l\},\\
[v < k] &= \{x \in E \mid v(x) < k\},\\
[l < v < k] &= \{x \in E \mid l < v(x) < k\}.
\end{aligned}$$

For $\rho > 0$ and $y \in \mathbf{R}^N$, denote by $B_\rho(y)$ the ball of radius ρ centered at y, and by $K_\rho(y)$ the cube of edge ρ, centered at y and with faces parallel to the coordinate planes. If y is the origin let $B_\rho(0) = B_\rho$, and $K_\rho(0) = K_\rho$.

For a Lebesgue measurable set $A \subset \mathbf{R}^N$ denote by $|A|$ its measure.

Lemma B.2.3 (DeGiorgi) *Let $v \in W^{1,1}(K_\rho(y))$, and let $k < l$ be real numbers. There exists a constant γ depending only upon N, p and independent of k, l, v, y, ρ, such that*

$$(l-k)|[v > l]| \le \gamma \frac{\rho^{N+1}}{|[v < k]|} \int_{[k<v<l]} |Dv| dx. \tag{B.2}$$

Remark B.2.4 *The conclusion of the lemma continues to hold for functions $v \in W^{1,1}(E)$ provided E is* convex. *It can be used for balls $B_\rho(y)$.*

The embedding (B.1) of Corollary B.1.2 gives a majorization of the $L^q(E)$-norm of u solely in terms of the $L^p(E)$-norm of its gradient. This is possible because u vanishes on ∂E in the sense of the traces.

A Poincaré-type inequality bounds some integral norm of a function $u \in W^{1,p}(E)$ in terms *only* of some integral norm of its gradient, provided some information is available on the set where u vanishes.

Proposition B.2.5 *Let $E \subset \mathbf{R}^N$ be bounded and convex and let $\phi \in C(\bar{E})$ satisfy*

$$0 \le \phi \le 1, \qquad \textit{and the sets } [\phi > k] \textit{ are convex for all } k \in \mathbf{R}^+.$$

Let $v \in W^{1,p}(E)$ and assume that the set

$$\mathcal{E} = [v = 0] \cap [\phi = 1]$$

has positive measure. There exists a constant C depending only upon N and p, and independent of v and ϕ, such that

$$\Big(\int_E \phi |v|^p dx \Big)^{\frac{1}{p}} \le C \frac{(diam(E))^N}{|\mathcal{E}|^{\frac{N-1}{N}}} \Big(\int_E \phi |Dv|^p dx \Big)^{\frac{1}{p}}. \tag{B.3}$$

Remark B.2.6 *Inequality* (B.2) *follows from this by applying* (B.3) *with* $\phi = 1$ *and* $p = 1$ *to the function*

$$w = \begin{cases} \min\{v, l\} - k & \text{if } v > k \\ 0 & \text{if } v \le k \end{cases}$$

By Lemma B.2.1 such a function is in $W^{1,1}(E)$.

B.3 Parabolic spaces and embeddings

For $0 < T < \infty$ let E_T denote the cylindrical domain $E \times (0, T]$. The space $L^{r,q}(E_T)$ for $q, r \ge 1$ is the collection of functions f defined and measurable in E_T such that

$$\|f\|_{q,r;E_T} = \left(\int_0^T \left(\int_E |f|^q dx \right)^{\frac{r}{q}} d\tau \right)^{\frac{1}{r}} < \infty.$$

Also $f \in L^{q,r}_{loc}(E_T)$, if for every compact subset $K \subset E$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_{t_1}^{t_2} \left(\int_K |f|^q dx \right)^{\frac{r}{q}} d\tau < \infty.$$

Whenever $q = r$ we set $L^{q,q}(E_T) = L^q(E_T)$. These definitions are extended in the obvious way when either q or r are infinity.

We introduce spaces of functions, depending on $(x, t) \in E_T$, that exhibit different behavior in the space and time variables. These are spaces where typically solutions to parabolic equations in divergence form are found.

Let $m, p \ge 1$ and consider the Banach spaces

$$\begin{aligned} V^{m,p}(E_T) &= L^\infty(0, T; L^m(E)) \cap L^p(0, T; W^{1,p}(E)), \\ V_0^{m,p}(E_T) &= L^\infty(0, T; L^m(E)) \cap L^p(0, T; W_0^{1,p}(E)), \end{aligned}$$

both equipped with the norm

$$\|v\|_{V^{m,p}(E_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{m,E} + \|Dv\|_{p,E_T}.$$

When $m = p$ set $V_0^{p,p}(E_T) = V_0^p(E_T)$ and $V^{p,p}(E_T) = V^p(E_T)$. Both spaces are embedded in $L^q(E_T)$ for some $q > p$. In a precise way we have

Proposition B.3.1 *There exists a constant* γ *depending only upon* N, p, m *such that for every* $v \in V_0^{m,p}(E_T)$

$$\begin{aligned} \iint_{E_T} |v(x,t)|^q dxdt \quad &\le \quad \gamma^q \left(\iint_{E_T} |Dv(x,t)|^p dxdt \right) \\ &\times \quad \left(\operatorname*{ess\,sup}_{0<t<T} \int_E v(x,t)|^m dx \right)^{\frac{p}{N}} \end{aligned} \tag{B.4}$$

where

$$q = p\frac{N+m}{N}.$$

Moreover

$$\|v\|_{q,E_T} \le \gamma \|v\|_{V^{m,p}(E_T)}. \tag{B.5}$$

Remark B.3.2 *The multiplicative inequality* (B.4) *and the embedding* (B.5) *continue to hold for functions* $v \in V^{m,p}(E_T)$ *such that*

$$\int_E v(x,t)dx = 0 \qquad \text{for a.e. } t \in (0,T)$$

provided ∂E *is piecewise smooth. In such a case the constant* γ *depends also on the structure of* ∂E*, but not on its size.*

The next corollary follows from Proposition B.3.1 by taking $m = p$ and by applying Hölder's inequality.

Corollary B.3.3 *Let* $p > 1$*. There exists a constant* γ*, depending only upon* N *and* p*, such that for every* $v \in V_0^p(E_T)$

$$\|v\|_{p,E_T}^p \le \gamma \, |[|v| > 0]|^{\frac{p}{N+p}} \, \|v\|_{V^p(E_T)}^p.$$

When $m = p$, Proposition B.3.1 takes the form

Proposition B.3.4 *There exists a constant* γ*, depending only upon* N *and* p*, such that for every* $v \in V_0^p(E_T)$

$$\|v\|_{q,r;E_T} \le \gamma \|v\|_V^p(E_T),$$

where the numbers $q, r \ge 1$ *are linked by*

$$\frac{1}{r} + \frac{N}{pq} = \frac{N}{p^2},$$

and their admissible range is

$$\begin{array}{lll} \text{if } N = 1, & q \in (p,\infty], & r \in [p^2,\infty); \\ \text{if } 1 \le p < N, & q \in [p, \frac{Np}{N-p}], & r \in [p,\infty]; \\ \text{if } 1 < N \le p, & q \in [p,\infty), & r \in (\frac{p^2}{N},\infty]. \end{array}$$

We conclude this section by stating a parabolic version of Lemma B.2.1 and Corollary B.2.2 concerning the truncated functions $(v-k)_\pm$.

Lemma B.3.5 *Let* $v \in V^{m,p}(E_T)$*. Then* $(v-k)_\pm \in V^{m,p}(E_T)$ *for all* $k \in \mathbf{R}$*. Assume in addition that* ∂E *is piecewise smooth and that the trace of* $v(\cdot,t)$ *on* ∂E *is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{\infty,\partial E} \le M \qquad \text{for some } M > 0.$$

Then $(v-k)_\pm \in V_0^{m,p}(E_T)$ *for all* $k \ge M$*.*

B.4 Some technical facts

B.4.1 A lemma on fast geometric convergence

Lemma B.4.1 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \le C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$$

then $\{Y_n\} \to 0$ as $n \to \infty$.

B.4.2 An interpolation lemma

Lemma B.4.2 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities*

$$Y_n \le C b^n Y_{n+1}^{1-\alpha},$$

where $C, b > 1$ and $\alpha \in (0, 1)$ are given constants. Then

$$Y_0 \le \left(\frac{2C}{b^{1-\frac{1}{\alpha}}} \right)^{\frac{1}{\alpha}}.$$

Remark B.4.3 *The lemma turns the qualitative information of equi-boundedness of the sequence $\{Y_n\}$ into a quantitative a priori estimate for Y_0.*

B.4.3 Steklov averages

Let $v \in L^1(E_T)$ and let $0 < h < T$; the Steklov averages $v_h(\cdot, t)$ and $v_{\bar{h}}(\cdot, t)$ are defined by

$$v_h = \begin{cases} \frac{1}{h} \int_t^{t+h} v(\cdot, \tau) d\tau & \text{for} \quad t \in (0, T-h], \\ 0 & \text{for} \quad t > T-h. \end{cases}$$

$$v_{\bar{h}} = \begin{cases} \frac{1}{h} \int_{t-h}^{t} v(\cdot, \tau) d\tau & \text{for} \quad t \in (h, T], \\ 0 & \text{for} \quad t < h. \end{cases}$$

Lemma B.4.4 *Let $v \in L^{q,r}(E_T)$. Then, as $h \to 0$, $v_h \to v$ in $L^{q,r}(E_{T-\epsilon})$ for every $\epsilon \in (0, T)$. If $v \in C(0, T; L^q(E))$, then $v_h(\cdot, t) \to v(\cdot, t)$ in $L^q(E)$ for every $t \in (0, T-\epsilon)$ for all $\epsilon \in (0, T)$.*

A similar statement holds for $v_{\bar{h}}$. The proof of the lemma is straightforward from the theory of L^p spaces.

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