CURVATURE AND CONTINUITY OF OPTIMAL TRANSPORT

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This abstract sketches a geometric framework proposed in [1] and its consequences concerning the general regularity theory for optimal mappings developed by Ma, Trudinger, Wang and Loeper, following pioneering work on special cost functions by (at least) Caffarelli, Delanoë, Huang, Guan, Gutierrez, Oliker, Urbas, and X-J Wang. Due to space limitations we do not attempt to cite the literature or give much historical context, referring the reader instead to our paper, except that we note a different approach to some of our results was discovered independently at about the same time by Trudinger & Wang in arXiv:math/0702807. For simplicity our assumptions here are more restrictive than required in [1].

Let M and $\bar{M}$ be domains with compact closure $\mathrm{cl}\, M \subset M'$ and $\mathrm{cl}\, \bar{M} \subset \bar{M}'$ in smooth manifolds M' and $\bar{M}'$. Suppose M and $\bar{M}$ are equipped with Borel probability measures ρ and $\bar{\rho}$, and let $s \in C^4(\Omega')$ be the *surplus* ($= -$ transportation cost) defined on the product space $\Omega' = M' \times \bar{M}'$. The optimal transportation problem of Kantorovich is then to find the measure $\gamma \geq 0$ on $M \times \bar{M}$ which achieves the supremum

$$-W_{-s}(\rho, \bar{\rho}) := \max_{\gamma \in \Gamma(\rho, \bar{\rho})} \int_{M \times \bar{M}} s(x, \bar{x}) d\gamma(x, \bar{x}). \tag{1}$$

Here $\Gamma(\rho, \bar{\rho})$ denotes the set of joint probabilities having the same left and right marginals as $\rho \otimes \bar{\rho}$. It is not hard to check that this maximum is attained; any maximizing measure $\gamma \in \Gamma(\rho, \bar{\rho})$ is then called *optimal*. Each feasible $\gamma \in \Gamma(\rho, \bar{\rho})$ can be thought of as a weighted relation pairing points x distributed like ρ with points $\bar{x}$ distributed like $\bar{\rho}$; optimality implies this pairing also maximizes the average value of the specified surplus $s(x, \bar{x})$ for transporting each point x to its destination $\bar{x}$.

The optimal transportation problem of Monge amounts to finding a Borel map $F : M \longrightarrow \bar{M}$, and an optimal measure γ vanishing outside $\mathrm{Graph}(F) := \{(x, \bar{x}) \in M \times \bar{M} \mid \bar{x} = F(x)\}$. When such a map F exists, it is called an *optimal map* between ρ and $\bar{\rho}$; in this case, the relation γ is single-valued, so that ρ-almost every point x has a unique partner $\bar{x} = F(x)$, and optimality can be achieved in (0.1) without subdividing the mass at such points x between different destinations. Although Monge's problem is more subtle to solve than Kantorovich's, when M is a smooth manifold and ρ vanishes on every Lipschitz submanifold of lower dimension, a *twist* condition (see **(A1)** below) on the surplus function $s(x, \bar{x})$ guarantees existence and

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uniqueness of an optimal map F, as well as uniqueness of the optimal measure γ. One can then ask about the smoothness of the optimal map $F : M \longrightarrow \bar{M}$.

For $\rho, \bar{\rho}$ smooth and bounded away from zero on their respective domains, Ma, Trudinger & Wang gave hypotheses on Euclidean domains $M, \bar{M} \subset \mathbf{R}^n$ and $s \in C^4(\Omega')$ which ensure an affirmative answer. Their hypotheses may appear daunting, but inspired by Loeper's discoveries on Riemannian manifolds we recast them geometrically as follows. Use local coordinates $x^1, \ldots, x^n$ on M' and $x^{\bar{1}}, \ldots, x^{\bar{n}}$ on $\bar{M}'$ to define an inner product $d\ell^2 := (\partial^2 s/\partial x^i \partial x^{\bar{j}})(dx^i \otimes dx^{\bar{j}} + dx^{\bar{j}} \otimes dx^i)/2$ of indefinite sign and a symplectic form $\omega := (\partial^2 s/\partial x^i \partial x^{\bar{j}}) dx^i \wedge dx^{\bar{j}}$ on the tangent bundle $T\Omega'$ of the product space. Repeated indices are summed from $1, \ldots, n$ or $n+1, \ldots, n+\bar{n}$ according to whether they are barred or unbarred. Assume these bilinear forms are *non-degenerate* **(A2)** and $n = \bar{n}$. Then $d\ell^2$ defines a pseudo-Riemannian metric on Ω' with as many timelike as spacelike directions, i.e. signature (n, n). A vector $P \in T_{(x,\bar{x})}\Omega'$ is called *null* if it is self-orthogonal with respect to this metric. The canonical splitting of a vector in the tangent space $T_{(x,\bar{x})}\Omega' = T_x M' \oplus T_{\bar{x}} \bar{M}'$ is denoted by $P = p \oplus \bar{p}$. The metric $d\ell^2$ induces a pseudo-Riemannian curvature tensor $R_{i'j'k'l'}$ on Ω', which we use to define sectional curvature

$$\sec_{(x,\bar{x})} P \wedge Q := \sum_{i'=1}^{2n} \sum_{j'=1}^{2n} \sum_{k'=1}^{2n} \sum_{l'=1}^{2n} R_{i'j'k'l'} P^{i'} Q^{j'} P^{k'} Q^{l'} \tag{2}$$

in the standard way, except that we do not attempt to normalize it for fear of dividing by zero in the case of most interest to us, namely the null vectors $P = p \oplus 0$ and $Q = 0 \oplus \bar{p}$ orthogonal to each other, or equivalently $p \oplus \bar{p}$ null.

The surplus function $s \in C^4(\Omega')$ is said to be *weakly regular* **(A3w)** if $d\ell^2$ is non-degenerate and

$$\sec_{(x,\bar{x})} (p \oplus 0) \wedge (0 \oplus \bar{p}) \geq 0 \tag{3}$$

for all $(x, \bar{x}) \in \Omega'$ and null-vectors $p \oplus \bar{p} \in T_{(x,\bar{x})}\Omega'$. It is said to be *strictly regular* **(A3s)** if, in addition, equality in (0.3) implies $p = 0$ or $\bar{p} = 0$. This terminology is motivated by the fact that weak regularity is known to be necessary [2] as well as sufficient for smoothness of optimal maps between nice probability measures. A set $\Lambda \subset \Omega'$ is *geodesically convex* if every pair of points $(x, \bar{x}), (y, \bar{y}) \in \Lambda$ is linked by a curve satisfying the geodesic equation for our pseudo-Riemannian metric. It is *vertically convex* if $\Lambda \cap (\{x\} \times \bar{M})$ is geodesically convex for each $x \in M$; *horizontally convex* if $\Lambda \cap (M \times \{\bar{x}\})$ is geodesically convex for each $\bar{x} \in \bar{M}$; and *bi-convex* if both hold. Our first main result is a maximum principle:

Theorem 1. *Let $s \in C^4(M' \times \bar{M}')$ be weakly regular. If $\Lambda \subset M' \times \bar{M}'$ is open, horizontally convex and $t \in [0,1] \longrightarrow (x, \bar{x}(t)) \in \Lambda$ is a geodesic then $\cup_{0 \leq t \leq 1} (y, \bar{x}(t)) \subset \Lambda$ implies $f(t,y) := s(y, \bar{x}(t)) - s(x, \bar{x}(t)) \leq \max\{f(0,y), f(1,y)\}$.*

Idea of proof. Vanishing of $f'(t_0) = 0$ gives the null condition for weak regularity to imply $f''(t_0) \geq 0$, with strict inequality in the strictly regular case. This precludes a local maximum and is obtained using horizontal convexity to integrate the identity

$$2 \frac{\partial^4}{\partial r^2 \partial t^2} s(y(r), \bar{x}(t)) = \sec_{(y(r), \bar{x}(t))} (y'(r) \oplus 0) \wedge (0 \oplus \bar{x}'(t)) \geq 0$$

along a geodesic from $(x, \bar{x}(t_0))$ (where all t derivatives of f vanish) to $(y, \bar{x}(t_0))$. $\square$

For $\Lambda = M \times \bar{M}$, a version of this theorem was originally deduced [2] under additional hypotheses by relying on a sophisticated result of Trudinger & Wang. But our theorem requires no additional hypotheses, not even that the surplus $s \in C^4(M' \times \bar{M}')$ be *twisted*, meaning **(A1)**: for each $\bar{y}, \bar{z} \in \bar{M}'$ the function $x \in M' \longrightarrow s(x, \bar{y}) - s(x, \bar{z})$ has no critical points. If, in addition the reflected surplus $s^*(\bar{x}, x) = s(x, \bar{x})$ is twisted on $\bar{M}' \times M'$, we say s is *bi-twisted*. Our theorem combines with a subtle yet elementary argument of Loeper to yield [2] [1]:

Theorem 2. *Let $s \in C^4(\Omega')$ be twisted and weakly regular on $\Omega' = \mathbf{R}^n \times \mathbf{R}^n$ and $M \times \bar{M} \subset \Omega'$ a bounded bi-convex domain. Suppose $u \in C(\mathrm{cl}\, M)$ and $\bar{u} \in C(\mathrm{cl}\, M)$ are continuous functions with $u(x) = \max_{\bar{x} \in \mathrm{cl}\, \bar{M}} s(x, \bar{x}) - \bar{u}(\bar{x})$ for each $x \in \mathrm{cl}\, M$. If there exist $(x, \bar{x}) \in M \times \mathrm{cl}\, \bar{M}$ such that $u(z) \geq u(x) + s(z, \bar{x}) - s(x, \bar{x})$ for all z close to x, then the same inequality holds for all $z \in \mathrm{cl}\, M$.*

For strongly regular, bi-twisted surpluses and probability densities $d\rho/d\,\mathrm{vol} \in L^\infty(M)$ and $d\,\mathrm{vol}/d\bar{\rho} \in L^\infty(\bar{M})$ on $M \times \bar{M} \subset \mathbf{R}^n \times \mathbf{R}^n$ bounded and bi-convex, powerful ideas of Loeper augmented by a few simplifications then yield a self-contained proof [1] of his Hölder continuity of optimal maps: $F \in C^{1,1/\max\{5,4n-1\}}_{loc}(M; \mathrm{cl}\, \bar{M})$. A future ambition is to extend his continuity result to more general geometries $M' \neq \mathbf{R}^n \neq \bar{M}'$. We must surrender smoothness of the cost to satisfy the twist condition as soon as the manifold M' is compact. Hölder continuity results from [2] for the restriction of $s(x, \bar{x}) = \log |x - \bar{x}|$ to the unit sphere $M = \bar{M} = \mathbf{S}^n$ in $\mathbf{R}^{n+1}$, and for the geodesic distance squared $s(x, \bar{x}) = -d^2(x, \bar{x})$ on the round sphere, are also recovered by our technique [1]. In our current work, they are extended to Riemannian submersions of geometries like the latter; (related work is in progress by Delanoë & Ge). We also explore products thereof.

Let us conclude by observing any s-optimal diffeomorphism $F : M \longrightarrow \bar{M}$ has a graph which is spacelike with respect to $d\ell^2$ and Lagrangian with respect to ω, and conversely, using results from Trudinger & Wang, that any diffeomorphism between suitable domains whose graph is $d\ell^2$-spacelike and ω-Lagrangian is in fact the s-optimal map between the measures $\rho := \pi_\#(\mathrm{vol}|_{\mathrm{Graph}(F)})$ and $\bar{\rho} := \bar{\pi}_\#(\mathrm{vol}|_{\mathrm{Graph}(F)})$ obtained by the canonical projections through $\pi(x, \bar{x}) = x$ and $\bar{\pi}(x, \bar{x}) = \bar{x}$ of the Riemannian volume vol induced by the pseudo-metric $d\ell^2$ on $\mathrm{Graph}(F) \subset \Omega'$. This reveals an unexpected connection between optimal transportation and symplectic or pseudo-Kähler geometry. There is related work of Wolfson and of Warren in the (pseudo-) Euclidean case with $s(x, \bar{x}) = x \cdot \bar{x}$.

References

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