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SYMPOSIUM

Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness

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Abstract Hedonic pricing with quasi-linear preferences is shown to be equivalent to stable matching with transferable utilities and a participation constraint, and to an optimal transportation (Monge–Kantorovich) linear programming problem. Optimal assignments in the latter correspond to stable matchings, and to hedonic equilibria. These assignments are shown to exist in great generality; their marginal indirect payoffs with respect to agent type are shown to be unique whenever direct payoffs vary smoothly with type. Under a generalized Spence-Mirrlees condition (also known as a

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twist condition) the assignments are shown to be unique and to be *pure*, meaning the matching is one-to-one outside a negligible set. For smooth problems set on compact, connected type spaces such as the circle, there is a topological obstruction to purity, but we give a weaker condition still guaranteeing uniqueness of the stable match.

Keywords Hedonic price equilibrium · Matching · Optimal transportation · Spence-Mirrlees condition · Monge–Kantorovich · Twist condition

JEL Classification C62 · C78 · D50

1 Introduction

The goal of this note is to establish and exploit a general, structural equivalence result between three families of models, two of which are familiar to economists while the third belongs in mathematics and operational research. Specifically, we consider a general framework for studying hedonic price problems with quasi-linear preferences, and show that it is equivalent to a matching model with transferable utilities. From a mathematical perspective, both problems can in turn be rephrased under the form of a linear program, in fact an optimal transportation problem of Monge–Kantorovich type. Secondly, we argue that, due to the wide body of knowledge about linear programming in general, and optimal transportation in particular (see for example Anderson and Nash 1987; Villani 2003), the reduction of the model to this form seems not only conceptually clearer, but better adapted to bringing powerful methods of theoretical and computational analysis to bear on the question.

As an illustration, we first provide a general existence result for the models under consideration. The result is valid for matching as well as hedonic pricing models. It applies to multidimensional problems, and does not require single crossing conditions à la Spence-Mirrlees.[1] In the smooth setting, we establish uniqueness of the marginal payoff with respect to type, even though the optimal matching can be non-unique.

We also clarify the role of the well known Spence-Mirrlees condition, also called the twist condition in the mathematical literature. In the one-dimensional setting usually considered by economists, the condition guarantees some form of assortative matching—which, in turn, implies that the equilibrium is both unique and pure, (*purity* meaning the matching is one-to-one for almost all agents). As we discuss below, the notions of purity and uniqueness generalize naturally to multi-dimensions, whereas the notion of assortative matching does not.

We first describe a generalization of the Spence-Mirrlees condition that is valid in general type spaces, does not require differentiability of the surplus function, allows for non-participation, and is not dependent on the coordinates (i.e. the parametrization) of the problem. We then show that this condition, while sufficient, is *not* necessary for uniqueness of the stable match. In particular, we discuss an example in which the

[1] Our approach can be viewed as a simplification of the more complex (but ultimately equivalent) formulation of the problem as a convex nonlinear program due to Ekeland (2005, 2009), and subsequently developed in his joint work with Carlier and Ekeland (2009).

stable match is unique although the Spence-Mirrlees condition is violated. In such a case, however, the solution fails to be pure. That is, when Spence-Mirrlees does not hold, it may be the case that identical agents on one side of the market are matched with different counterparts, a situation that might be interpreted in terms of mixed strategies. Lastly, we provide a new and weaker condition that guarantees uniqueness of the stable match in the matching model (or of the equilibrium in the hedonic model) even in the absence of pure matching.

In both hedonic models and matching (or assignment) models, much of the intuition economists have developed is restricted to models in which either there is a finite number of types or in which the agents in the model can be described by a one-dimensional characteristic under a single-crossing property. Much of the discussion in the theoretical and empirical literature focuses on whether there is positive assortative matching. The optimal transportation approach, initially introduced by Shapley and Shubik (1972) and extended to a continuous setting by Gretsky et al. (1992), opens up the study of hedonic and matching models with multidimensional characteristics, general surplus functions, and general distributions of types. The present paper reviews the relevant results from this literature showing how they can be applied in these economic settings. Further, it highlights some significant issues related to the geometry and topology of the type spaces which have not previously been explored, neither in the economics nor the mathematics literature. For example, when agent types are located on a circle, a sphere, or products thereof (such as a periodic square), no *smooth* generalization of the Spence-Mirrlees condition can hold, and stable matchings (or assignments) are not generally pure. The *subtwist* criterion we introduce resolves assignment uniqueness in some of these settings, but leaves others as open challenges. An interesting question that we do not discuss relates to the continuity or smoothness of the dependence of the buyer's characteristics on those of the seller with whom he chooses to match. Significant recent progress on this question is surveyed by Villani (2009).

Our work builds upon and extends several existing contributions in economics and in mathematics. Gretsky et al. (1992, 1999) also study the matching of buyers and sellers in an economy with potentially a continuum of agents. In their economy, buyers and sellers who match are not free to trade any contract. Rather each seller is endowed with a single contract that they can sell or not. Their economy is thus a *hedonic endowment* economy, while ours can be seen as *hedonic production* one. Gretsky et al. (1992) shows that equilibrium in the endowment economy is equivalent to an optimal transportation problem and to a matching problem. They also prove that equilibrium exists. Gretsky et al. (1999) analyze the equilibrium in the endowment economy and focus on its links with perfect competition. They prove that in the continuum economy, perfect competition (the inability of individuals to influence price) obtains when the social gains function (i.e. the value of the primal program) is differentiable, or equivalently when the solution to the dual is unique. They also prove that perfect competition is generic and provide a sufficient condition for uniqueness of the dual solution. They do not analyze uniqueness of the optimal assignment or purity of the solution.

Our approach can also be viewed as a simplification of the more complex (but ultimately equivalent) formulation of the problem as a convex nonlinear program due to Ekeland (2005, 2009), and subsequently developed in his joint work, Carlier and Ekeland (2009). In particular, Ekeland (2005, 2009) also points out the similarity between hedonic models and optimal transportation problems. He proves existence of equilibrium under conditions very similar to ours and uniqueness and purity under an analogous version of the multidimensional Spence-Mirrlees condition. However, he does not consider our weaker, sufficient condition for uniqueness.

On the mathematical side, multi-dimensional generalizations of the Spence-Mirrlees condition developed through work of many authors, including Brenier, Caffarelli, Gangbo, McCann, Carlier, Ambrosio, Rigot, Ma, Trudinger, Wang, Bernard, Buffoni, Bertrand, Agrachev, Lee, Figalli and Rifford as surveyed by Villani (2003, 2009). Special cases of costs satisfying the subtwist condition were investigated by Uckelmann (1997), McCann (1999), Gangbo and McCann (2000), Plakhov (2004), and Ahmad (2004).

In the absence of a Spence-Mirrlees criterion, our uniqueness assignment result relies on Hestir and Williams' sufficient condition for extremality among doubly stochastic measures (Hestir and Williams 1995), and a variant thereon from Ahmad et al. (2009).

2 The basic framework

2.1 The competitive hedonic model

Consider a competitive spot market in which sellers produce and buyers acquire objects or contracts z which come in a wide range of qualities $z \in Z_0$. What is peculiar to many competitive hedonic markets, including those for housing, workers, vegetables, automobiles, pensions, insurance contracts, and many others, is that in the spot market for these contracts, a large number of buyers and sellers trade fixed quantities, often small, of contracts whose value (to buyers and/or sellers) depend on quantifiable qualities, or characteristics.[2] These "hedonic" characteristics are known to the buyers and/or sellers at the time of the transaction and as a result are reflected in the equilibrium market price.

Assuming buyer and seller preferences have been specified, the problem posed by such a market is to decide how supply equilibrates with demand to determine the set of contracts actually exchanged on the market (or the set of commodities actually produced and consumed), and the price $P(z)$ at which each type of contract is traded. Note that such an equilibrium implicitly defines a pairing or matching of buyers with sellers who choose to enter into this market by agreeing to contract or exchange with each other.

Standing hypotheses The sets X_0, Y_0, Z_0, of buyer, seller, and contract types, may be modeled as subsets of complete separable metric spaces, possibly multidimensional. To allow for the possibility that some agents choose not to participate, we augment the spaces $X := X_0 \cup \{\emptyset_X\}$, $Y := Y_0 \cup \{\emptyset_Y\}$ and $Z = Z_0 \cup \{\emptyset_Z\}$ by including an isolated point in each: a partner $\emptyset_X$ for any unmatched sellers, a partner $\emptyset_Y$ for any

[2] These models also apply to markets where prices are nonlinear in quantities because different quantities are not perfect substitutes and cannot be freely traded.

unmatched buyers, and the null contract $\emptyset_Z$. Preferences are encoded into functions representing the utility $u(x,z)$ of product $z \in Z$ to buyer $x \in X$, and the utility (disutility or cost) $v(y,z)$ of product $z \in Z$ to seller $y \in Y$. These utility functions $u : X \times Z \longrightarrow \mathbf{R} \cup \{-\infty\}$ and $v : Y \times Z \longrightarrow \mathbf{R} \cup \{+\infty\}$ are specified a priori, along with non-negative Borel measures μ_0 on X_0 and ν_0 on Y_0 of finite total mass representing the distribution of buyer and seller types throughout the population. The utility functions are constrained so that neither the dummy buyer type $\emptyset_X$ nor the dummy seller type $\emptyset_Y$ can participate in any exchange save the null contract:

$$u(\emptyset_X, z) = \begin{cases} 0 & \text{if } z = \emptyset_Z, \\ -\infty & \text{else;} \end{cases} \qquad v(\emptyset_Y, z) = \begin{cases} 0 & \text{if } z = \emptyset_Z, \\ +\infty & \text{else,} \end{cases} \tag{1}$$

while the measures μ_0 and ν_0 are extended to X and Y by assigning mass $\nu_0(Y_0) + 1$ and $\mu_0(X_0) + 1$ to the points $\emptyset_X$ and $\emptyset_Y$, respectively:

$$\mu := \mu_0 + (\nu_0(Y_0) + 1)\delta_{\emptyset_X} \quad \nu := \nu_0 + (\mu_0(X_0) + 1)\delta_{\emptyset_Y}. \tag{2}$$

The augmented measures balance $\mu[X] = \nu[Y] < \infty$, so we can renormalize them to be probability measures (i.e. have unit mass) without loss of generality.[3]

To guarantee the convergence of various integrals, and attainment of various suprema and infima, we assume throughout (and tacitly hereafter) that $u(x,z) < \infty$ extends upper semicontinuously to the completion of $X \times Z$ and $v(y,z) > -\infty$ lower semicontinuously to the completion of $Y \times Z$. We normalize the utility of the null-contract to be zero

$$u(x, \emptyset_Z) = 0 = v(y, \emptyset_Z), \tag{3}$$

which can be achieved without loss of generality if the reserve utilities $u(x, \emptyset_Z) \in L^1(X, d\mu)$ and $v(y, \emptyset_Z) \in L^1(Y, d\nu)$ are continuous and integrable, by subtracting them from u and v.

Define the pairwise surplus function

$$s(x,y) = \sup_{z \in Z} u(x,z) - v(y,z). \tag{4}$$

We assume that for each pair the supremum is attained. Further, in case u or v is discontinuous or Z fails to be compact (Hildenbrand 1974), we assume the set of contracts

$$Z(x,y) = \arg\max_{z \in Z} u(x,z) - v(y,z) \tag{5}$$

that maximize the surplus (4) is non-empty, compact, and depends upper hemicontinuously on $(x,y) \in X \times Y$. It is well-known (Hildenbrand 1974) that there exists

[3] The excessive mass on $\delta_{\emptyset_X}$ and $\delta_{\emptyset_Y}$ ensures that at least some null types match with each other and obtain a pairwise surplus of zero.

a measurable selection, i.e., a Borel function $z_0 : X \times Y \to Z$ such that $z_0(x, y)$ is contained in $Z(x, y)$ for all (x, y).

In case u or $-v$ fails to be bounded, we assume there exist $\bar{q} \in L^1(X, d\mu)$ and $\bar{r} \in L^1(Y, d\nu)$ which extend to real-valued lower semi-continuous functions on the completions cl X and cl Y of their domains such that

$$\sup_{x \in \mathrm{cl}\, X} u(x, z) - \bar{q}(x) \leq \inf_{y \in \mathrm{cl}\, Y} v(y, z) + \bar{r}(y) \tag{6}$$

for all $z \in Z$. This is roughly equivalent to the existence of prices on Z which make the indirect utilities integrable. Given any μ-measurable map $f : \mathrm{Dom}\, f \longrightarrow Z$ on a subset $\mathrm{Dom}\, f \subset X$, we define a measure on Z, called the *push-forward* of μ through f, by the formula $(f_\#)\mu(B) = \mu(f^{-1}(B))$ for Borel $B \subset Z$. Here μ-measurability simply means $f^{-1}(B)$ differs from a Borel set by set of μ outer-measure zero.

Suppose $P : Z \longrightarrow \mathbf{R} \cup \{\pm\infty\}$ denotes the competitive market price of quality $z \in Z$. To allow non-participation, it is subject to the constraint $P(\emptyset_Z) = 0$. We assume that buyer utility is linear in price so that in such a market, the indirect utility available to buyer type $x \in X$ is defined by the quasi-linear utility maximization

$$U(x) = \sup_{z \in Z} \{u(x, z) - P(z)\}. \tag{7}$$

Here $U(x) \geq 0$ is non-negative since $\emptyset_Z \in Z$; each buyer x retains the right not to consume. Similarly, we assume seller utility is linear in price so that the indirect utility available to seller type $y \in Y$ is given by the utility maximization

$$V(y) = \sup_{z \in Z} \{P(z) - v(y, z)\}, \tag{8}$$

with $V(y) \geq 0$ and vanishing in the case of non-participation. We make the conventions $(-\infty) - (-\infty) = -\infty$ and $\infty - \infty = -\infty$ to resolve ambiguities in (7) and (8).

Let α be a non-negative measure on $X \times Y \times Z$. The support of α refers to the smallest closed set $\mathrm{Spt}(\alpha) \subseteq X \times Y \times Z$ of full mass. The measure α represents an assignment of buyers and sellers to each other and to products. We use the push-forward notation to denote its marginal projections $\pi^X_\# \alpha$ and $\pi^Y_\# \alpha$ under mappings such as $\pi^X(x, y, z) = x$ and $\pi^Y(x, y, z) = y$ on $X \times Y \times Z$.

The pair (α, P) is an hedonic *equilibrium* if these projections coincide with the initial measures on each set:

$$\begin{aligned} \pi^X_\# \alpha &= \mu \\ \pi^Y_\# \alpha &= \nu \end{aligned} \tag{9}$$

and if, for α-almost all points $(x, y, z) \in \mathrm{Spt}\, \alpha$, we have that

$$\begin{aligned} U(x) &= u(x, z) - P(z) \\ V(y) &= P(z) - v(y, z). \end{aligned} \tag{10}$$

In such an equilibrium, each triple $(x, y, z) \in \operatorname{Spt} \alpha$ represents a mutually agreeable exchange of contract z between seller y and buyer x, where z is a contract most favored by both seller y and buyer x independently, given market prices P. The prices are market clearing, in the sense that the assignment α is consistent with the utility maximization of both buyers and sellers (10) while simultaneously balancing supply with demand (9). Since the prices of untraded commodities potentially affect the indirect utilities $U(x)$ and $V(y)$, prices for these commodities are subject to upper and lower bounds in a market at equilibrium. We use the term *market clearing* pair synonymously with equilibrium pair. This notion of equilibrium allows for the possibility that some agents are indifferent between multiple qualities in Z. Indeed, when α assigns a buyer x to multiple sellers or contracts, we may interpret the conditional distribution implied by α as a mixed strategy for buyer x. In such an equilibrium, the assignment α must still ensure that the number of buyers and sellers of each contract type are compatible in the sense of (9).

2.2 The associated matching problem

Similarly, models of one-to-one matching with transferable utility are used to analyze marriage markets, labour markets and the matching of students to schools to understand who matches with whom in an equilibrium stable matching. In these models the partners on each side of the matching have characteristics that affect the surplus that may be attained by any matched pair. The characteristics of the agents matched are reflected in the equilibrium matching and in the utility payoffs that each agent obtains. Formally, a matching model is defined by two spaces X and Y defined as above and an upper semicontinuous mapping $s\colon \mathrm{cl}(X \times Y) \longrightarrow [0, \infty[$ which represents the surplus that can be generated by any pair (x, y) in $X \times Y$ if matched together. Under transferable utility, for any match (x, y) the surplus $s(x, y)$ can be distributed between the partners: i.e., x receives some $u(x)$ and y receives some $v(y)$ with $u(x) + v(y) = s(x, y)$.

As stated in introduction, there is a natural, one-to-one correspondence between hedonic models and matching problems. We first characterize the pairwise matching problem derived from the hedonic price model just described.

Characterization The basic idea is very simple. For each pair $(x, y) \in X \times Y$, recall the pairwise surplus function s defined in (4). In words, whenever a buyer x is matched with a seller y, they generate together the total surplus $s(x, y)$, defined as a maximum over the set Z of possible commodities. Then $s : X \times Y \longrightarrow [0, \infty[$ is upper semicontinuous by our assumptions (3), and the set $Z(x, y)$ where the supremum is attained (5) is non-empty, compact-valued, and upper hemicontinuous. Our normalizations (1)–(3) permit either buyer or seller to go unmatched (to match with a null type) and force the utility of the unmatched state to be zero:

$$s(x, \emptyset_Y) = u(x, \emptyset_Z) = 0$$
$$s(\emptyset_X, y) = -v(\emptyset_Z, y) = 0.$$

One can then define a pairwise matching problem by the set of *buyers* (X, μ), the set of *sellers* (Y, ν), and the pairwise surplus defined by the surplus function s. An *assignment* (or a *matching*) is defined as a measure γ on $X \times Y$, the marginals of which coincide with μ and ν. Using the same notations as above, we thus write that

$$\begin{aligned} \pi^X_{\#}\gamma &= \mu \\ \pi^Y_{\#}\gamma &= \nu \end{aligned} \tag{11}$$

where the projection mappings $\pi^X(x, y) = x$ and $\pi^Y(x, y) = y$ this time are defined on $X \times Y$. If $(x, y) \in (X_0 \times Y_0) \cap \operatorname{Spt}(\gamma)$, we say that x and y are *matched*. A buyer may be matched to multiple sellers and vice versa. If $x \in X_0$ and there is no $y \in Y_0$ such that $(x, y) \in \operatorname{Spt}(\gamma)$, we say that x is *unmatched* (and similarly for y).

A *payoff* corresponding to γ is a pair of functions $\bar{U} : X \to \mathbf{R}$ and $\bar{V} : Y \to \mathbf{R}$ with the normalization $\bar{U}(\emptyset_X) = 0$ such that for γ-a.e. $(x, y) \in \operatorname{Spt}(\gamma)$,

$$\bar{U}(x) + \bar{V}(y) \leq s(x, y). \tag{12}$$

Finally, an *outcome* is defined as a triple $(\gamma, \bar{U}, \bar{V})$ where $(\bar{U}, \bar{V})$ is a payoff corresponding to γ.

We have thus showed how one can associate, to any hedonic problem, a matching model. Note that the converse is also true: for every matching problem defined by the upper semicontinuous surplus function $s(x, y)$, one can trivially construct a hedonic problem from a suitable choice of utility functions.

For example $Z = Y$, $u = s$, with $v(y, y) = 0$ and $v(x, z) = +\infty$ for all $z \neq y$. Smoother examples are more involved to articulate but also possible; in fact, every upper semicontinuous surplus function $s(x, y) \geq 0$ and continuous assignment $z(x, y)$ can be shown to arise from a hedonic model.

Stability Following the literature,[4] we define stability by:

Definition 1 An outcome $(\gamma, \bar{U}, \bar{V})$ is stable if for any $(x, y) \in X \times Y$,

$$\bar{U}(x) + \bar{V}(y) \geq s(x, y). \tag{13}$$

Note, that this definition implies that a stable outcome satisfies

$$\begin{aligned} \bar{U}(x) &\geq s(x, \emptyset_Y) = 0 \\ \bar{V}(y) &\geq s(\emptyset_X, y) = 0 \end{aligned}$$

for all $x \in X$ and for all $y \in Y$. In words: a match is stable if two conditions are fulfilled:

[4] See for instance Roth and Sotomayor (1990).

1. No matched agent would be better off unmatched.
2. No two agents x and y, who are not matched together, would both prefer being matched together than their current situation.

To see the link between the formal and informal definitions, consider an outcome $\left(\gamma, \bar{U}, \bar{V}\right)$ that satisfies (13). The functions $\bar{U}(x)$ and $\bar{V}(y)$ can be interpreted as the utilities derived by x and y from the outcome at stake. As noted above, restriction (13) immediately implies condition one. In addition, restriction (13) along with the definition of a payoff implies that $\bar{U}(x)+\bar{V}(y) = s(x, y)$ for γ-a.e. $(x, y) \in \mathrm{Spt}(\gamma)$. Finally, restriction (13) guarantees that any two agents $(x, y) \notin \mathrm{Spt}(\gamma)$ who are not matched with each other, cannot generate a surplus larger than $\bar{U}(x)+\bar{V}(y)$. Indeed, if x and y were such that $s(x, y) > \bar{U}(x) + \bar{V}(y)$, then it would be the case that (i) they are not matched together in the outcome under consideration, and (ii) they can both improve their utility by leaving their current situation and rematching together. But such a situation would violate the definition of stability.

Finally, a matching γ is stable if there exists a payoff $\left(\bar{U}, \bar{V}\right)$ such that the outcome $\left(\gamma, \bar{U}, \bar{V}\right)$ is stable.

A well known result, in our transferable utility context, is that a matching is stable if and only if it maximizes total surplus.[5] Define for each matching γ the total surplus

$$\gamma[s] = \int_{X\times Y} s(x, y)\, \mathrm{d}\gamma(x, y).$$

Then

Proposition 1 (Gretsky et al. 1992) *A matching γ of (X, μ) with (Y, ν) is stable if and only if there exists no other matching γ' such that*

$$\gamma'[s] > \gamma[s].$$

It follows that the matching problem is itself equivalent to a linear programming problem of the *optimal transportation* type, as we next discuss.

2.3 The transportation problem

We claim that in fact both hedonic pricing and stable matching lead to the problem of pairing *buyers* (X, μ) with *sellers* (Y, ν) so as to optimize the average (or total) of the surplus function $s(x, y)$. This problem can be expressed as a linear program:

Program (MK) (Monge–Kantorovich) Given an upper semi-continuous function $s : X \times Y \longrightarrow [0, \infty]$ on two probability spaces (X, μ) and (Y, ν), solve

$$\max_{\gamma\in\Gamma(\mu,\nu)} \gamma[s] \tag{14}$$

[5] Shapley and Shubik (1972) prove this result in the matching problem with a finite number of types. Gretsky et al. (1992) extend the Shapley Shubik result to the economy with a continuum of types.

over the set of measures

$$\Gamma(\mu,\nu)=\{0\le\gamma \text{ on } X\times Y \mid \pi^X_{\#}\gamma=\mu,\quad \pi^Y_{\#}\gamma=\nu\}. \tag{15}$$

with prescribed marginals. Here the π are the projections: $\pi^X(x,y)=x$ and $\pi^Y(x,y)=y$.

Dual program (MK′) This surplus maximization can be interpreted as an optimal transportation problem of Monge–Kantorovich form (Monge 1781; Kantorovich 1942; Rachev and Rüschendorf 1998; Villani 2003). The dual linear program, found by Kantorovich and his collaborators, is posed as follows. Define

$$\mu[q]=\int_X q(x)\,\mathrm{d}\mu(x)$$

and

$$\nu[r]=\int_Y r(y)\,\mathrm{d}\nu(y).$$

Then the Kantorovich dual program is:

$$\min_{(q,r)\in\mathrm{Lip}_s(\mu,\nu)}\{\mu[q]+\nu[r]\} \tag{16}$$

where $\mathrm{Lip}_s(\mu,\nu)$ consists of all pairs of functions $q\in L^1(X,\mathrm{d}\mu)$ with $q(\emptyset_X)=0$ and $r\in L^1(Y,\mathrm{d}\nu)$ which satisfy the constraint[6]

$$q(x)+r(y)\ge s(x,y)\quad\forall(x,y)\in X\times Y. \tag{17}$$

Interestingly, the dual constraints (17) exactly reproduce the stability conditions (13) of the matching problem. Indeed, for any stable match, the dual variables $q(x)$ and $r(y)$ can be interpreted as a payoff.

A key property of the primal–dual pair is that for all γ that are feasible for (MK) and for all pairs (q,r) feasible for (MK′)

$$\gamma[s]\le\mu[q]+\nu[r]. \tag{18}$$

Moreover, a feasible triple (γ,q,r) produces equality in (18) (if and) only if γ maximizes (MK) and the pair (q,r) minimize (MK′). The *only if* statement is obvious and plays a crucial role hereafter; the *if* statement is the basic duality result from linear programming (see e.g., Anderson and Nash 1987; Gretsky et al. 1992 or Villani 2003); it can also be recovered as a special instance of the existence of a Nash

[6] The choice $q(\emptyset_X)=0$ costs no generality. Theorem 1 then implies $r(\emptyset_Y)=0$ for any pair (q,r) achieving the infimum (16), in view of our normalization (2).

equilibrium in an infinite-dimensional, two-player, zero-sum, bilinear, mixed-strategy game (von Neumann 1953).

3 Stable matching and hedonic pricing via optimal transportation

3.1 The matching problem: an existence result

A first outcome of the previous arguments is a general existence result for the optimal transportation problem, therefore for the matching problem. Specifically, the upper semi-continuity of $s(x, y) \geq 0$ guarantees that the maximum (14) is attained. It has a finite value if the dual problem is feasible in which case the minimum (16) is also attained (Kellerer 1984). To summarize, we quote Villani (2009, Theorem 5.10), giving the obvious extension from complete separable metric spaces to subsets X and Y thereof, whose closures will be denoted $\mathrm{cl}\, X$ and $\mathrm{cl}\, Y$:

Theorem 1 (Existence and duality) *Let subsets X and Y of complete, separable metric spaces be equipped with Borel probability measures μ and ν and an upper semicontinuous function $s\colon \mathrm{cl}(X \times Y) \longrightarrow [0, \infty[$. Assume some feasible $(\bar{q}, \bar{r}) \in \mathrm{Lip}_s(\mu, \nu)$ extend to real-valued lower semicontinuous functions satisfying $\bar{q}(x)+\bar{r}(y) \geq s(x, y)$ on $\mathrm{cl}\, X \times \mathrm{cl}\, Y$. Then the maximum (14) is attained by some $\gamma \in \Gamma(\mu, \nu)$ and the minimum (16) by some $(q, r) \in \mathrm{Lip}_s(\mu, \nu)$. Moreover, $\gamma[s] = \mu[q] + \nu[r] < \infty$, and γ assigns zero outer measure to the complement of the zero set*

$$S := \{(x, y) \in X \times Y \mid q(x) + r(y) - s(x, y) = 0\}. \tag{19}$$

Proof Since $\tilde{X} := \mathrm{cl}\, X$ and $\tilde{Y} = \mathrm{cl}\, Y$ are themselves complete separable metric spaces, the theorem follows immediately from Villani (2009, Theorem 5.10) assuming X and Y are complete. When X and Y are incomplete, μ extends to a Borel probability measure $\tilde{\mu}$ on the closure $\tilde{X}$ which assigns zero outer measure to $\tilde{X} \setminus X$, so that X is $\tilde{\mu}$-measurable even if it is not Borel. Similarly, ν extends to a Borel probability measure $\tilde{\nu}$ on $\tilde{Y}$. Villani asserts the existence of optimizers $\tilde{\gamma} \in \Gamma(\tilde{\mu}, \tilde{\nu})$ and $(\tilde{q}, \tilde{r}) \in \mathrm{Lip}_s(\tilde{\mu}, \tilde{\nu})$. Moreover $\tilde{\gamma}$ assigns zero outer measure to the complement of $X \times Y$, hence restricts to the γ-measurable set $X \times Y$ and induces a Borel measure $\gamma \in \Gamma(\mu, \nu)$ there, with $\gamma[s] = \tilde{\gamma}[s]$. Similarly, the $\tilde{\mu}$-measurable and $\tilde{\nu}$-measurable functions $(\tilde{q}, \tilde{r})$ restrict to μ- and ν-measurable functions $(q, r) \in \mathrm{Lip}_s(\mu, \nu)$ which satisfy the conclusions of the theorem. □

Notice our assumptions (3)–(6) on the utilities $u(x, z)$ and $v(y, z)$ imply the surplus $s(x, y)$ defined by (4) satisfies all hypotheses of this theorem.

Remark 1 (s-convex payoff functions) Define

$$r^s(x) = \sup_{y \in Y} s(x, y) - r(y) \tag{20}$$

$$q^{\tilde{s}}(y) = \sup_{x \in X} s(x, y) - q(x) \tag{21}$$

in which case we say q is *s-convex* and r is *$\tilde{s}$-convex* (Rachev and Rüschendorf 1998; Villani 2003). It is important to note that any feasible pair (q, r) in (16) can be replaced by (r^s, r) and hence $(r^s, r^{s\tilde{s}})$ without increase in cost. Since $r^{s\tilde{s}s} = ((r^s)^{\tilde{s}})^s = r^s$, it costs no generality to take $(q, r) = (r^s, q^{\tilde{s}})$, meaning $q = q^{\tilde{s}s}$ and $r = r^{s\tilde{s}}$. Any minimizing pair from the theorem above therefore satisfies $(q, r) = (r^s, r^{s\tilde{s}})$ on a set of full $\mu \times \nu$ measure—which implies the μ-measurability of the s-convex minimizer r^s (and the ν-measurability of $r^{\tilde{s}\tilde{s}s}$) as required. If s is actually continuous then all s-convex functions are lower semi-continuous (hence Borel measurable) whether or not they are minimizers. These well known facts play a key role in the proof that the minimum is attained, in the developments to come, and in computational strategies to approximate a solution to the minimization.

The geometry of the set S defined in (19) takes center stage in the analysis which follows, since this set determines which buyers can match with which sellers at equilibrium. For example, it is well known that in the one-dimensional matching model with $D^2_{xy}s(x, y) > 0$, there is a unique optimal assignment that involves positive assortative matching. In this case, the set S is the graph of a strictly increasing function $y = f(x)$.

Economically, the solutions (q, r) of the dual problem are also important because they represent the utility payoffs obtained by each type. Even for $x \notin \mathrm{Spt}\,\mu$, the range of allowed values for $q(x)$ has economic relevance, since it bounds the payoff available when a few new buyers of type x choose to enter the established market; similarly, at $y \notin \mathrm{Spt}\,\nu$ the range of values for $r(y)$ bounds the payoff available when a few sellers of type y enter the established market.

3.2 Existence of an hedonic equilibrium

It remains to show that the existence result obtained in the matching problem implies the existence of an hedonic equilibrium. Given the structure of the relationship between the two problems, it is clear that if buyer x and seller y are matched in the matching problem, they will trade some common quality z in an hedonic equilibrium. What has to be constructed is a price schedule $P(z)$ that supports those trades.

Recall the definition of $Z(x, y)$ given in (5) and let $z_0(x, y) \in Z(x, y)$ be a measurable selection. The main result is the following:

Proposition 2 (Equilibrium prices) *Let γ solve the primal program (14) and (q, r) solve the dual program (16). Then there exist a price function $P : Z \longrightarrow \mathbf{R} \cup \{\pm\infty\}$ satisfying*

$$P_{\max}(z) := \inf_{y \in Y} \{v(y, z) + r(y)\} \geq P(z) \geq \sup_{x \in X} \{u(x, z) - q(x)\} =: P_{\min}(z). \quad (22)$$

With $\alpha \equiv (id_X \times id_Y \times z_0)_{\#}\gamma$, any such P forms an equilibrium pair (α, P).

Note that the left side of (22), $P_{\max}(z)$, is the minimum equilibrium willingness to accept of all sellers in the market: no sellers will trade z unless $P(z) \geq P_{\max}(z)$. Similarly, the right side of (22), $P_{\min}(z)$, is the maximum equilibrium willingness to pay of all buyers. No buyers will trade z unless $P(z) \le P_{\min}(z)$. When $P_{\max}(z) > P_{\min}(z)$ no trade takes place. When $P_{\max}(z) = P_{\min}(z)$, an exchange may be made by the set of buyers and sellers who attain the infimum and supremum.

Proof Combining (17) with the definition of the surplus s:

$$q(x) + r(y) \ge s(x, y) \ge u(x, z) - v(y, z) \quad \text{on } X \times Y \times Z,$$

hence

$$v(y, z) + r(y) \ge u(x, z) - q(x) \quad \text{on } X \times Y \times Z$$

and

$$\inf_{y \in Y} \{v(y, z) + r(y)\} \ge \sup_{x \in X} \{u(x, z) - q(x)\} \quad \text{on } Z.$$

which shows that $P_{\max}(z) \ge P_{\min}(z)$. Now, choose any function $P(z)$ satisfying (22); the infimum or supremum themselves would suffice.

The basic duality result from linear programming asserts $\gamma[s] = \mu[q] + \nu[r] = \gamma[q + r]$, since $\gamma \ge 0$ has μ and ν for marginals. Thus equality holds for γ-a.e. (x, y) in the inequality (17); i.e. whenever γ matches buyer x with seller y. Consider $\bar{x}$ who is matched with $\bar{y}$, in the sense that they belong to the Borel set S defined in (19). This is the set of full γ measure where equality holds in the dual inequality constraints. Since the pair $(\bar{x}, \bar{y})$ agree on their preferred contracts $\bar{z} \in Z(\bar{x}, \bar{y})$ attaining (4),

$$q(\bar{x}) + r(\bar{y}) = s(\bar{x}, \bar{y}) = u(\bar{x}, \bar{z}) - v(\bar{y}, \bar{z}),$$

and we have

$$v(\bar{y}, \bar{z}) + r(\bar{y}) = P(\bar{z}) = u(\bar{x}, \bar{z}) - q(\bar{x})$$

on the set $T := \{(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z \mid \bar{z} \in Z(\bar{x}, \bar{y})\}$. Upper hemicontinuity of $Z(\bar{x}, \bar{y})$ implies T is closed, while $z_0(\bar{x}, \bar{y}) \in Z(\bar{x}, \bar{y})$ implies $\operatorname{Spt} \alpha \subset T$. Our choice (22) of price now yields

$$u(\bar{x}, z) - P(z) \le q(\bar{x}) = u(\bar{x}, \bar{z}) - P(\bar{z}) \quad \forall z \in Z$$

and

$$P(z) - v(y, z) \le r(\bar{y}) = P(\bar{z}) - v(\bar{y}, \bar{z}) \quad \forall z \in Z$$

so that $\bar{z}$ maximizes both $u(\bar{x}, z) - P(z)$ and $P(z) - v(\bar{y}, z)$. Since the equalities hold for $(\bar{x}, \bar{y}, \bar{z})$ in a set $(S \times Z) \cap \operatorname{Spt} \alpha$ of full measure for α, we conclude that (α, P) is a market-clearing hedonic equilibrium pair. □

The result implies that to any stable match corresponds an hedonic equilibrium. Therefore, the existence result derived in the previous section has the immediate, following consequence:

Corollary 1 *The hedonic model described in Sect. 1 has an equilibrium.*

It is important to note that existence obtains in a general context. No restriction is imposed on the dimension of the spaces at stake nor on the measures describing the distributions of types. Both discrete and continuous distributions are allowed. Moreover, no specific assumptions are made on u and v beyond the standard ones. In particular, we do *not* assume any Spence-Mirrlees condition. Our result thus establishes the existence of hedonic equilibria in a fully general context.[7]

Finally, it is interesting to note that the converse is also true: to any hedonic equilibrium, one can associate a stable match, as asserted by the following result:

Lemma 1 *Let (α, P) be a hedonic equilibrium pair. If $u \in C(X \times Z)$ and $v \in C(X \times Z)$ are continuous, then the indirect utilities $U(x)$ and $V(y)$ from (7) to (8) minimize Kantorovich's dual problem (16), while $\gamma = \left(\pi^X \times \pi^Y\right)_{\#} \alpha$ maximizes the primal problem (14). Here $\pi^X(x, y, z) = x$ and $\pi^Y(x, y, z) = y$.*

Proof First observe that equilibrium condition (9) states that γ has μ and ν for marginals, hence is a feasible competitor in the Monge–Kantorovich primal program (14). The definitions (7) and (8) of $U(x)$ and $V(y)$ imply

$$U(x) + V(y) \geq u(x, z) - v(y, z) \tag{23}$$

for all $z \in Z$. Taking the supremum over $z \in Z$ implies that (U, V) is a feasible pair for the Kantorovich dual program (16); they are lower semi-continuous due to continuity of u and of v. Moreover, equilibrium condition (10) forces equality in (23) for α-a.e. $(x, y, z) \in \mathrm{Spt}(\alpha)$, hence

$$U(x) + V(y) = s(x, y).$$

The lower bounds $U, V \geq 0$ permit this to be integrated against α, yielding

$$\int U(x)\, \mathrm{d}\mu(x) + \int V(y)\, \mathrm{d}\nu(y) = \int s(x, y)\, \mathrm{d}\gamma(x, y).$$

Hence γ maximizes the primal program whilst the pair (U, V) minimizes the dual program. $\square$

In other words, (i) the hedonic pricing problem with quasi-linear utility, (ii) the stable matching problem with transferable utility, and (iii) the optimal transportation problem are equivalent; none is more nor less general than the others. Moreover, approximate solutions can be computed using linear programming techniques. This

[7] Ekeland (2005, 2009) presents an alternative proof based on convex programming instead of linear programming. Gretsky et al. (1992) prove existence in a version of the model in which sellers are endowed with z. That is, $v(y, z) = +\infty$ unless $y = z$, so seller utility is simply $v(z)$.

opens up the study of these problems in empirical settings in which the type spaces are high dimensional, have both discrete and continuous elements, and have different dimensions on the buyer and seller side of the market. More work needs to be done to study these problems in these applied settings.

In theoretical settings, one obtains necessary and sufficient conditions for the optimal assignment or the stable matching, now shown to exist, via the Kuhn–Tucker conditions from linear programming. The form of these conditions in the optimal transportation context is well-understood (Rachev and Rüschendorf 1998; Villani 2009). In a suitably weak topology, one could also show that the solution depends continuously on the data in the sense that the limit of a sequence of solutions to different problems is a solution of the limiting problem. However, to make concrete statements about uniqueness of the solution or the form of the optimal measure γ, one requires additional structure on the problem. This is the topic of the next section.

4 Uniqueness and purity

4.1 Pure solutions

We consider two properties of the equilibrium, namely uniqueness and purity. Gretsky et al. (1992) study whether the dual has unique solutions; i.e. whether the equilibrium payoffs to agents are unique. They derive a condition equivalent to this in terms of differentiability of the social gains function, prove genericity of this property, and provide a sufficient condition for uniqueness of payoffs in a certain range of environments.

When empirical issues are at stake, however, uniqueness of payoffs may not be sufficient. Two issues should be considered here. First, uniqueness of the matching itself is an important property; if not satisfied, more sophisticated econometric techniques are needed to account for the possible multiplicity of equilibria. Second, empirical works devoted to identification (particularly in the hedonic framework) usually assume that the equilibrium is pure, in the sense that the mapping between producers and buyers is deterministic (or, equivalently, the support of the optimal measure in the $X \times Y$ space is born by the graph of a function). Intuitively, an equilibrium is pure if (almost) all agents have a pure strategy at equilibrium, i.e., for each agent there exists one trading partner that she chooses with probability one. In the opposite case of a non-pure equilibrium, a non-null set of agents are either indifferent between several partners, or indifferent between action and inaction; then equilibrium may require randomization or mixed strategies. Again, such situations require the use of econometric techniques specifically designed for set identification. On both issues, the reader is referred to the paper by Galichon and Henry in this issue.

In the smooth setting, we first complement existing results by giving alternative sufficient conditions for uniqueness of marginal payoffs in terms of exogenous parameters of the hedonic model. We then focus our attention on purity, and show that a generalized version of the standard, Spence-Mirrlees condition guarantees both uniqueness and purity of the optimal solution. Finally, we provide a weaker condition that is sufficent for uniqueness, but not for purity, of the solution.

A couple of well-known examples illustrate these ideas. We start with a standard situation in which the Spence-Mirrlees condition guarantees a unique, pure equilibrium.

Example 1 (Positive assortative matching) Consider a hedonic economy in which $X_0 = Y_0 = Z_0 \subseteq \mathbf{R}$ are intervals. Assume $s(x, y)$ is twice differentiable in the interior of $X_0 \times Y_0$ and extends to a continuous bounded function on its closure. If $D^2_{xy} s(x, y) > 0$, then there is a unique equilibrium, it involves positive assortative matching, and all but countably many agents have a pure strategy optimum. The same facts are true if $s(x, y)$ is supermodular, in the sense that whenever $x > x'$ and $y > y'$ then

$$s(x, y) + s\left(x', y'\right) > s\left(x', y\right) + s\left(x, y'\right)$$

Both $D^2_{xy} s(x, y) > 0$ and supermodularity of $s(x, y)$ are versions of a Spence-Mirrlees or single-crossing condition.

At the opposite end of the spectrum, we may have models in which a continuum of (pure and non-pure) equilibria exist:

Example 2 (Orthogonal surplus) Consider a plane, and let X_0 be the interval $[0, 1]$ on the horizontal axis, and Y_0 be the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ on the vertical axis; both sets are equipped with the uniform distribution. Finally, consider the surplus $s(x, y) = 2 - x^2 - y^2$; i.e., any match generates a surplus of two, from which a transportation cost equal to the square of the distance between the two points is withdrawn. Then the maximum aggregate surplus, equal to $5/12$, is obtained by uncountably many measures γ, including pure solutions (e.g., the uniform distribution over the graph of functions like $f(x) = 1/2 - x$ or $f(x) = x - 1/2$) and non-pure solutions (e.g. the uniform distribution over the square $[0, 1] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$).

These examples suggest interesting conclusions. First, additional restrictions are clearly needed to guarantee either uniqueness or purity. In the first example, a standard Spence-Mirrlees condition produces assortative matching, which in turn guarantees uniqueness and purity. Note, however, that in this example (as in the second one) the sets X_0 and Y_0 are one-dimensional.

When we move away from the one-dimensional matching model, the concept of assortative matching is not well-defined. In such economies, in which X_0 and Y_0 are not subsets of the real line and the surplus s need not be differentiable, a condition more general than the Spence-Mirrlees conditions above is required. In this section we recall such a condition. The generalized Spence-Mirrlees condition (or 'twist condition', as it is known in the mathematics literature) is sufficient for both uniqueness and purity. We develop a version of this condition that is valid in general type spaces, does not require differentiability of the surplus function, allows for non-participation, and is not dependent on the coordinates (i.e. the parametrization) of the problem. This condition also need only apply to either the buyers or the sellers.

Finally, we emphasize that uniqueness and purity are different concepts. For instance, a unique equilibrium may fail to be pure, as we illustrate in an example. Therefore, we introduce a condition (called *subtwist* below) weaker than generalized Spence-Mirrlees, and we show that this condition is sufficient for uniqueness but not purity. Let us define the concept of pure matchings formally.

Definition 2 (Pure) Let $X = X_0 \cup \{\emptyset_X\}$ and $Y = Y_0 \cup \{\emptyset_Y\}$ be subsets of complete, separable metric spaces augmented with isolated points $\emptyset_X$, $\emptyset_Y$, and equipped with Borel probability measures μ and ν. A feasible (but not necessarily optimal) solution $\gamma \in \Gamma(\mu, \nu)$ to (MK) program (14) is *pure* if there exists a function $f : X_0 \longrightarrow Y$ such that γ is concentrated on the graph of f, in the sense that γ assigns zero outer measure to the set $\{(x, y) \in X_0 \times Y \mid y \neq f(x)\}$.

In words, if the solution is pure, then there exists a well-defined function f such that any $x \in X_0$ is matched with probability one to $y = f(x)$. The set of buyers who remain indifferent between action or inaction, or between two or more preferred sellers, forms a set of measure zero; almost every buyer has a pure (as opposed to mixed) preference for whether he wishes to buy, and if so from whom. Such a pure solution will entail a pure matching of buyers and sellers to products if $Z(x, f(x)) := \arg\max_{z \in Z} \{u(x, z) - v(f(x), z)\}$ is a singleton. Note that most empirical studies consider only solutions which are pure, and for which $Z(x, f(x))$ consists of a single contract (for μ almost all $x \in X_0$).

4.2 A generalized Spence-Mirrlees (twist) condition

A standard tool in economic approaches to matching or hedonic problems is the Spence-Mirrlees condition—also known as the *twist* condition in the mathematics literature (Villani 2009).

Though the Spence-Mirrlees condition has been generalized to multidimensional type spaces—see Gangbo (1995), Carlier (2003) or Ma et al. (2005)—one may notice that the vast majority of economic studies still adopt a one-dimensional version of the condition.

Let us first specialize to the *Lipschitz-buyer* setting, meaning the space of buyers X_0 is an n-dimensional manifold (smooth without loss of generality), and the surplus function $s(x, y)$ and distribution $d\mu_0(x)$ of buyers enjoy a sufficiently smooth dependence on $x \in X_0$, as we now make precise. We describe this setting as Lipschitz-buyer to emphasize that Y_0 and Z_0 may or may not be smooth manifolds, and could even be finite spaces as when a continuum of buyers match with finitely many sellers. In this and subsequent definitions (of the twist and subtwist conditions, and of numbered limb systems), we go to some trouble to define notions which are independent of local choices of coordinates on the manifold X_0.

The reason for this is the following. Imagine a model which matches workers with varying skill levels $x \in X_0$ with firms which employ different technologies $y \in Y_0$. Obviously the skill level of the workers can be assessed (or parameterized) in many different ways. However, the question of whether the surplus function $s(x, y)$ is *Lipschitz*, *semiconvex*, *twisted* or *subtwisted* should be independent of the methodology used to assess the worker's skill levels, at least among methodologies which provide equivalent information. This principle of parametrization independence also plays a striking role in the theory addressing smoothness of the assignment $y = f(x)$ of workers to tasks (Kim and McCann 2009). The reader may prefer to skip the formal definitions, consulting instead the examples of relevance immediately thereafter.

Definition 3 (Lipschitz and semiconvex functions) Let X_0 be a smooth n-dimensional manifold and $X = X_0 \cup \{\emptyset_X\}$. Then $s : X \times Y \longrightarrow \mathbf{R}$ is said to be *Lipschitz on* X_0 *uniformly in* Y if for each (nonmaximal) coordinate ball $B_R \subset X_0 \backslash \partial X_0$, there is a constant C_B depending on the coordinates and the ball, but independent of y, such that all $x, \bar{x} \in B_R$ satisfy

$$|s(x, y) - s(\bar{x}, y)| \leq C_B \|x - \bar{x}\|, \tag{24}$$

where $\|x\| = \langle x, x\rangle^{1/2}$ denotes the distance and inner product in coordinates. Similarly, $s(x, y)$ is *semiconvex on* X_0 *uniformly in* Y if for each nonmaximal coordinate ball $B_R \subset X_0 \backslash \partial X_0$, there is a function $\omega_B(r)$ depending on the coordinates and the ball, but independent of $y \in Y$, such that $0 = \lim_{r \to 0} \omega_B(r)/r$, and for each $\bar{x} \in B_R$ there exists $\bar{p} \in \mathbf{R}^n$ such that

$$s(x, y) \geq s(\bar{x}, y) + \langle \bar{p}, x - \bar{x}\rangle + \omega_B(\|x - \bar{x}\|) \tag{25}$$

holds for all $x \in B_R$ and $y \in Y$. A function $q : X \longrightarrow \mathbf{R}$ is said to be Lipschitz on X_0, or semiconvex on X_0, if $s(x, y) = q(x)$ independent of $y \in Y$ satisfies the corresponding definitions above.

This definition of semiconvexity—which is sometimes called locally uniform subdifferentiability—is weaker than the standard one, in which $2\omega_B(r) = C_B r^2$, but has been chosen for consistency with usage in Villani (2009, Proposition 10.12). It might be appropriate to add the adjective *local* to the definitions of *Lipschitz* and *semiconvex* given above, as Villani does, but since differentiable manifolds are defined by local charts, there is no good definition for what it might mean for a function thereon to be globally Lipschitz, so we omit the adjective *local* for brevity whenever we feel confusion cannot arise. For this purpose, we overlook the fact that X_0 was assumed to be a metric space at the outset.

Definition 4 (Lipschitz-buyer and semiconvex-buyer settings) Assume, in addition to the hypotheses of Theorem 1, that X_0 is a smooth n-dimensional manifold and μ a Borel probability measure on $X := X_0 \cup \{\emptyset_X\}$. The setting is *Lipschitz-buyer* if μ concentrates no mass on subsets of X_0 which have zero volume, and if moreover the surplus function $s \in C(X \times Y)$ is locally Lipschitz on X_0 uniformly in Y. Similarly, the setting is *semiconvex-buyer* if μ concentrates no mass on *countably rectifiable hypersurfaces*[8] in X_0, and the surplus function $s \in C(X \times Y)$ is locally semiconvex on X_0 uniformly in Y.

[8] Here sets of zero volume refer to sets which are Lebesgue negligible in any and hence all coordinate charts on X_0. A countably rectifiable hypersurface refers to a set which is contained in a countable union of Lipschitz hypersurfaces; more precisely, it is countably $(n-1)$-rectifiable in local coordinates on X_0, in the sense of Definition 10.47 of Villani (2009).

As the next examples show, our model falls into the semiconvex-buyer setting whenever the buyer's utility $u(x, z)$ or the surplus function $s(x, z)$ is sufficiently smooth. Although its description appears more technical, the semi-convex buyer setting has the advantage that the measure μ may be more concentrated than the Lipschitz-buyer setting allows. In particular, a measure μ_0 on the interval $X_0 = [0, 1]$ satisfies the semiconvex-buyer hypothesis as long as it assigns zero mass $\mu_0(\{x\}) = 0$ to each type $x \in [0, 1]$; it need not be absolutely continuous with respect to Lebesgue measure, as the Lipschitz-buyer hypothesis would require. This improvement can be traced back to McCann (1995) and Gangbo and McCann (1996).

Example 3 (Lipschitz-buyer) If X_0 is a smooth manifold and Y is compact, any surplus function $s(x, y)$ locally Lipschitz on $X_0 \times Y$ also satisfies (24). Similarly if X_0 is a smooth manifold, Z is compact, and $v : Y \times Z \longrightarrow \mathbf{R}$ is arbitrary, any utility function $u(x, z)$ locally Lipschitz on $X_0 \times Z$ induces a surplus (4) satisfying the Lipschitz-buyer hypothesis (24).

Example 4 (Semiconvex-buyer) If X_0 and Y_0 are smooth manifolds and Y_0 is compact, any surplus function $s \in C^2(X_0 \times Y_0)$ also satisfies (25). If X_0 and Z_0 are smooth manifolds and Z_0 is compact, and $v{:}Y \times Z \longrightarrow \mathbf{R}$ is arbitrary, any utility function $u \in C^2(X_0 \times Z_0)$ induces a surplus (4) satisfying the semiconvex-buyer hypotheses, despite the fact that $s(x, y)$ will not generally be differentiable.

Although the surplus function (4) may fail to be differentiable, the Lipschitz-buyer setting guarantees the surplus $s(x, y)$ is locally Lipschitz[9] with respect to $x \in X_0$, with Lipschitz constant independent of $y \in Y$. This in turn guarantees any s-convex function $q = q^{\tilde{s}s}$ will be locally Lipschitz on X_0, hence (by Rademacher's theorem) differentiable on a set Dom $Dq \subset X_0$ of full measure. The derivative $Dq(x)$ is a vector in the cotangent space $T^*_x X_0$ to X_0 at the point $x \in \mathrm{Dom}\, Dq$. Given $q{:}X_0 \longrightarrow \mathbf{R}$ locally Lipschitz, we define its *superdifferential* $\partial q(x_0)$ at $x_0 \in X_0$ to consist of the set of covectors $w \in T^*_{x_0} X_0$ such that

$$q(x) \leq q(x_0) + \langle w, x - x_0 \rangle + o(\|x - x_0\|) \text{ as } x \to x_0, \tag{26}$$

with the error term allowed to depend on x_0 and on the coordinates chosen. For fixed $y \in Y$, we define the superdifferential $\partial^x s(x_0, y) \subset T^*_{x_0} X$ of $s(x, y)$ with respect to x analogously.

Before proceeding, let us state a uniqueness proposition which does not require further assumptions. This proposition asserts μ-a.e. uniqueness of the marginal payoff Dq with respect to buyer type, which may or may not determine the payoff $q(x)$ uniquely depending on the connectivity properties of Spt $\mu_0 \subset X_0$, and whether the participation constraint is active. Still, this proposition yields a point of contact between our work and that of Gretsky et al. (1999), by giving alternative sufficient conditions on the measures and surplus function to enforce uniqueness of marginal payoffs. Note however, in the absence of further assumptions such as the twist condition from Definition 5, our proof of this proposition may not extend to the Lipschitz-buyer setting.

[9] See Theorem 10.26 of Villani (2009). The same technique validates the claims made in Examples 3–4.

Our proposition is inspired by a pressure uniqueness result of Brenier concerning fluid mechanics (Brenier 1993).

Proposition 3 (A semiconvex buyer's marginal payoffs are unique) *Let $s : X \times Y \longrightarrow [0, \infty[$ be defined on probability spaces (X, μ) and (Y, ν) in the semiconvex-buyer setting. If both $(q, r) = (r^s, q^{\tilde{s}})$ and $(\tilde{q}, \tilde{r}) = (\tilde{r}^s, \tilde{q}^{\tilde{s}}) \in Lip_s(\mu, \nu)$ minimize (16), then q and $\tilde{q}$ are (locally) semiconvex and $Dq = D\tilde{q}$ holds μ-almost everywhere on $X \backslash \{\emptyset_X\}$.*[10]

Proof Suppose $(q, r) = (r^s, q^{\tilde{s}}) \in \mathrm{Lip}_s(\mu, \nu)$ minimizes (16). Since $q = r^{\tilde{s}}$, q is lower semicontinuous by the continuity assumption $s \in C(X \times Y)$. It is locally semiconvex and hence differentiable except on a countably rectifiable hypersurface in $\{x \in X_0 \mid q(x) < \infty\}$ by Villani (2009, Theorems 10.8(iii) and 10.26). Let

$$S := \{(x, y) \in X \times Y \mid q(x) + r(y) - s(x, y) = 0\}. \tag{27}$$

denote the closed set where the lower semicontinuous non-negative function $q(x) + r(y) - s(x, y)$ vanishes. Since μ concentrates no mass on the Lipschitz hypersurfaces where differentiability of q fails, all joint measures $\gamma \in \Gamma(\mu, \nu)$ assign full mass to $A = \mathrm{Dom}\, Dq \times Y$ in the semiconvex-buyer setting, noting the convention $\emptyset_X \in \mathrm{Dom}\, Dq$. Moreover, at least one optimizer $\gamma \in \Gamma(\mu, \nu)$ exists, and its support is contained in the closed set S, according to Theorem 1.

If X and Y are complete separable metric spaces, let K denote the σ-compact carrying the full mass of γ provided e.g., by Dudley (2002, p. 255) or Villani (2006, Theorem I-55), so that μ vanishes outside the σ-compact projection of $K \cap \mathrm{Spt}\, \gamma$ through π^X. Now suppose $x_0 \in \pi^X(K \cap \mathrm{Spt}\, \gamma) \cap \mathrm{Dom}\, Dq \backslash \{\emptyset_X\}$. Then there exists $(x_0, y_0) \in \mathrm{Spt}\, \gamma \subset S$, whence the first-order condition for vanishing in (27) implies superdifferentiability of $s(x, y_0)$ at x_0 with $Dq(x_0) \in \partial^x s(x_0, y_0)$. On the other hand, semiconvexity implies subdifferentiability and hence differentiability of $x \in X_0 \longrightarrow s(x, y_0)$ at x_0, and its super- and subdifferentials must both then coincide with $\{D_x s(x_0, y_0)\}$, as in Gangbo and McCann (1996). Thus $Dq(x_0) = D_x s(x_0, y_0)$. Notice the right hand side depends only on $(x_0, y_0) \in \mathrm{Spt}\, \gamma$ such that $x_0 \in \pi^X(K \cap \mathrm{Spt}\, \gamma) \cap \mathrm{Dom}\, Dq \setminus \{\emptyset_X\}$, and is otherwise independent of q. If a second semiconvex function $\tilde{q}$ minimized (16), we would similarly have $D\tilde{q}(x_0) = D_x s(x_0, y_0) = Dq(x_0)$ on the set $\pi^X(K \cap \mathrm{Spt}\, \gamma) \cap \mathrm{Dom}\, Dq \cap \mathrm{Dom}\, D\tilde{q}$ of full μ_0 measure, to establish the proposition.

If X and Y are merely subsets of complete separable metric spaces, we use their completions $\tilde{X}$ and $\tilde{Y}$ to find a σ-compact set $\tilde{K} \subset \tilde{X} \cap \tilde{Y}$ carrying the full mass of γ, and establish the result on the intersection of the σ-compact set $\pi^{\tilde{X}}(\tilde{K} \cap \mathrm{Spt}\, \gamma)$ with $\mathrm{Dom}\, Dq \subset X$, which still carries the full mass of μ. □

We now state a generalization of the Spence-Mirrlees condition appropriate to the Lipschitz-buyer setting.

[10] It costs no generality to assume $(q, r) = (r^s, q^{\tilde{s}})$. Indeed, from Remark 1, we know $((r^s)^{\tilde{s}})^s = r^s$ quite generally, and any minimizing pair (q, r) agrees with $\left(r^s, r^{s\tilde{s}}\right) \in \mathrm{Lip}_s(\mu, \nu)$ up to $L^1(\mu) \times L^1(\nu)$ negligible distinctions.

Definition 5 (Twisted-buyer condition) In the Lipschitz-buyer setting, a surplus function $s : X \times Y \longrightarrow [0, \infty[$ is said to be *twisted-buyer* if there is a set $X_L \subset X_0$ of zero volume such that $\partial^x s(x_0, y_1)$ is disjoint from $\partial^x s(x_0, y_2)$ for all $x_0 \in X_0 \setminus X_L$ and $y_1 \neq y_2$ in Y. The same definition applies in the semiconvex-buyer setting, except that X_L must then lie in a μ-negligible set, such as a countably rectifiable hypersurface.

Example 5 A surplus differentiable with respect to x is twisted-buyer if and only if there is a negligible set $X_L \subset X_0$ of buyers such that: for each distinct pair of sellers, any critical points of the function $x \in X_0 \longrightarrow s(x, y_1) - s(x, y_2)$ lie in X_L.

For instance, the surplus function $s(x, y) = 2 - |x - y|^2$ on disjoint open sets $X_0, Y_0 \subset \mathbf{R}^n$ is both twisted-buyer and twisted-seller. We must insist on disjointness of X_0 and Y_0 since for $y \in X_0$ the function $x \in X_0 \longrightarrow s(x, y) - s(x, \emptyset_Y) = 2 - |x - y|^2$ has $x = y$ as a critical point. This fact has been exploited in matching problems with optional participation, as in Caffarelli and McCann (2009). On the other hand, with the surplus function of Example 2, neither the twisted-buyer nor the twisted-seller condition is satisfied, since $s(x, y_1) - s(x, y_2) = (y_2)^2 - (y_1)^2$ does not depend on x.

The twisted-buyer condition has two consequences which are well-known (see Carlier 2003; Gangbo 1995 or Ma et al. 2005) provided that at equilibrium, participation is complete. It guarantees the Monge–Kantorovich maximization (14) is attained by a unique assignment γ of buyers with sellers. Moreover, it also implies this unique maximizer is *pure*, meaning there is a mapping $f{:}X \rightarrow Y$ defined μ-almost everywhere such that $\gamma = (id_X \times f)_{\#}\mu$. The following theorem confirms that the twisted-buyer condition formulated above guarantees uniqueness and purity of the mixed solution even when the situation is complicated by the presence of the isolated point $\emptyset_X$ in $X = X_0 \cup \{\emptyset_X\}$ representing the null buyer. The proof makes use of results found in Hestir and Williams (1995) and Ahmad et al. (2009), which allow us to establish a unique representation of the equilibrium measure simply by showing that almost all sellers have pure preferences at equilibrium. In Appendix A we recapitulate those results and in Appendix B give the full proof of Theorem 2 not only for the sake of completeness, but also to illustrate the efficacy of Lemma 3 and Theorem 4.

Theorem 2 (Twisted-buyers induce pure and unique assignments) *Let* $s : X \times Y \longrightarrow [0, \infty[$ *be a twisted-buyer surplus function, defined on probability spaces* (X, μ) *and* (Y, ν) *in either the Lipschitz-buyer or semiconvex-buyer setting. Then the maximizer* γ *of (14) on* $\Gamma(\mu, \nu)$ *is unique. Moreover, there is a* μ_0 *measurable map* $f{:}X_0 \longrightarrow Y$ *such that* $\gamma = \gamma_0 + \gamma_1$ *where* $\gamma_1 = (id_{X_0} \times f)_{\#}\mu_0$ *and* $\gamma_0 = (\emptyset_X \times id_Y)_{\#}(\nu - \pi^Y_{\#}\gamma_1)$.

Proof See Appendix B.

As anticipated, the set (19) takes center stage in the analysis there.

4.3 Examples of twisted-buyer costs

Since the surplus depends on the utility functions, it is useful to have criteria on $u(x, z)$ and $v(y, z)$ which guarantee $s(x, y)$ is twisted. One such criterion is given by the following example.

Example 6 Consider the Tinbergen (1956) model. Here, $X = Y = Z = \mathbf{R}^n$ with $u(x,z) = \frac{1}{2}(x-z)' A (x-z)$ and $v(y,z) = \frac{1}{2}(y-z)' B (y-z)$ with A and B symmetric and with $A - B < 0$. Then

$$s(x,y) = \frac{1}{2}\left(x'Ax - y'By\right) - \frac{1}{2}(Ax - By)'(A-B)^{-1}(Ax - By)$$

and

$$D_x\left(s(x,y_1) - s(x,y_2)\right) = -\frac{1}{2}(A-B)^{-1} B\left((y_1 - y_2)\right) A'.$$

This only equals zero when $y_1 = y_2$ so $s(x,y)$ satisfies the both buyer and seller twist condition.

In this model, the hedonic equilibrium is unique. Buyers' willingness to pay is

$$\text{WTP} = \frac{1}{2}(x-z)' A (x-z) - q(x).$$

If $A < 0$ and $B > 0$, buyers with smaller values of $|x - z|$ are willing to pay more for z and sellers with smaller values of $|y - z|$ are willing to accept more. The willingness to pay curves of different buyers never cross. The exact balance of buyers and sellers across locations depends on the distributions of buyer and seller types in the economy.

As a more general example in which the twist condition is satisfied, consider the following lemma.

Lemma 2 (Utilities yielding a twisted surplus) *Let $X_0, Y_0, Z_0 \subseteq \mathbf{R}^n$ be open domains with X_0 and Y_0 convex. Take $u \in C^2(X_0 \times Z_0)$ and $v \in C^2(Y_0 \times Z_0)$ with $D_x u(x,z) \neq \mathbf{0}$ on $X_0 \times Z_0$. For $\mu_0 \times \nu_0$ a.e. (x,y), assume $Z(x,y) = \arg\max_{z \in Z}\{u(x,z) - v(y,z)\} = \{z_0(x,y)\}$ is a singleton, and for all $(x,y) \in X_0 \times Y_0$ assume $M + M^t > 0$, where*

$$M = D^2_{xz}u(x,z_0)(D^2_{zz}u(x,z_0) - D^2_{zz}v(y,z_0))^{-1} D^2_{zy}v(y,z_0)$$

and $z_0 = z_0(x,y)$. Then $s(x,y)$ satisfies the twisted-buyer condition.

Proof Ignoring $\emptyset_Z$, the surplus function is given by

$$s(x,y) = \max_{z \in Z_0}\{u(x,z) - v(y,z)\}.$$

Since $z_0(x,y)$ is unique and on the interior of Z_0 by assumption, and since u and v are differentiable, $z_0(x,y)$ satisfies

$$D_z u(x,z_0) - D_z v(y,z_0) = 0. \tag{28}$$

The envelope theorem then implies differentiability of s at (x,y), and

$$D_x s(x,y) = D_x u(x, z_0(x,y)). \tag{29}$$

If for each $x \in X_0$, $y \in Y \longrightarrow D_x s(x, y)$ is injective, then $s(x, y)$ satisfies the twist condition. Since Y is convex, a sufficient condition for this is the positive (or negative) definiteness of the quadratic form $D^2_{xy} s(x, y)$. Since $z_0 = z_0(x, y)$ maximize surplus, $D^2_{zz} u(x, z_0) - D^2_{zz} v(y, z) \leq 0$, and the inequality is strict by hypothesis. The implicit function theorem then implies continuous differentiability of $z_0(x, y)$ in (28). Differentiating (29) with respect to y yields

$$D^2_{xy} s(x, y) = D^2_{xz} u(x, z_0(x, y)) D_y z_0(x, y), \tag{30}$$

while differentiating (28) with respect to y yields

$$(D^2_{zz} u(x, z_0) - D^2_{zz} v(y, z_0)) D_y z_0(x, y) - D^2_{zy} v(y, z_0) = 0. \tag{31}$$

These combine to give

$$D_y z_0(x, y) = (D^2_{zz} u(x, z_0) - D^2_{zz} v(y, z_0))^{-1} D^2_{zy} v(y, z_0) \tag{32}$$

which, substituted into (30) yields

$$D^2_{xy} s(x, y) = D^2_{xz} u(x, z_0(x, y))(D^2_{zz} u(x, z_0) - D^2_{zz} v(y, z_0))^{-1} D^2_{zy} v(y, z_0). \tag{33}$$

By hypothesis, this matrix is positive definite as required. □

When $n = 1$, this lemma gives the setting whose empirical properties are studied in Heckman et al. (2005). When $n = 1$, this also reduces to the usual Spence-Mirrlees conditions $D^2_{xz} u \neq 0 \neq D^2_{yz} v$ on u and v separately plus strict concavity with respect to z of the difference $u(x, z) - v(y, z)$.

4.4 The subtwist: a weaker condition for uniqueness

A priori, there does not seem to be any economic reason why the twist condition should be expected to hold.[11] Whether or not twisting is necessary to guarantee purity of assignments for general measures μ_0 and ν_0 in the Lipschitz-buyer setting is an open question. The good news, however, is that it is certainly not necessary to guarantee uniqueness of the assignment γ. Though it is frequently assumed to be satisfied in applications where the spaces of buyers X_0 and sellers Y_0 are subsets of $\mathbf{R}^n$, this is not always the case. There are also important settings where twisting cannot be satisfied. Taking $X_L = \emptyset$ for simplicity, no differentiable surplus function satisfies the twist condition on a compact space X_0 such as the circle or sphere $\mathbf{S}^n := \{\|x\|^2 = 1 \mid x \in \mathbf{R}^{n+1}\}$—or the periodic cube $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$—since $x \in X_0 \longrightarrow s(x, y_1) - s(x, y_2)$ obviously has critical points where its maximum and minimum are attained. If the buyers and sellers were distributed continuously over the surface of the planet or

[11] For many common economic models, the twist condition cannot actually hold. For instance, it is typically violated in models of horizontal differentiation on a circle (see Example 4.20 for an illustration).

around locations on an expressway encircling a city, there would be no hope of twisting. This situation is not much improved by assuming X_L non-empty: no surplus $s(Rx, Ry) = s(x, y) \in C^1(\mathbf{S}^n \times \mathbf{S}^n)$ invariant under all rotations R of the sphere $X_0 = \mathbf{S}^n = Y_0$ can be twisted, since no negligible set $X_L \neq \emptyset$ is rotationally invariant. Similarly, no surplus $s(x + k, y + k) = s(x, y) \in C^1(\mathbf{T}^n \times \mathbf{T}^n)$ invariant under all translations $k \in \mathbf{R}^n$ can be twisted in the periodic setting. Clearly there are topological obstructions to twisting. It is a fundamental open question to understand when uniqueness of equilibria can be expected to persist in such settings. We give a sufficient condition which resolves this question in settings such as the circle and sphere—where $x \in X_0 \longrightarrow s(x, y_1) - s(x, y_2)$ has only two critical points. This would not be the case in the periodic setting $\mathbf{T}^2$, and we do not know a single example of a smooth surplus function for which the (MK) solution to program (14) with $\mu_0 \ll \mathrm{vol}$ can generally be expected to be unique in this geometry.

The following theorem guarantees uniqueness of the optimal assignment γ. Even when all buyers elect to participate, there are many examples where the unique assignment will not be pure, meaning a positive fraction of buyers remain indifferent between two or more preferred sellers at equilibrium.

Definition 6 (Subtwist condition) In the Lipschitz-buyer setting, a surplus function $s : X \times Y \longrightarrow [0, \infty[$ is said to be *subtwisted* if there is a set $X_L \subset X_0$ of zero volume such that whenever $\partial^x s(x_0, y_1)$ intersects $\partial^x s(x_0, y_2)$ for some $x_0 \in X_0 \backslash X_L$ and $y_1 \neq y_2 \in Y$, then x_0 is either the unique global maximum or the unique global minimum of $s(x, y_1) - s(x, y_2)$ on $X = X_0 \cup \{\emptyset_X\}$. The same definition applies in the semiconvex-buyer setting, except that X_L must then lie in a μ-negligible set, such as a countably rectifiable hypersurface.

Example 7 A surplus differentiable with respect to x is subtwisted if and only if there is a negligible set $X_L \subset X_0$ of buyers such that: for each distinct pair of sellers, the function $x \in X_0 \backslash X_L \longrightarrow s(x, y_1) - s(x, y_2)$ has no critical points except for at most one global maximum and at most one global minimum.

Theorem 3 (Unique equilibria with mixed assignments) *If probability spaces (X, μ) and (Y, ν) and a surplus function $s(x, y)$ satisfy the subtwist condition in the Lipschitz-buyer or semiconvex-buyer setting, then the maximizer γ of (14) is unique. Moreover, γ is supported on a numbered limb system with three limbs, as in Definition 7.*

Proof See Appendix C. □

4.5 Circular assignment: example of a subtwisted cost

The preceding theorem generalizes results of Gangbo and McCann (2000) and Ahmad (2004). A special case of these earlier results yields the following illustrative example, as in Ahmad et al. (2009). In Appendix D, we describe an algorithm for computing an approximate solution for this example.

Example 8 (School districts on a ringroad) Consider a simple model of spatial matching in which a continuum of students and a continuum of schools are located at points on a circular expressway around a city. The pairwise surplus from matching a student to a school is a decreasing function of distance due to commuting costs; in particular, each student x would prefer to be matched to a school with the same location as hers to minimize transportation expenses.

Formally, thus, let $X_0 = Y_0 = \mathbf{S}^1$ and $s(x, y) = 1 + \cos(2\pi(x - y))$ where each x represents a student and each y represents a school. The rate at which the surplus decreases is increasing for $|x - y| \leq \frac{1}{4}$. However, it is decreasing for $\frac{1}{4} \leq |x - y| \leq \frac{1}{2}$. Note that $s(x, y) \geq 0$, so that participation is complete. Also, the model does not satisfy the twist condition. Indeed, the surplus s is differentiable, but for any (y_1, y_2) the function $s(x, y_1) - s(x, y_2)$ admits $x = \frac{y_1+y_2}{2} \pm \frac{1}{4}$ as its global maximum and minimum; in particular, they are critical points.

Now, assume first that $\mu = \nu$, meaning students and schools have the same distribution on the circle. Then the unique solution of the primal surplus maximization problem would have support on the graph $y = x$. Every student would travel a maximum distance of zero. Any pair (q, r) of non-negative constants $q(x) = q_0$ and $r(y) = r_0$ such that $q_0 + r_0 = 2$ would solve the dual problem. This is a case in which the assignment is unique and pure, despite the fact that the twist condition does not apply. Note however, that the equilibrium price (22) is not uniquely determined until a choice of utility transferred $q_0 \in [0, 2]$ is made. This ambiguity in price would be resolved in scenarios where some students or schools choose not to participate, either because of a net imbalance between supply and demand, or due to a uniform increase in the commuting costs.

However, the model becomes much more interesting when the densities associated with μ and ν are different. Assume μ and ν are those detailed in Appendix D and shown in Fig. 1. Specifically, the distribution of students is concentrated around $x = \frac{1}{4}$ while the distribution of schools is concentrated around $y = \frac{3}{4}$. That is, most of the students live on the north side of the city while most of the schools are located on the south side. In this case, the optimal matching is still unique; but it is very different from the previous case. Indeed, it is impossible to match each student to a school near to their residence. The support of the unique optimal measure, computed using the method described in Appendix D, is shown in Fig. 2. All students $x \in \left[0, \frac{1}{8}\right] \cup \left[\frac{3}{8}, 1\right]$, are matched to a single school near to their home. For example, $x = 0.1$ is matched to $y = 0.867$. All students $x \in \left(\frac{1}{8}, \frac{3}{8}\right)$ are matched to two schools; one at a distance less than or equal to $\frac{1}{4}$ and one at a distance greater than $\frac{1}{4}$. In the equilibrium these students are indifferent between the two locations.[12]

Students from location x, obtain a surplus equal to $q(x)$. Schools in location y, obtain a surplus $r(y)$. The surplus functions of the students and schools are displayed in Fig. 3. The students and schools that are in scarce supply, $x = \frac{3}{4}$ and $y = \frac{1}{4}$, obtain the highest surplus. Those who are abundant, $x = \frac{1}{4}$ and $y = \frac{3}{4}$, obtain the lowest. The optimal measure assigns a fraction of each of the abundant students and schools

[12] The apparent indifference of certain students to three or more schools is due to the limited resolution of our computation in Fig. 2. Otherwise, the solution would violate a theorem of Gangbo and McCann (2000). We do not know whether the true solution has the triple junctions suggested by the figure, or gaps separating the increasing from the decreasing curves. However, Ahmad (2004) asserts the number of triple junctions, if any, cannot exceed two.

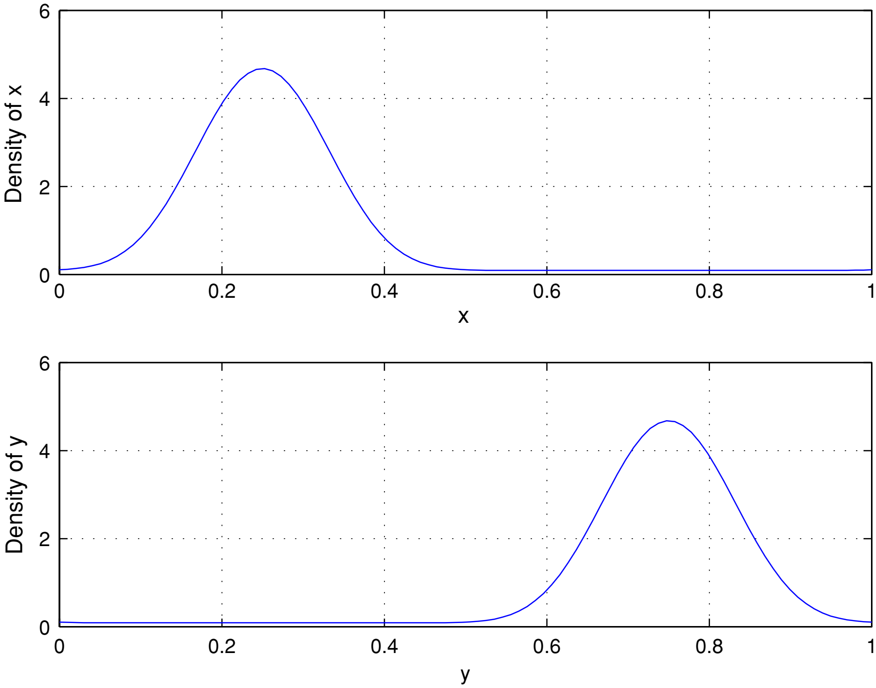

Fig. 1 Densities of students and schools on the circle in Example 8

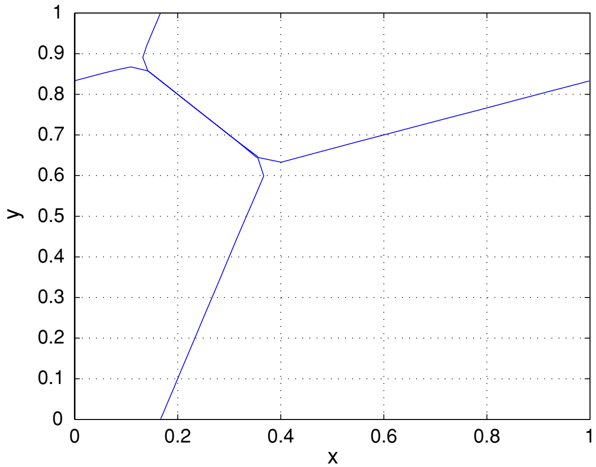

Fig. 2 Numerical support of optimal assignment in Example 8

to each of two locations, one less than a distance of $\frac{1}{4}$, one greater than this distance. Because there is such a large number of students near $x = \frac{1}{4}$ and schools near $y = \frac{3}{4}$, there is a social benefit from having some students travel a great distance. Technically, the measure γ on each branch is calculated as follows. Figure 2 depicts two limbs of a numbered limb system:

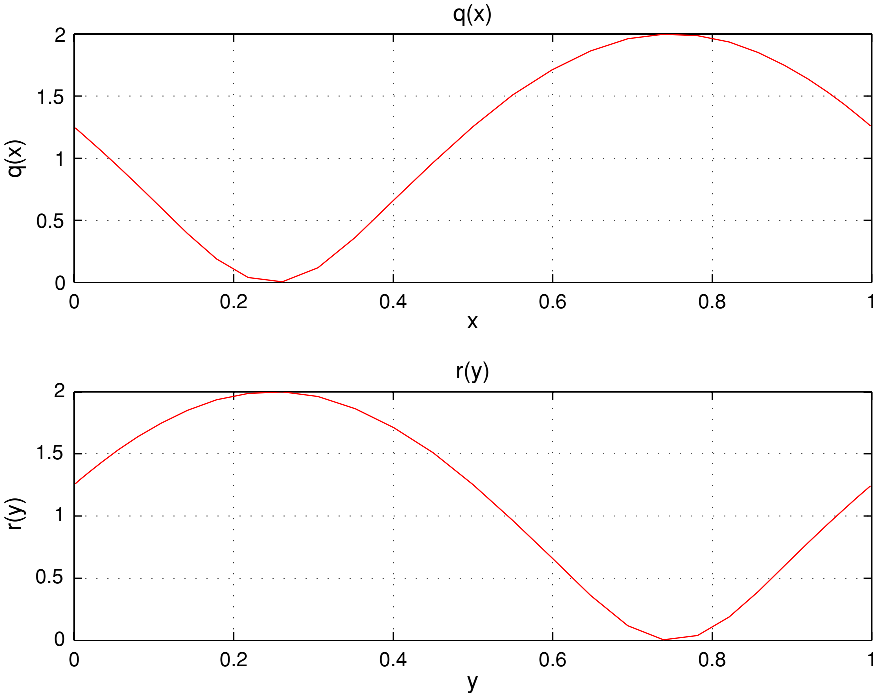

Fig. 3 Indirect utilities of students and schools computed numerically

$$y = f_2(x) \quad \text{for } x \in [0, 1] \quad \text{and } y \in \left(\tfrac{5}{8}, \tfrac{7}{8}\right)$$
$$x = f_3(y) \quad \text{for } y \in \left[0, \tfrac{5}{8}\right] \cup \left[\tfrac{7}{8}, 1\right].$$

The third limb $f_1(y) : \left(\frac{5}{8}, \frac{7}{8}\right) \to \{\emptyset_X\}$ is not displayed. These limbs define the support of the optimal measure. The optimal measure is given by

$$\gamma_3 = (f_3 \times id_Y)_{\#}\, \nu\big|_{\mathrm{Dom}\, f_3}$$
$$\gamma_2 = (id_X \times f_2)_{\#} \left(\mu - \pi^X_{\#}\gamma_3\right)\Big|_{\mathrm{Dom}\, f_2}$$
$$\gamma_1 = (f_1 \times id_Y)_{\#} \left(\nu - \pi^Y_{\#}\gamma_2\right)\Big|_{\mathrm{Dom}\, f_1} = 0.$$

In the example, we see the matches $(x, y) \in X \times Y_0$ between γ-a.e. participating pair can be found in the graph of one of two mappings $g : \mathrm{Dom}\, g \subset Y_0 \longrightarrow X$ or $f : \mathrm{Dom}\, f \subset X_0 \longrightarrow Y_0$, with range of f disjoint from $\mathrm{Dom}\, g$. This should be contrasted with the Spence-Mirrlees (twisted) case, where the matches lie on the graph of a single map $f : X_0 \longrightarrow Y$, à la Monge. It can also be compared with the necessary and nearly sufficient condition given in Hestir and Williams (1995) for a doubly stochastic measure $\gamma \in \Gamma(\lambda, \lambda)$ on the square $X_0 = Y_0 = [0, 1]$ to be extremal, which asserts that the support of γ must lie in a numbered limb system, with at most countably many limbs; see also Appendix A. In our theorem the system consists of three limbs, while in the twisted-buyer case it consists of two limbs. We do not know of any convenient condition on the surplus function which could lead to unique matches γ concentrated on a system with four or more numbered limbs. However, developments so far suggest the maximal number of limbs must generally be linked to the complexity of the (Morse) critical point structure of the function $x \in X_0 \longrightarrow s(x, y_1) - s(x, y_2)$.

Alternately, taking student assignments to schools to be fixed, Example 8 can parlayed into an example set on the periodic square $\mathbf{T}^2$ instead of the circle, by allowing students without cars to contract with students who drive to school to achieve desirable carpooling arrangements. There are then two kinds of students, and the type space of each is two-dimensional, consisting of a residential and a school location. In this case, topology forces even the subtwist condition to fail, leaving uniqueness an unresolved issue for all smooth surpluses!

5 Multiple-agent contracts

The hedonic pricing and matching problems we have discussed admit a natural generalization to the setting in which each contract z requires the participation of k agents chosen from different type spaces $(X_1, \mu_1), \ldots, (X_k, \mu_k)$. Carlier and Ekeland (2009) study this problem and establish existence of equilibrium. In this section, we show how to formulate their multiple agent contract problem as a linear program, instead of as a convex program.

The case $k = 2$ has been discussed above, but for $k > 2$ we assume the utility of contract $z \in Z$ to agent $x \in X_i$ is given by an upper semicontinuous function $u_i : \mathrm{cl}(X_i \times Z) \longrightarrow \mathbf{R} \cup \{-\infty\}$ plus any compensation $P_i(z)$ he receives. Thus the indirect utility available to him is

$$U_i(x) = \sup_{z \in Z}\{u_i(x, z) + P_i(z)\},$$

with the usual convention $-\infty + \infty = -\infty$. The payments $P_i(z)$ are assumed to satisfy a *frictionless* trading condition $0 = \sum_{i=1}^{k} P_i(z)$ on Z which prevents arbitrage and neglects friction. Payments corresponding to the null contract must vanish $P_i(\emptyset_Z) = 0$. As before, each type space $X_i = X_i^0 \cup \{\emptyset_i\}$ includes an isolated dummy agent type of mass

$$\mu_i(\emptyset_i) = 1 + \sum_{j \neq i} \mu_j \left(X_j^0\right),$$

and satisfies

$$u_i(x, z) = \begin{cases} 0 & \text{if } z = \emptyset_Z \quad \text{and} \quad x \in X_i, \\ -\infty & \text{if } z \in Z \setminus \{\emptyset_Z\} \quad \text{and} \quad x = \emptyset_i. \end{cases} \tag{34}$$

A joint measure α on $X_1 \times \cdots \times X_k \times Z$ together with frictionless payment schedules $P_i : Z \longrightarrow \mathbf{R} \cup \{\pm\infty\}$ represent a *market clearing equilibrium* if it has marginals $\pi^{X_i}_{\#}\alpha = \mu_i$ for each $i = 1, \ldots, k$, and

$$U_i(x_i) = u_i(x_i, z) + P_i(z)$$

holds for each $i = 1, \ldots, k$ and α-a.e. $(x_1, \ldots, x_k, z) \in \operatorname{Spt}\alpha$. Here, $\pi^{X_i}(x_1, \ldots, x_k, z)x_i \in X_i$ for each $i = 1, \ldots, k$.

Define the non-negative surplus function

$$s(x_1, \ldots, x_k) = \max_{z \in Z} \sum_{i=1}^{k} u_i(x_i, z).$$

The same arguments presented above show equivalence of this hedonic pricing problem to the linear program

$$\max_{\gamma} \gamma[s],$$

where the maximum is taken over all joint measures $\gamma \geq 0$ on $X_1 \times \cdots \times X_k$ having prescribed marginals $\mu_i = \pi^{X_i}_{\#}\gamma$. The dual infimum

$$\min_{q_i : X_i \longrightarrow [0,\infty]} \sum_{i=1}^{k} \mu_i[q_i] \tag{35}$$

is taken over functions q_i satisfying $s(x_1, \ldots, x_k) \leq \sum_{i=1}^{k} q_i(x_i)$ on $X_1 \times \cdots \times X_k$, normalized so $q_1(\emptyset_1) = 0$. Duality still holds $\gamma[s] \leq \sum \mu_i[q_i]$, with equality if and only if γ is a maximizer and $(q_1, \ldots, q_k)$ minimizes. As before, it follows from $q_1(\emptyset_1) = 0$ that $q_i(\emptyset_i) = 0$ for each $i \leq k$. The existence and characterization of maximizers, minimizers, and equilibria is identical, but the literature exploring conditions on the surplus which guarantee uniqueness of the maximizing assignment or assortative matching is much more limited in the multiple marginal case; see Rachev and Rüschendorf (1998) and Gangbo and Święch (1998) for references. The frictionless transfer payment $P_j(z)$ required by the agent playing the jth role in the contract $z \in Z$ is related to the equilibrium payoffs $q_i(x_i)$ of all types by

$$\begin{aligned} P_j(z) &= T_j(z) - \frac{1}{k}\sum_{j=1}^{k} T_i(z), \\ T_i(z) &= \inf_{x_i \in X_i} q_i(x_i) - u_i(x_i, z). \end{aligned} \tag{36}$$

Let us also observe that the uniqueness of marginal payoffs proved in Proposition 3 extends immediately to the multiple agent problem. Thus if μ_1 vanishes on all countably rectifiable hypersurfaces of a smooth manifold $X^0_1 := X_1 \backslash \{\emptyset_1\}$, and the surplus function $s \in C(X_1 \times \cdots X_k)$ is semiconvex on X_1^0, uniformly in the other $k-1$ variables, and $(q_1, \ldots, q_k)$ minimizes (35) with q_1 semiconvex, then Dq_1 is uniquely determined μ_1 almost everywhere on X_1^0.

Appendix

A Supports of extremal doubly stochastic measures

The uniqueness of optimal assignments is established above using Hestir and William's (1995) characterization of extremal doubly stochastic measures in terms of their supports, or more precisely a variant of this characterization formulated by Ahmad et al. (2009). We summarize hereafter the characterization in the form that we need.

An important property of pure solutions is that the equilibrium is uniquely determined by the profile of buyers and their strategies. The first result that we shall use from Ahmad et al. (2009) allows us to establish a unique representation of the equilibrium measure simply by showing that almost all sellers have pure preferences at equilibrium, as in Theorem 2. What separates the following lemma from antecedents such as Lemma 2.4 of Gangbo and McCann (2000) is that μ_0-measurability of f is a consequence and not a hypothesis. This improvement was derived using an argument from Villani (2009, Theorem 5.28).

Lemma 3 [Pure measures are push-forwards (Ahmad et al. 2009)] *Let X_0 and Y_0 be subsets of complete separable metric spaces, and $\gamma \geq 0$ a σ-finite Borel measure on the product space $X_0 \times Y_0$. Denote the left marginal of γ by $\mu_0 := \pi_{\#}^{X_0}\gamma$. If γ is concentrated on the graph of $f : X_0 \longrightarrow Y_0$, meaning $\{(x, y) \in X_0 \times Y_0 \mid y \neq f(x)\}$ has zero outer measure, then f is μ_0-measurable and $\gamma = (id_{X_0} \times f)_{\#}\mu_0$.*

The preceding lemma shows that any measure that is concentrated on a graph is uniquely determined by its marginals. This would be the case for optimal measures in the twisted-buyer setting. Uniqueness, however, does not require the twist condition to be satisfied; as the next result demonstrates, the sufficient conditions given by Lemma 3 and Theorem 2 are far from necessary for uniqueness of the equilibrium—a peculiarity of continuous type spaces X.

Given a map $f : D \longrightarrow Y$ on $D \subset X$, we denote its graph, domain, range, and the graph of its (multivalued) inverse by

$$\begin{aligned} \mathrm{Graph}(f) &:= \{(x, f(x)) \mid x \in D\}, \\ \mathrm{Dom}\, f &:= \pi^X(\mathrm{Graph}(f)) = D, \\ \mathrm{Ran}\, f &:= \pi^Y(\mathrm{Graph}(f)), \\ \mathrm{Antigraph}(f) &:= \{(f(x), x) \mid x \in \mathrm{Dom}\, f\} \subset Y \times X. \end{aligned}$$

More typically, we will be interested in the Antigraph$(g) \subset X \times Y$ of a map $g : \mathrm{Dom}\, g \subset Y \longrightarrow X$. Following Hestir and Williams (1995) we define:

Definition 7 (Numbered limb system) Let X_0 and Y_0 be subsets of complete separable metric spaces. A relation $S \subset X_0 \times Y_0$ is a *numbered limb system* if there is a countable disjoint decomposition of $X_0 = \cup_{i=0}^{\infty} I_{2i}$ and of $Y_0 = \cup_{i=0}^{\infty} I_{2i+1}$ with a sequence of maps $f_{2i} : \mathrm{Dom}(f_{2i}) \subset X_0 \longrightarrow Y_0$ and $f_{2i+1} : \mathrm{Dom}(f_{2i+1}) \subset Y_0 \longrightarrow X_0$ such that $S \subset \cup_{i=1}^{\infty} \mathrm{Graph}(f_{2i}) \cup \mathrm{Antigraph}(f_{2i-1})$, with $\mathrm{Dom}(f_k) \cup \mathrm{Ran}(f_{k+1}) \subset I_k$ for each $k \geq 0$. The system has (at most) N limbs if $\mathrm{Dom}(f_k) = \emptyset$ for all $k > N$.

Notice the map f_0 is irrelevant to this definition though I_0 is not; we may always take $\mathrm{Dom}(f_0) = \emptyset$, but require $\mathrm{Ran}(f_1) \subset I_0$. The point of this definition is the following sufficient condition for extremality in $\Gamma(\mu, \nu)$. In case $\mathrm{Graph}(f_{2i})$ and $\mathrm{Antigraph}(f_{2i-1})$ are (Borel) measurable subsets of $X_0 \times Y_0$ for each $i \geq 1$, the sufficiency of this condition was established by Hestir and Williams (1995). The following variant of this result was formulated in Ahmad et al. (2009), where a direct proof has also been given. It plays a key role in Theorem 3. If γ assigns zero outer measure to the complement of S in $X_0 \times Y_0$, we say γ *vanishes outside of* S. In this case S is γ-measurable, meaning it belongs to the completion of the Borel σ-algebra with respect to the measure γ.

Theorem 4 (Numbered limb systems support unique equilibria) *Let X_0 and Y_0 be subsets of complete separable metric spaces, equipped with σ-finite Borel measures μ on X_0 and ν on Y_0. Suppose there is a numbered limb system $S \subset \cup_{i=1}^{\infty} \mathrm{Graph}(f_{2i}) \cup \mathrm{Antigraph}(f_{2i-1})$ with the property that $\mathrm{Graph}(f_{2i})$ and $\mathrm{Antigraph}(f_{2i-1})$ are γ-measurable subsets of $X_0 \times Y_0$ for each $i \geq 1$ and for every $\gamma \in \Gamma(\mu, \nu)$ vanishing outside of S. If the system has finitely many limbs or $\mu[X_0] < \infty$, then at most one $\gamma \in \Gamma(\mu, \nu)$ vanishes outside of S. If such a measure exists, it is given by $\gamma = \sum_{k=1}^{\infty} \gamma_k$ where*

$$\gamma_{2i} = (id_{X_0} \times f_{2i})_{\#}\eta_{2i}, \qquad \gamma_{2i-1} = (f_{2i-1} \times id_{Y_0})_{\#}\eta_{2i-1}, \tag{37}$$

$$\eta_{2i} = \left(\mu - \pi_{\#}^{X_0}\gamma_{2i+1}\right)\Big|_{\mathrm{Dom}\, f_{2i}}, \qquad \eta_{2i-1} = \left(\nu - \pi_{\#}^{Y_0}\gamma_{2i}\right)\Big|_{\mathrm{Dom}\, f_{2i-1}}. \tag{38}$$

Here f_k is measurable with respect to the η_k completion of the Borel σ-algebra. If the system has $N < \infty$ limbs, then $\gamma_k = 0$ for $k > N$, and η_k and γ_k can be computed recursively from the formulae above starting from $k = N$.

Measurability of the graphs and antigraphs is required by Theorem 4 only to decompose each candidate γ into countably many pieces, to which Lemma 3 can then be applied.

The application of these results to deduce Theorem 3 also requires an elementary measurability lemma from point set topology:

Lemma 4 *Let A and B be topological spaces and $Z \subset A \times B$ be closed. If $g : A \times B \longrightarrow \mathbf{R} \cup \{-\infty\}$ is upper semi-continuous, and B is σ-compact, then*

$$h(a) := \sup_{\{b \in B | (a,b) \in Z\}} g(a, b)$$

is Borel.

Proof Exhaust $B = \cup B_k$ using countably many compact sets $B_k \subset B_{k+1}$. Set

$$h_k(a) = \sup_{b \in B} g_k(a,b) = \max_{b \in B_k} g_k(a,b)$$

where

$$g_k(a,b) := \begin{cases} g(a,b) & \text{if } (a,b) \in Z \cap (A \times B_k), \\ -\infty & \text{otherwise,} \end{cases}$$

is upper semi-continuous.

Defining the levels sets $H_k^\lambda = \{a \in A \mid h_k(a) \geq \lambda\}$ and $G_k^\lambda = \{(a,b) \in A \times B_k \mid g_k(a,b) \geq \lambda\}$ yields $H_k^\lambda = \pi^A(G_k^\lambda)$. The projection H_k^λ of the closed set G_k^λ is easily seen to be closed using compactness of B_k. Thus $h_k \leq h_{k+1}$ is a sequence of upper semi-continuous functions increasing monotonically to a Borel limit $h = \sup_k h_k$ on A. □

B Proof of Theorem 2

Theorem 1 and Remark 1 provide a non-negative s-convex minimizing pair $(q,r) = (r^s, q^{\tilde{s}})$ to the dual problem (16). Recall that q is then locally Lipschitz on X_0, by Villani (2009, Theorem 10.26), and $r = q^{\tilde{s}}$ is lower semi-continuous in (21) by the continuity assumption $s \in C(X \times Y)$. The same theorem shows q to be locally semi-convex in the semiconvex-buyer setting. Let

$$S := \{(x,y) \in X \times Y \mid q(x) + r(y) - s(x,y) = 0\}.$$

denote the closed set where the non-negative function $q(x) + r(y) - s(x,y)$ vanishes. Since μ concentrates no mass on subsets of zero volume and $X_0 \backslash \operatorname{Dom} Dq$ has zero volume, all joint measures $\gamma \in \Gamma(\mu,\nu)$ assign full mass to $A := (\operatorname{Dom} Dq \backslash X_L) \times Y$ in the Lipschitz-buyer setting, with the convention $\emptyset_X \in \operatorname{Dom} Dq$. The same conclusion is true in the semiconvex-buyer setting, since μ is then assumed to vanish on the countably rectifiable hypersurface where differentiability of q fails, recalling Theorem 10.8(iii) of Villani (2009). Moreover, all optimizers $\gamma \in \Gamma(\mu,\nu)$ vanish outside S, according to Theorem 1. At least one optimizer exists, due to the (upper semi-) continuity of $s \in C(X \times Y)$. The next step of the proof will be to show that $S \cap A$ is contained in one of numbered limb systems of Definition 7. After this is established, Theorem 4 will be used to infer there is only one measure in $\Gamma(\mu,\nu)$ that vanishes outside $S \cap A$, hence the optimizer is unique.

Set $I_2 = Y_0$, $I_1 = Y$ and $I_0 = \{\emptyset_X\}$. Set $f_1(y) = \emptyset_X$ for all $y \in \operatorname{Dom} f_1 = Y$. It remains to show that $\pi^X(x,y) = x$ gives an injective map from $S_2 := (S \cap A) \cap (X_0 \times Y)$ to $\operatorname{Dom} f_2 := \pi^X(S_2)$. Once this injectivity has been shown, f_2 can be defined to make $id_X \times f_2 : \operatorname{Dom} f_2 \longrightarrow S_2$ invert $\pi^X|_{S_2}$, and a comparison with Definition 7 then reveals that S is contained in a numbered limb system. To prove the required injectivity, suppose (x_0, y_1) and (x_0, y_2) both belong to $S_2 \subset S \cap A$. The function $q(x) + r(y) - s(x,y) \geq 0$ vanishes at all points in S, hence enjoys $\mathbf{0}$ as a subgradient there. If this function is differentiable with respect to x at $x_0 \in \mathrm{Dom}\, Dq \backslash X_L$, we have $D_x s(x_0, y_1) = Dq(x) = D_x s(x_0, y_2)$; otherwise $Dq(x_0) \in \partial^x s(x_0, y_1) \cap \partial^x s(x_0, y_2)$. In either case, the twisted-buyer condition yields $y_1 = y_2$, whence π^X is injective on S_2.

To invoke Theorem 4, it remains only to establish the γ-measurability of Antigraph (f_1) and of Graph(f_2) for each $\gamma \in \Gamma(\mu, \nu)$ vanishing outside their union. Since Antigraph$(f_1) = \{\emptyset_X\} \times Y$ is Borel and disjoint from Graph(f_2), the γ-measurability of Graph(f_2) follows from the complement of Antigraph$(f_1) \cup$ Graph(f_2) having zero γ outer-measure. The μ_0-measurability of $f_2\colon \mathrm{Dom}\, f_2 \longrightarrow Y$ and special form $\gamma = \gamma_2 + \gamma_1$ with $\gamma_2 = (id_{X_0} \times f)_{\#}\mu_0$ and $\gamma_1 = (\emptyset_X \times id_Y)_{\#}(\nu - \pi^Y_{\#}\gamma_2)$ are both consequences of Theorem 4, since the set of optimizers in $\Gamma(\mu, \nu)$ is non-empty. Thus the theorem is established. $\square$

C Proof of Theorem 3

Theorem 1 and Remark 1 provide a non-negative s-convex minimizing pair $(q, r) = (r^s, q^{\tilde{s}})$ to the dual problem (16). As in the proof of that theorem, it costs no generality to replace X and Y by σ-compact sets in the completions cl X and cl Y carrying the full mass of μ and ν, respectively. Lower semi-continuity of r (and of q) follows from the same remark since $s \in C(X \times Y)$. Recalling that $\mu_0 \ll \mathrm{vol}$ by hypothesis and that q is locally Lipschitz (Villani 2009), let $\mathrm{Dom}\, Dq \supset \{\emptyset_X\}$ denote the Borel subset of X with full μ-measure where q is differentiable. Even in the semiconvex-buyer setting Dom Dq has full μ-measure, because q is locally semiconvex, hence differentiable except on a countably rectifiable hypersurface, to which μ_0 assigns zero mass; see McCann (1995) and Gangbo and McCann (1996) or Theorems 10.8(iii) and 10.26 of Villani (2009). Taking $X_L \subset X_0$ as in the subtwist condition, set $X_R = \mathrm{Dom}\, Dq \cap X \backslash X_L$ and let

$$S := \{(x, y) \in X_R \times Y \mid q(x) + r(y) - s(x, y) = 0\}$$

denote the set where the non-negative function $q(x) + r(y) - s(x, y)$ vanishes. Lower semi-continuity of this function implies S is closed in $A := X_R \times Y$.

The cross-difference (McCann 1999)

$$\Delta(x, y, x', y') := s(x, y) + s(x', y') - s(x, y') - s(x', y)$$

of the surplus is a continuous function on $(X \times Y)^2$. Before embarking on the proof, recall $\Delta \geq 0$ on S^2, i.e., any two equilibrium assignments (x, y) and (x', y') in S satisfy

$$\Delta(x, y, x', y') \geq 0. \tag{39}$$

This intuitive claim of Smith and Knott (1992) dates partly back to Monge (1781), and can be deduced by summing the inequalities

$$
\begin{aligned}
0 \le q(x') + r(y) - s(x', y) &= q(x') + s(x, y) - q(x) - s(x, y') \\
0 \le q(x) + r(y') - s(x, y') &= q(x) + s(x', y') - q(x') - s(x, y').
\end{aligned}
$$

All measures γ with left marginal μ assign full mass to $A = X_R \times Y$, since X_L is μ-negligible as a consequence of the subtwist hypotheses; this implies the γ-measurability of A. Theorem 1 asserts that all optimizers $\gamma \in \Gamma(\mu, \nu)$ vanish outside S, and by (upper semi-) continuity of $s(x, y)$ that at least one optimal measure γ exists. The proof will proceed by showing there exist maps $f_3\colon \mathrm{Dom}\, f_3 \longrightarrow X_0$ and $f_1\colon \mathrm{Dom}\, f_1 \longrightarrow \{\emptyset_X\}$ on disjoint subsets of $Y = (\mathrm{Dom}\, f_1) \cup (\mathrm{Dom}\, f_3)$ and $f_2\colon \mathrm{Dom}\, f_2 \subset X_0 \longrightarrow \mathrm{Dom}\, f_1$ such that $S \subset \mathrm{Antigraph}(f_1) \cup \mathrm{Graph}(f_2) \cup \mathrm{Antigraph}(f_3)$. Once this assertion is established, the results follow immediately from Theorem 4 after identifying $I_k = \mathrm{Dom}(f_k)$ for $k = 1, 3$, $I_2 = X_0$ and $I_0 = \{\emptyset_X\}$, and verifying the required measurability of $\mathrm{Graph}(f_2)$ and $\mathrm{Antigraph}(f_3)$.

Relative closedness of S in A and σ-compactness of $B = X \times Y$ imply

$$
h(x_1, y_1) := \sup_{\{(x, y_2) \in X \times Y \mid (x_1, y_2) \in S\}} \Delta(x_1, y_1, x, y_2)
$$

Borel on $A = X_R \times Y$, according to Lemma 4. Taking $y_2 = y_1$ implies $h \ge 0$ on S. A point $(x_1, y_1) \in S$ is said to be *marked* if $x_1 \in X_0$ and $h(x_1, y_1) = 0$, i.e.

$$
s(x, y_1) - s(x, y_2) \ge s(x_1, y_1) - s(x_1, y_2) \tag{40}
$$

for all $x \in X$ and $(x_1, y_2) \in S$.

Let $S_1 \subset S$ denote the marked points in S, and $S_2 = (X_0 \times Y) \cap S \backslash S_1$ the unmarked points. We claim $(x_1, y_1) \in S_2$ and $(x_2, y_1) \in S$ forces $x_1 = x_2$; in other words, the part of S which projects to $\mathrm{Dom}\, f_3 := \pi^Y(S_2)$ lies in an antigraph of unmarked points. The proof of this claim is inspired by the sole-supplier lemma (Gangbo and McCann 2000). Fix $(x_1, y_1) \in S_2$. Then $x_1 \in X_0 \cap \mathrm{Dom}\, Dq \backslash X_L$ and $q(x)+r(y)-s(x, y) \ge 0$ is minimized at (x_1, y_1), so the first order condition for a minimum implies $Dq(x_1) = D_x s(x_1, y_1)$ if the latter exists, and $Dq(x_1) \in \partial^x s(x_1, y_1)$ in any case. Since (x_1, y_1) is unmarked, there exist some $(x_1, y_2) \in S$ and $x \in X$ which violate (40). Obviously, $Dq(x_1) \in \partial^x s(x_1, y_2)$ so the superdifferentials of s intersect. The subtwist condition now guarantees x_1 is the unique maximizer of $s(\cdot, y_1)-s(\cdot, y_2)$; it cannot be the minimizer due to the presumed violation of (40). Any $(x_2, y_1) \in S$ therefore satisfies $s(x_2, y_1) - s(x_2, y_2) < s(x_1, y_1) - s(x_1, y_2)$, or else $x_2 = x_1$. The strict inequality violates (39), establishing the claim $x_1 = x_2$.

On $\mathrm{Dom}\, f_3 := \pi^Y(S_2)$ define $f_3(y) := x$ for each $(x, y) \in S_2$. The preceding claim shows the part of S which projects to $\mathrm{Dom}\, f_3$ coincides precisely with $\mathrm{Antigraph}(f_3) = S_2$.

We next show that both (x_1, y_1) and (x_1, y_0) in S_1 implies $y_0 = y_1$, so $S_1 = \mathrm{Graph}(f_2)$, where $f_2 : \mathrm{Dom}\, f_2 \longrightarrow Y$ is defined on $\mathrm{Dom}\, f_2 := \pi^X(S_1)$ by

$$f_2(x_1) = y_1 \quad \text{if } (x_1, y_1) \in S_1. \tag{41}$$

Before addressing this claim, note our definitions of $S_1, S_2 \subset X_0 \times Y$ ensure $\mathrm{Dom}\, f_2 \cup \mathrm{Ran}\, f_3 \subset X_0$. Also, as argued above, $(x_1, y_1) \in S \backslash S_2$ precludes $(x_2, y_1) \in S_2$, so (41) guarantees $\mathrm{Ran}\, f_2 \subset \mathrm{Dom}\, f_1 := Y \backslash \mathrm{Dom}\, f_3$. Finally, defining $f_1(y) := \emptyset_X$ ensures $\mathrm{Ran}\, f_1 \subset I_0$, where $I_0 := X \backslash X_0 = \{\emptyset_X\}$. Also $(x_1, y_1) \in S_0 := S \backslash (S_1 \cup S_2)$ implies $x_1 = \emptyset_X = f_1(y_1)$ while precluding $(x_2, y_1) \in S_2$, so $y_1 \notin \mathrm{Dom}\, f_3$ and $S_0 \subset \mathrm{Antigraph}(f_1)$.

Since $S_2 = \mathrm{Antigraph}(f_3)$ and $S_1 = \mathrm{Graph}(f_2)$ we have verified most the hypotheses for the unique representation of given by Theorem 4. It remains only to show suitable measurability of these graphs, and that $f_2(x_1)$ is well-defined by (41). To contradict the latter point, suppose $(x_1, y_1) \neq (x_1, y_0)$ both belong to S_1. According to (40), this means that $s(x, y_1) - s(x, y_0)$ and its negative $s(x, y_0) - s(x, y_1)$ are both minimized at $x = x_1$. But then $s(x, y_1) - s(x, y_0) = \text{const}$ independent of x, which violates the subtwist condition since then $\partial^x s(x, y_0) = \{\mathbf{0}\} = \partial^x s(x, y_1)$ for all $x \in X$. We conclude $f_2(x_1)$ is well-defined by (41), and therefore that any optimizer $\gamma \in \Gamma(\mu, \nu)$ is supported on a numbered limb system S with three limbs.

Since $A = X_R \times Y$ and hence S are γ-measurable for each $\gamma \in \Gamma(\mu, \nu)$, and $h : A \longrightarrow \mathbf{R} \cup \{-\infty\}$ is Borel, γ-measurability of the disjoint sets $S_2 = S \cap \{h > 0\} \cap (X_0 \times Y)$ and $S_1 = S \cap \{h = 0\} \cap (X_0 \times Y)$ and $S_0 = S \backslash (S_1 \cup S_2)$ follow. Now suppose $\gamma \in \Gamma(\mu, \nu)$ assigns zero outer measure to the complement of $S = S_0 \cup S_1 \cup S_2$. Then $S_2 = \mathrm{Antigraph}(f_3)$ and $S_1 = \mathrm{Graph}(f_2)$ are both γ-measurable, as is $S_0 \subset \mathrm{Graph}(f_1) \subset \{\emptyset_X\} \times Y$ since γ vanishes on $(\{\emptyset_X\} \times Y) \backslash S_0$. Because all optimal measures $\gamma \in \Gamma(\mu, \nu)$ vanish outside of S, Theorem 4 asserts uniqueness of optimizer and concludes the proof. □

D Computation

We use Example 8 to illustrate a computational algorithm for approximating solutions to the optimal transportation problem. In this example the surplus function is

$$s\,(x, y) = 1 + \cos\,(2\pi\, |x - y|)$$

and the densities of x and y if ϕ denotes the standard normal density:

$$\mathrm{d}\mu\,(x) = \frac{10\phi\,(10z\,(x + 0.25)) + 1}{11}$$
$$z\,(x + 0.25) = \ln\left(\frac{\mathrm{mod}\,(x + 0.25)}{1 - \mathrm{mod}\,(x + 0.25)}\right).$$

and

$$
\begin{aligned}
\mathrm{d}\nu\left(y\right) &= \frac{10\phi\left(10z\left(y-0.25\right)\right)+1}{11} \\
z\left(y-0.25\right) &= \ln\left(\frac{\operatorname{mod}\left(y-0.25\right)}{1-\operatorname{mod}\left(y-0.25\right)}\right).
\end{aligned}
$$

These densities are depicted in Fig. 1. The densities are periodic, bounded away from zero and symmetric with modes at $\frac{1}{4}$ and $\frac{3}{4}$, respectively.

To discretize the problem, we search for solutions to the dual in a space of Fourier series.[13] Let $a_j \in \mathbf{R}$ for all $j \in \{0, \ldots, 2n\}$ and define

$$
\mathbf{F}_n = \left\{ f\left(x\right) : f\left(x\right) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos\left(2\pi jx\right) + \sum_{j=n+1}^{2n} a_j \sin\left(2\pi jx\right) \right\}
$$

Rather than solve the linear program directly, we solve the following:

$$
\min_{q,r\in\mathbf{F}_n} \int q\left(x,a\right)\mathrm{d}\mu + \int r\left(y,b\right)\mathrm{d}\nu \tag{42}
$$

subject to

$$
\begin{aligned}
q\left(x,a\right) &\geq \max_{y\in Y} s\left(x,y\right) - r\left(y,b\right) \quad \text{for } x \in \left\{x_1, \ldots, x_{n_1}\right\} \\
r\left(y,b\right) &\geq \max_{x\in X} s\left(x,y\right) - q\left(x,a\right) \quad \text{for } y \in \left\{y_1, \ldots, y_{n_2}\right\}.
\end{aligned}
$$

In the solution shown in Figs. 1, 2, 3, we used $n = 7$ for both q and r and computed the integrals in (42) using a mixture of Gauss-Hermite and Gauss-Legendre rules each with 31 nodes. The constraints were imposed at the 31 Gauss-Hermite integration nodes. Matlab code to implement this algorithm is available from the authors upon request.

This method has several benefits. First, the resulting solutions are nearly s-convex and $\tilde{s}$-convex. Second, the discretization has $n_1 + n_2$ nonlinear constraints rather than the $n_1 \cdot n_2$ linear constraints that would result from a pointwise discretization of the dual linear program constraints. Moreover, the algorithm concentrates computational effort on regions of $X \times Y$ where the constraints are binding; that is, near the zero set

$$
S = \left\{\left(x,y\right) | q\left(x,a\right) + r\left(y,b\right) = s\left(x,y\right)\right\}.
$$

[13] We also used spline basis functions. However, in this example, the Fourier basis functions produced more stable and more accurate approximations for a given number of terms in the approximation.

Finally, the algorithm produces an estimate of the support of the measure γ that maximizes the primal problem. For each x, this set valued map is

$$y = m(x) = \arg\max_{\bar{y} \in Y} s(x, \bar{y}) - r(\bar{y}, b).$$

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