Geometric Theory of Parshin Residues.

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
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Abstract

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2010

In the early 70's Parshin introduced his notion of the multidimensional residues of meromorphic top-forms on algebraic varieties. Parshin's theory is a generalization of the classical one-dimensional residue theory. The main difference between the Parshin's definition and the one-dimensional case is that in higher dimensions one computes the residue not at a point but at a complete flag of irreducible subvarieties $X = X_n \supset \cdots \supset X_0$, $\dim X_k = k$. Parshin, Beilinson, and Lomadze also proved the Reciprocity Law for residues: if one fixes all elements of the flag, except for X_k, where $0 < k < n$, and consider all possible choices of X_k, then only finitely many of these choices give non-zero residues, and the sum of these residues is zero.

Parshin's constructions are completely algebraic. In fact, they work in very general settings, not only over complex numbers. However, in the complex case one would expect a more geometric variant of the theory.

In my thesis I study Parshin residues from the geometric point of view. In particular, the residue is expressed in terms of the integral over a smooth cycle. Parshin-Lomadze Reciprocity Law for residues in the complex case is proved via a homological relation on these cycles.

The thesis consists of two parts. In the first part the theory of Leray coboundary operators for stratified spaces is developed. These operators are used to construct the cycle and prove the homological relation. In the second part resolution of singularities techniques are applied to study the local geometry near a complete flag of subvarieties.

Acknowledgements

We would like to thank Askold Khovanskii for raising the question and helpful discussions. We also thank Pierre Milman and Edward Bierstone for useful comments on the resolutions of singularities.

Contents

Introduction.

Overview.

Let X be a compact complex algebraic curve and ω be a rational 1-form on X. Let $x \in X$ be a point in X, such that X is locally irreducible at x. In an open neighborhood of x one can write

$$\omega = f(t)dt,\ f(t) = \sum_{i>N} \lambda_i t^i,$$

where t is a local normalizing parameter at x and N is an integer. The coefficient λ_{-1} in the above series does not depend on the choice of parameter t and is called the residue of ω at x. We denote it $res_x\omega$.

Note, that this definition is purely algebraic and works over any algebraically closed field.

Let now X be locally reducible at x. Let $X^1, X^2, \ldots, X^k$ be the local irreducible components of X at x. Following the above procedure, one can define the residue of ω at x along each X^i. We denote such a residue $res_{\{x,X^i\}}\omega$. One defines the residue of ω at x by taking the sum of residues along local irreducible components:

$$res_x\omega = \sum_{i=1}^{k} res_{\{x,X^i\}}\omega.$$

The well-known residue formula says that the total sum of residues of ω is zero:

$$\sum_{x\in X} res_x\omega = 0.$$

More precisely, the theorem says that only finitely many summands in this sum are not equal zero, and the sum of non-zero residues is zero.

One can prove this theorem for curves over any algebraically closed field, not necessarily complex numbers (see, for example, [S], [T]).

However, in the complex case one can prove the residue formula in a very simple topological way:

Let $\Sigma \subset X$ be the subset consisting of singularities of X and poles of ω. Note, that Σ is finite. The proof can be done in the following steps:

1. Let X^i be a local irreducible component of X at $x \in X$. One can consider a small cycle $\gamma_{\{x,X^i\}} \subset X \backslash \Sigma$, going one time around x on X^i counterclockwise. Then
$$res_{\{x,X^i\}}\omega = \frac{1}{2\pi i} \int\limits_{\gamma_{\{x,X^i\}}} \omega.$$
We denote by γ_x the sum of cycles γ_{x,X^i} over all local irreducible components X^i of X at x.

2. If $x \notin \Sigma$, then γ_x is homologous to zero in $X \backslash \Sigma$. Indeed, γ_x is the boundary of a small neighborhood of x in $X \backslash \Sigma$.

3. The sum of cycles γ_x over all points $x \in \Sigma$ is homologous to zero in $X \backslash \Sigma$. Indeed, for every $x \in \Sigma$, γ_x is the boundary of a small neighborhood U_x of x in X. Then $X \backslash \bigcup\limits_{x \in \Sigma} U_x$ is a 2-chain in $X \backslash \Sigma$, which boundary is exactly $-(\sum\limits_{x \in \Sigma} \gamma_x)$.

Since in $X \backslash \Sigma$ the form ω is holomorphic and $d\omega = 0$, the residue formula now follows from the Stokes Theorem.

In the late 70's A. Parshin introduced his notion of multidimensional residue for a rational n-form ω on an n-dimensional algebraic variety V_n. (In [P1] Parshin mostly deals with the two-dimensional case, then A. Beilinson and V. Lomadze in [B] and [L] generalized Parshin's ideas to the multidimensional case). The main difference between the Parshin residue and the classical one-dimensional residue is that in higher dimensions one computes the residue not at a point but at a complete flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$. Similar to the one-dimensional case, the flag F could be "locally reducible". In that case, one defines the residue at every "local irreducible component" of F and then sum up over these components. We call the "local irreducible components" of the flag F *Parshin points.*

Parshin, Beilinson, and Lomadze proved the Reciprocity Law for multidimensional residues, which generalizes the classical residue formula:

Fix a partial flag of irreducible subvarieties $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, where V_k is omitted $(0 < k < n)$. Then

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} res_{V_n \supset \cdots \supset X \supset \cdots \supset X_0}(\omega) = 0,$$

where the sum is taken over all irreducible k-dimensional subvarieties X, such that $V_{k+1} \supset X \supset V_{k-1}$. Similarly to the one-dimensional case, the precise statement of the theorem is that there are only finitely many summands which are not zeros, and the sum of non-zero summands is zero.

In addition, if V_1 is proper (compact in the complex case), then one has the same relation for $k = 0$:

$$\sum_{x \in V_1} res_{V_n \supset \cdots \supset X_1 \supset \{x\}}(\omega) = 0.$$

Again, there are finitely many non-zero summands and their sum is zero.

We review the definition of the Parshin residues and the formulation of the Reciprocity Law in the section 1.2.1.

The methods used by Parshin, Beilinson and Lomadze are purely algebraic and applicable in very general situations, not only over complex numbers. However, in the complex case one would expect a more geometric variant of the theory. J.-L. Brylinski and D. A. McLaughlin give a more topological treatment of the complex case in [BrM]. Given a flag $F = \{V_n \supset \cdots \supset V_0\}$ they introduce *flag-localized* homology groups $H_*^{V_i}(V_n; F)$ and a homology class $k_F \in H_n^{V_n}(V_n; F)$, such that

$$res_F\omega = \frac{1}{(2\pi i)^n} \int\limits_{k_F} \omega$$

for any meromorphic n-form ω. The class k_F is obtained from of the fundamental class $c_{V_0} \in H_{2n}(V_n, V_n \backslash V_0)$ by applying the boundary homomorphisms in the appropriate flag-localized homology groups n times. J.-L. Brylinski and D. A. McLaughlin mention, that the class k_F could be constructed in a more geometric way, so that it is naturally represented by a union of certain real n-tori. However, they only give such a construction in the case when all elements of the flag F are smooth.

Multidimensional residues are widely used in complex geometry. However, Parshin residues are almost unknown for the complex geometers. Instead, complex geometers use Grothendieck residues. The reason is that the original Parshin's theory is completely algebraic. The topological description of the residue is missing. We think that the Parshin residue is a very natural object, the direct generalization of the classical one-dimensional residue. We hope that this thesis will open the door for the complex geometric applications of Parshin residues.

Residues, computed at flags of subvarieties, appeared in complex geometric literature. In particular, V. V. Schechtman and A. N. Varchenko ([SV]) used similar construction in the context of the arrangements of hyperplanes in a complex affine space. They showed, that the homology group $H_k(U)$ of the complement U of an arrangement is generated by cycles associated to flags $L^1 \supset L^2 \supset \cdots \supset L^k$ of edges of the arrangement (here $codim L^m = m$). Moreover, they showed, that the only relations on these generators are given by the same formula as in the Reciprocity Law.

Note, that in this case one only considers flags consisting of smooth submanifolds. One can argue that the resolution of singularities techniques should allow one to reduce the general case, where the elements of the flag might be singular, to the case of smooth flags. Indeed, it is possible. However, understanding geometric properties of such resolutions require some effort. We construct the resolution of singularities for a flag $F = \{V_n \supset \cdots \supset V_0\}$ and study its' properties in the Chapter 2.2.

The resolution of singularities techniques allow one to replace an arbitrary meromorphic form by a form, which divisor is a normal crossing divisor, and an arbitrary flag of subvarieties by a flag of smooth submanifolds. However, the relations on the residues in the resolved space, corresponding to the Reciprocity Law, are not local any more. They are rather total sums of residues along certain compact complex curves. It is still possible to use the resolution of singularities to prove the Reciprocity Law. We found this proof recently and are planning to include it in a separate paper.

Part I.

In the first part of the thesis we use the geometry of stratified spaces (see [MJ]) to generalize the geometric construction of the class k_F to the case of singular flags. This allows us to generalize the topological proof of the residue formula to the multidimensional case.

Let $\Sigma \subset V_n$ be the union of the singular locus of V_n and the poles of ω. Note, that $V_n \backslash \Sigma$ is smooth, ω is holomorphic in $V_n \backslash \Sigma$, and $d\omega = 0$ in $V_n \backslash \Sigma$ (by dimension). Our proof follows exactly the same steps as in the one-dimensional case:

1. Let $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$ be a complete flag of irreducible subvarieties of V_n. We construct a smooth cycle $\gamma_F \subset V_n \backslash \Sigma$, such that
$$res_F \omega = \frac{1}{(2\pi i)^n} \int_{\gamma_F} \omega.$$
More precisely, γ_F is the union of smooth real n-dimensional tori, one for each "local irreducible component" of the flag F. (Theorem 1.2.2.)

2. Consider any Whitney Stratification $\mathbf{S}$ of V_n, such that Σ is a union of strata. If at least one element of the flag F is not the closure of a stratum from $\mathbf{S}$, then γ_F

is homologous to zero in $V_n \backslash \Sigma$. (Theorem 1.2.3.)

3. Fix a partial flag of irreducible subvarieties $F^p = \{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, $n > k > 0$. Suppose that all elements of F^p are closures of strata from $\mathbf{S}$. Let $S_1, \ldots, S_k$ be all k-dimensional strata from $\mathbf{S}$, such that $V_{k+1} \supset \overline{S_i} \supset V_{k-1}$. Let $F_1, \ldots, F_k$ be complete flags of irreducible subvarieties in V_n, such that the k-dimensional element of F_i is $\overline{S_i}$, and all other elements for all F_i coincide with the elements of the partial flag F^p. Then the sum of cycles $\sum_{i=1}^{k} \gamma_{F_i}$ is homologous to zero in $V_n \backslash \Sigma$. In addition, if V_1 is compact, then the same relation holds for $k = 0$. (Theorems 1.1.5 and 1.2.2.)

Similar to the one-dimensional case, the Reciprocity Law now follows from the Stokes Theorem.

It follows from the step 2, that for a given n-form ω there could be only finitely many non-trivial Parshin residues.

Theorem 1.1.5, which is the main ingredient in our proof of the Reciprocity Law, is applicable to any abstract stratified space, not only complex variety. In a sense, it generalizes the Reciprocity Law to abstract stratified spaces.

Part II.

In the second part of the thesis we use the resolution of singularities techniques to study the analytic properties of multidimensional Laurent expansions of meromorphic functions near an "irreducible component" of a flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$. The Parshin's local parameters $(u_1, \ldots, u_n)$ (involved in the original Parshin's constructions, see Definition 1.2.3) play the role of coordinates in such a neighborhood.

We define a neighborhood of an "irreducible component" of F to be a holomorphic map $\phi : U \to V_n$, where U is a polydisk in $\mathbb{C}^n$ with center at the origin, satisfying the following properties:

1. The flag of coordinate subspaces in U is mapped to F (i.e. $\phi(\{x_n = \cdots = x_{k+1} = 0\}) \subset V_k$, where $(x_1, \ldots, x_n)$ is a fixed system of coordinates in U);

2. The restriction of ϕ to the complement to the union of coordinate hyperplanes $U^n = \{(x_1, \ldots, x_n) \in U : x_1 x_2 \ldots x_n \neq 0\}$ is an isomorphism to the image;

3. The inverse map ϕ^{-1} is given (where defined) by an n-tuple of monomials in local parameters $(u_1, \ldots, u_n)$;

4. A technical condition, which distinguishes different "irreducible components" of the flag F (see Section 2.2.3 for more details).

We show that for any meromorphic function f, any "irreducible component" of F, and any Parshin's local parameters $(u_1, \ldots, u_n)$ there exists a neighborhood $\phi : U \to V_n$ of this irreducible component, such that f is holomorphic in $\phi(U^n)$. It follows, that the pull-back $f \circ \phi$ of f is holomorphic in U^n. Therefore, f expands into a Laurent series in $(u_1, \ldots, u_n)$, normally converging in $\phi(U^n)$ (Theorem 2.2.4 and Corollary).

It follows, that for a meromorphic form $\omega = f du_1 \wedge \cdots \wedge du_n$ the residue of ω at the "irreducible component" of F is equal to the coefficient of the expansion of f at $u_1^{-1} u_2^{-1} \ldots u_n^{-1}$. Alternatively, one can choose a small enough real positive ϵ, so that the equations $|u_1| = \cdots = |u_n| = \epsilon$ define a real smooth n-torus τ in $\phi(U^n)$. Then the residue is equal to $\frac{1}{(2\pi i)^n} \int\limits_{\tau} \omega$.

The second part consists of two Sections. In the Section 2.1 we introduce the standard model of a neighborhood of a Parshin point. We need to incorporate monomial changes of coordinates preserving the flag of coordinate subspaces in this model. Therefore, it is natural to consider affine toric varieties up to toric changes of coordinates, preserving the lexicographic order on monomials.

It would be enough to consider smooth affine toric varieties (isomorphic to $\mathbb{C}^n$) to prove the result discussed above. However, if one wants to capture more geometric information about the Parshin point, one needs the non-normal version of the theory.

In the Section 2.2 we use the resolution of singularities techniques to replace an arbitrary flag $F = \{V_n \supset \cdots \supset V_0\}$ of subvarieties by a flag $\overline{F} = \{\overline{V}_n \supset \cdots \supset \overline{V}_0\}$ of smooth submanifolds, and an arbitrary meromorphic n-form by a form which divisor is a normal crossing divisor. It turns out that the birational types of the elements of the resolved flag doesn't depend on the resolution (Lemma 2.2.6). In particular, the points of $\overline{V}_0$ are in one-to-one correspondence with the "irreducible components" of the flag F. We use these results to prove the existence of a neighborhood of a Parshin point.

We further use the systems of non-normal toric varieties, introduced in the Section 2.1, to obtain a ramified version of neighborhoods of Parshin points, which we call Toric Neighborhoods (see Theorem 2.2.5). They allow one to capture more geometric information about the Parshin point.

Chapter 1

Leray Coboundary Operators for Stratified Spaces and the Reciprocity Law for Residues.

1.1 Leray coboundary operators for stratified spaces.

1.1.1 Whitney Stratifications and Mather's Abstract Stratified Spaces Reviewed.

Definition 1.1.1. Let M be a smooth manifold. Let V be a locally closed subset of M. By a *Whitney stratification* $\mathbf{S}$ of V, we mean a subdivision of V into smooth strata, such that:

1. It is *locally finite* - each point of V has an open neighborhood which intersects only finitely many strata.

2. *Condition of the frontier* - for each stratum $X \in \mathbf{S}$ its boundary $(\overline{X} \setminus X) \cap V$ is a union of strata.

3. Each pair (X, Y) of strata satisfies *Whitney conditions* **a** and **b**:

   a: For any $x \in X$ and any sequence $\{y_n\} \in Y$, such that $y_n \to x$, if the sequence of tangent planes $T_{y_n}Y$ converges to some plane $\tau \subset T_xM$ (in the appropriate Grassmanian bundle over M) then $T_xX \subset \tau$.

   b: For any $x \in X$, any sequence $\{y_n\} \in Y$, and any sequence $\{x_n\} \in X$, such that $y_n \to x$ and $x_n \to x$, if the sequence of tangent planes $T_{y_n}Y$ converges to some plane $\tau \subset T_xM$, and the sequence of secants $\overline{x_n y_n}$ converges to some line l (in some smooth coordinate system in M), then $l \subset \tau$.

Remark. Actually, condition **b** implies condition **a**, so it is enough to require condition **b**.

One can prove, that if a pair of strata (X, Y) satisfies condition b and $\overline{Y} \cap X \neq \emptyset$, then $\dim X < \dim Y$.

Notation. We say, that $X < Y$ if $\overline{Y} \cap X \neq \emptyset$. One can see that this defines a partial order on the set of strata $\mathbf{S}$.

Example. Consider the surface in $\mathbb{C}^3$ given by the equation $y^2 + x^3 - z^2x^2 = 0$. The singular locus of the surface coincide with the z-axis. Thus, the z-axis and its complement gives a subdivision of the surface in two smooth pieces. Easy to prove that this pair satisfy the condition **a,** but doesn't satisfy condition **b** at the origin. Note, that the small neighborhood of the origin looks very different from the neighborhood of any other point of the z-axis.

It is easy to improve the subdivision in such a way that it satisfies condition **b**: one only needs to consider the origin as a separate stratum.

Whitney showed that if conditions **a** and **b** are satisfied for the pair (X, Y), then Y "behaves regularly" along X.

Theorem 1.1.1. *Let V be a complex analytic subset in a smooth manifold M. Let Σ be a locally finite family of complex analytic subsets in V. Then there exists a Whitney* *stratification of the set V such that each element of Σ is a union of strata and all strata are analytic.*

Detailed review of the theory of Whitney stratifications can be found in [GM].

Definition 1.1.2. An abstract stratified set is a triple $\{V, \mathbf{S}, \mathbf{J}\}$ satisfying the following axioms:

(A1) V is a Hausdorff, locally compact topological space with a countable basis for its topology.

(A2) $\mathbf{S}$ is a family of locally closed connected subsets of V, such that V is the disjoint union of the members of $\mathbf{S}$.

Members of $\mathbf{S}$ are called strata of V.

(A3) Each stratum of V is a topological manifold (in the induced topology), provided with a smoothness structure.

(A4) The family $\mathbf{S}$ is locally finite.

(A5) The family $\mathbf{S}$ satisfies the axiom of the frontier: if $X, Y \in \mathbf{S}$ and $X \cap \overline{Y} \neq \emptyset$, then $X \subset \overline{Y}$.

If $X \subset \overline{Y}$ and $Y \neq X$, we write $X < Y$. Easy to see that this defines a partial order on $\mathbf{S}$.

(A6) $\mathbf{J}$ is a triple $\{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$, where for each $X \in \mathbf{S}$, U_X is an open neighborhood of X in V, π_X is a continuous retraction of U_X onto X, and $\rho_X : U_X \to [o, \infty)$ is a continuous function.

We call U_X the tubular neighborhood of X, π_X the projection to X and ρ_X the tubular function of X.

(A7) $X = \{v \in U_X : \rho_X(v) = 0\}$.

For any strata X and Y, let $U_{X,Y} = U_X \cap Y$, $\pi_{X,Y} = \pi_X|_{U_{X,Y}}$ and $\rho_{X,Y} = \rho_X|_{U_{X,Y}}$.

(A8) For any strata X and Y the mapping

$$(\pi_{X,Y}, \rho_{X,Y}) : U_{X,Y} \to X \times (0, \infty)$$

is a smooth submersion.

(A9) For any strata X, Y and Z we have

$$\pi_{X,Y}\pi_{Y,Z}(v) = \pi_{X,Z}(v)$$

$$\rho_{X,Y}\pi_{Y,Z}(v) = \rho_{X,Z}(v)$$

whenever both sides of these equations are defined.

Definition 1.1.3. We say that two stratified sets $\{V, \mathbf{S}, \mathbf{J}\}$ and $\{V', \mathbf{S}', \mathbf{J}'\}$ are equivalent if the following conditions hold:

(a) $V = V'$, $\mathbf{S} = \mathbf{S}'$, and for each stratum X of $\mathbf{S} = \mathbf{S}'$, the two smoothness structures on X given by the two stratifications are the same.

(b) If $\mathbf{J} = \{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$ and $\mathbf{J}' = \{\{U'_X\}, \{\pi'_X\}, \{\rho'_X\}\}$, then for each stratum X, there exist a neighborhood U''_X of X in $U_X \cap U'_X$ such that $\rho_X|_{U''_X} = \rho'_X|_{U''_X}$ and $\pi_X|_{U''_X} = \pi'_X|_{U''_X}$.

It is easy to prove, that any stratified set is equivalent to one which satisfies the following conditions:

(A10) If X, Y are strata and $U_{X,Y} \neq \emptyset$, then $X < Y$.

(A11) If X, Y are strata and $U_X \cap U_Y \neq \emptyset$, then X and Y are comparable ($X < Y$ or $Y < X$).

Definition 1.1.4. The Triple $\mathbf{J} = \{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$ is called *control data.*

Definition 1.1.5. Let V be a subset in a smooth manifold M with a fixed Whitney stratification $\mathbf{S}$. A map $f : M \to P$, where P is a manifold, is called submersion on V if its restriction to each stratum $X \in \mathbf{S}$ is a submersion.

Definition 1.1.6. Let $\{V, \mathbf{S}, \mathbf{J}\}$ be a stratified space. A map $f : V \to P$, where P is a smooth manifold, is called *controlled submersion* if its restriction to each stratum is a submersion, and $f \circ \pi_X(y) = f(y)$ for all $X \in \mathbf{S}$, $y \in U_X$.

Theorem 1.1.2. *Let V be a subset in a smooth manifold M with a fixed Whitney stratification $\mathbf{S}$. Let $f : M \to P$ be a submersion on V. Then there exist a control data $\mathbf{C}$, such that $\{V, \mathbf{S}, \mathbf{C}\}$ is an abstract stratified space and f is a controlled submersion. Tubular functions and projections are restrictions of norms and projections in tubular neighborhoods of strata in M (one has to choose appropriate isomorphisms between tubular neighborhoods and normal bundles, endowed with Euclidian structure).*

Theorem 1.1.3. *Let P be a manifold and $f : V \to P$ be a proper, controlled submersion of an abstract stratified space. Then f is a locally trivial fibration and its restrictions to strata are differentiable fibrations.*

The following Thom's First Isotopy Lemma follows from Theorems 1.1.2 and 1.1.3:

Theorem 1.1.4. *(First Isotopy Lemma) Let V be a subset in a smooth manifold M with a fixed Whitney stratification $\mathbf{S}$. Let $f : M \to P$ be a smooth map to a smooth manifold P. Let $f|_V$ be a proper submersion. Then $f|_V$ is a locally trivial fibration.*

Control data were introduced by J.Mather in [MJ].

After shrinking the tubular neighborhoods U_X and rescaling the tubular functions ρ_X if necessary, one can assume the following additional conditions:

(A12) For any $X \in \mathbf{S}$ the map $(\pi_X, \rho_X) : U_X \backslash X \to X \times \mathbb{R}_+$ is a proper controlled submersion. Moreover, $\pi_X|_{U_X^{\leq 1}} : U_X^{\leq 1} \to X$ is proper (here $U_X^{\leq 1} = \{\rho_X \leq 1\}$).

(A13) For any $X < Y$ one has $\pi_Y(U_X^{\leq 1} \cap U_Y^{\leq 1}) \subset U_Y$.

Conditions (A12)-(A13) has some corollaries, which will be important to us.

Let $N_X := \{y \in U_X : \rho_X(y) = 1\}$.

Corollary 1. *N_X has natural structure of an abstract stratified space. The restriction $\pi_X|_{N_X} : N_X \to X$ is a proper controlled submersion, and, therefore, is a locally trivial fibration. Moreover, the fiber of this fibration is compact.*

Corollary 2. *There exist a homemorphism $\varphi_X : U_X \backslash X \to N_X \times \mathbb{R}_+$, such that:*

1. *φ_X preserves the stratifications and the restriction of φ_X to any stratum is a diffeomorphism to a stratum;*

2. *φ_X respects the projection π_X and the tubular function ρ_X in the following sense:*

$$\rho_X = \pi_2 \circ \varphi_X$$

$$\pi_X = (\pi_X|_{N_X}) \circ \pi_1 \circ \varphi_X$$

where π_1 and π_2 are the projections to the first and the second factors in $N_X \times \mathbb{R}_+$.

1.1.2 Leray coboundary operators and relations.

Let $f : M \to N$ be a smooth fibration with compact oriented k-dimensional fiber F. Then one can define the Gysin homomorphism on homology $f^* : H_*(N) \to H_{*+k}(M)$. Basically, one just set $f^*(a) = [f^{-1}(A)]$, where A is a representative of the homology class $a \in H_*(N)$.

Remark. We use the following convention about the orientations. Let $y \in M$ and $x = f(y) \in N$. Let $A \subset N$ be a smooth representative of a homology class $a \in H_*(N)$ and $y \in A$. Let the differential form ω_A on N be such that its restriction to A defines the orientation of A at x and the differential form ω_F on M be such that its restriction to the fiber F_x defines the orientation of F_x at y. Then the orientation of the preimage $f^{-1}(A) \subset M$ at the point y is given by the restriction of the form $f^*(\omega_A) \wedge \omega_F$.

Let now M be an oriented manifold with boundary, and $f : M \to N$ be a proper map to an oriented manifold N, such that its restriction both to the boundary $\partial M \subset M$ and the interior $\breve{M} \subset M$ are submersions. Then, by the Ehresmann Lemma for manifolds with boundary, f is a locally trivial fibration and its restrictions to ∂M and $\breve{M}$ are smooth fibrations.

Let $\phi := (f|_{\partial M})^* : H_*(N) \to H_{*+\dim M - \dim N - 1}(\partial M)$ be the Gysin homomorphism.

Lemma 1.1.1. *$i_* \circ \phi = 0$, where $i : \partial M \hookrightarrow M$ is the embedding.*

Proof. One can generalize the Gysin homomorphism to the described above case, when the fiber of f is a manifold with boundary. The only difference is that now the homomorphism lands in the relative homology group: $f^* : H_*(N) \to H_{*+k}(M, \partial M)$, where $k = \dim F = \dim M - \dim N$. Then one immediately see that $\phi = \partial \circ f^*$, where $\partial : H_*(M, \partial M) \to H_{*-1}(\partial M)$ is the boundary homomorphism from the long exact sequence of the pair $(M, \partial M)$. However, by the long exact sequence, $i_* \circ \partial = 0$. □

We apply the above constructions to the stratified spaces.

Let all the strata of a stratified space V be oriented.

Let $X \in \mathbf{S}$ be a stratum. According to the condition (A12), the restriction of the retraction $\pi_X : U_X \to X$ to $N_X = \{y \in U_X | \rho_X(y) = 1\}$ is a locally trivial fibration. Moreover, for any stratum Y such that $X < Y$ the restriction to $N_{X,Y} = N_X \cap Y = \{y \in U_{X,Y} | \rho_X(y) = 1\}$ is a smooth fibration.

Definition 1.1.7. Let $X < Y$ be two strata. We say that $X < Y$ are *consecutive strata* if there is no such Z that $X < Z < Y$.

Lemma 1.1.2. *Let $X < Y$ be consecutive strata. Then the fiber of $\pi_X|_{N_{X,Y}} : N_{X,Y} \to X$ is compact.*

Proof. Since $X < Y$ are consecutive strata, it follows that $N_{X,Y} = Y \cap N_X$ is a closed stratum of N_X (indeed, otherwise the closure of $N_{X,Y}$ in N_X would contain a smaller stratum). The fiber of the restriction of π_X to $N_{X,Y}$ is the intersection of the fiber of the restriction of π_X to N_X and $N_{X,Y}$. Therefore, it is compact as a closed subset of a compact set. □

Note, that $N_{X,Y}$ is orientable. Indeed, it is the level set of a smooth function $\rho_{X,Y}$ in $U_{X,Y} \subset Y$. Let us fix the orientation of $N_{X,Y}$ given as follows: we say that the restriction a differential $(\dim Y - 1)$-form $\omega_{N_{X,Y}}$ on Y defines the positive orientation of $N_{X,Y}$ if the form $d\rho_{X,Y} \wedge \omega_{N_{X,Y}}$ defines the positive orientation of Y.

Let $\dim X = n$ and $\dim Y = k$.

Definition 1.1.8. The *Leray coboundary operator* $\phi_{X,Y} : H_*(X) \to H_{*+k-n-1}(Y)$ is given by the composition $\phi_{X,Y} = i_* \circ \phi$, where $i : N_{X,Y} \hookrightarrow Y$ is the embedding and $\phi : H_*(X) \to H_{*+k-n-1}(N_{X,Y})$ is the Gysin homomorphism.

Theorem 1.1.5. *Let $X < Y$ be two strata. Let $Z_1, \dots, Z_n$ be all strata such that $X < Z_i < Y$. Suppose that $Z_1, \dots, Z_n$ are incomparable. Then*

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \cdots + \phi_{Z_k,Y} \circ \phi_{X,Z_k} = 0.$$

Proof. We want to apply the Lemma 1.1.1. Consider $D_i := N_{X,Y} \cap N_{Z_i,Y} = \{y \in Y | \rho_{Z_i}(y) = \rho_X(y) = 1\}$. Note, that $D_i = (\pi_{Z_i}|_{N_{Z_i,Y}})^{-1}(N_{X,Z_i})$. Therefore, $\pi_{Z_i}|_{D_i}$ is a smooth fibration over N_{X,Z_i}. Denote $p_i := \pi_X|_{N_{X,Z_i}} \circ \pi_{Z_i}|_{D_i} : D_i \to X$. According to the construction of the Leray coboundary operators, we have

$$\phi_{Z_i,Y} \circ \phi_{X,Z_i} = i_* \circ \phi_i,$$

where $i : D_i \hookrightarrow Y$ is the embedding and $\phi_i : H_*(X) \to H_{*+\dim Y - \dim X - 2}$ is the Gysin homomorphism of $p_i : D_i \to X$. Here we fix the orientation of D_i given in the following way: we say that the restriction of a differential $(\dim Y - 2)$-form ω_{D_i} on Y defines the positive orientation of D_i if the form $d\rho_{Z_i,Y} \wedge d\rho_{X,Y} \wedge \omega_{D_i}$ defines the positive orientation of Y.

Consider now $N_{X,Y} = \{y \in Y | \rho_X(y) = 1\}$. The restriction $\pi_X|_{N_{X,Y}}$ is a smooth fibration. However, the fibers of this fibration are not compact. On the other side, if we consider the restriction of π_X to the union $N_{X,Y \cup Z_1 \cup \cdots \cup Z_k} := N_{X,Y} \cup N_{X,Z_1} \cup \cdots \cup N_{X,Z_k} = N_X \cap (Y \cup Z_1 \cup \cdots \cup Z_k)$, then the fibers are compact.

$D_i \subset N_{X,Y}$ can be thought of as the boundary of the neighborhood $U_i = \{y \in N_{X,Y} \cap U_{Z_i} | \rho_{Z_i}(y) < 1\}$ of N_{X,Z_i} in $N_{X,Y \cup Z_1 \cup \cdots \cup Z_k}$. Denote $M = N_{X,Y} \backslash (U_1 \cup \cdots \cup U_k)$. By Ehresmann Lemma for manifolds with boundary, the restriction $\pi_X|_M : M \to X$ is a locally trivial fibration. Indeed, $\pi_X|_M$ is proper, because M is a closed subset of $N_{X,Y \cup Z_1 \cup \cdots \cup Z_k}$ and $\pi_X|_{N_{X,Y \cup Z_1 \cup \cdots \cup Z_k}}$ is a fibration with compact fibers; the restrictions of π_X to the interior of M and the boundary $\partial M = D_1 \cup \cdots \cup D_k$ are submersions.

To conclude the proof by Lemma 1.1.1, one needs to check, that the orientation of D_i as a piece of the boundary of M coincide (or is the opposite) with the orientation of D_i used in the first part of the proof. Indeed, we fixed the orientation of D_i in such a way, that if $\omega_{D_i}|_{D_i}$ gives the orientation of D_i then $d\rho_{Z_i,Y} \wedge d\rho_{X,Y} \wedge \omega_{D_i}$ gives the orientation of Y. Let $\omega_{N_{X,Y}} := -d\rho_{Z_i,Y} \wedge \omega_{D_i}$. According to our convention about the orientation of $N_{X,Y}$, $\omega_{N_{X,Y}}$ gives the positive orientation of $N_{X,Y}$. Therefore, the orientation of D_i as a piece of the boundary of M is given by $-\omega_{D_i}$. □

1.1.3 Dual Homomorphism and Spectral Sequence.

In this chapter the coefficient ring is always $\mathbb{R}$. For simplicity, we skip it in the notations.

Theorem 1.1.6. *Leray coboundary operator $\phi_{X,Y} : H_m(X) \to H_{m+k-n-1}(Y)$ (*$\dim X = n$, $\dim Y = k$*) is Poincare dual to the boundary homomorphism $\partial_{Y,X} : H^{BM}_{n-m+1}(Y) \to H^{BM}_{n-m}(X)$, where H^{BM} are Borel-Moore homologies (relative homologies of the one point compactification modulo infinity point).*

Proof. By Poincare duality, the intersection form $H_*(M) \times H^{BM}_{d-*}(M) \to \mathbb{R}$ is well defined and non-degenerate (here M is a smooth oriented manifold and $\dim M = d$). Therefore, the only thing we need to check is that for any classes $a \in H_n(X)$ and $b \in H^{BM}_{m-n+1}(Y)$,

$$< \partial_{Y,X} b, a >=< b, \phi_{X,Y}(a) > .$$

Note, that

$$H^{BM}_{n-m+1}(Y) = H^{BM}_{n-m+1}(Y \cup X, X)$$

(that's how one constructs the homomorphism $\partial_{Y,X}$).

Let $i : U_{X,Y} \hookrightarrow Y$ be the embedding. According to the definition of the Leray coboundary operator, $\phi_{X,Y}$ can be factored: $\phi_{X,Y} = i_* \circ \phi_{X,U_{X,Y}}$, where $\phi_{X,U_{X,Y}} : H_m(X) \to H_{m+k-n-1}(U_{X,Y})$ is the Leray coboundary operator for the stratified space with two strata:

X and $U_{X,Y}$. On the other side, the boundary homomorphism $\partial_{Y,X}$ also can be factored: $\partial_{Y,X} = \partial_{U_{X,Y},X} \circ i^*$ (here $i^* : H^{BM}(Y) \to H^{BM}(U_{X,Y})$ is the homomorphism induced by the inclusion i). Therefore, it is enough to assume that $Y = U_{X,Y}$.

According to the corollary 2 from the condition (A12), $U_{X,Y}$ is diffeomorphic to $N_{X,Y} \times \mathbb{R}_+$. Therefore, there is an isomorphism $\theta : H_*^{BM}(U_{X,Y}) \xrightarrow{\sim} H_{*-1}^{BM}(N_{X,Y})$, given by taking a representative, transversal to $N_{X,Y}$ and intersecting it with $N_{X,Y}$. The inverse isomorphism θ^{-1} is given by multiplying a representative by $\mathbb{R}_+$ in the product structure given by the corollary 2.

Remark. One should be careful with the orientations. We want the following condition to be satisfied: if $A \subset U_{X,Y}$ is a cycle, transversal to $N_{X,Y}$, $B := N_{X,Y} \cap A$, and $\omega|_B$ gives the orientation of B at some point, then $d\rho_{X,Y} \wedge \omega_B$ should give the positive orientation of A at this point. This means that we rather take $N_{X,Y} \cap A$ then $A \cap N_{X,Y}$. Recall, that the orientation of the transversal intersection of two oriented manifolds $K = M \cap N$ is defined in the following way: $\omega_K|_K$ defines the positive orientation of K at some point whenever $\omega_M \wedge \omega_K \wedge \omega_N$ defines the positive orientation of the ambient space at the same point, where ω_M and ω_N are such that $(\omega_M \wedge \omega_K)|_M$ defines the positive orientation of M at this point, and $(\omega_N \wedge \omega_K)|_N$ defines the positive orientation of N at this point.

With the above orientation conventions one gets that

$$< b, \phi_{X,Y}(a) >=< \theta(b), \phi(a) >,$$

where $\phi : H_*(X) \to H_{*+k-n-1}(N_{X,Y})$ is the Gysin homomorphism and the intersection on the right is taken inside $N_{X,Y}$. Moreover,

$$\partial_{U_{X,Y},X} = (\pi_X|_{N_{X,Y}})_* \circ \theta.$$

Therefore, we only need to check, that the Gysin homomorphism is dual to the $(\pi_X|_{N_{X,Y}})_* : H_{n-m}(N_{X,Y}) \to H_{n-m}(X)$, which is obvious.

□

Corollary. Leray coboundary operator $\phi_{X,Y}$ doesn't depend on the choice of the control data at least modulo torsion.

Consider the following filtration of topological spaces: $X \subset (X \cup Z_1 \cup \cdots \cup Z_p) \subset (X \cup Z_1 \cup \cdots \cup Z_p \cup Y)$. One can consider the spectral sequence for the Borel-Moore homologies of this filtration. Note, that all three terms of the filtration are locally compact. Therefore, one can easily show that the first term of the spectral sequence is given by

$E^1_{0,i} = H^{BM}_i(X)$,

$E^1_{1,i} = H^{BM}_{i+1}(X \cup Z_1 \cup \cdots \cup Z_p, X) = H^{BM}_{i+1}(Z_1) \oplus \cdots \oplus H^{BM}_{i+1}(Z_p))$,

$E^1_{2,i} = H^{BM}_{i+2}(X \cup Z_1 \cup \cdots \cup Z_p \cup Y, X \cup Z_1 \cup \cdots \cup Z_p) = H^{BM}_{i+2}(Y)$,

$\partial^1_{1,*} = \bigoplus_j \partial_{Z_j,X}$,

$\partial^1_{2,*} = \bigoplus_j \partial_{Y,Z_j}$.

$H^{BM}_0(Y)$	$H^{BM}_1(Y)$	$H^{BM}_2(Y)$	$H^{BM}_3(Y)$	$H^{BM}_4(Y)$	...
0	$\bigoplus H^{BM}_0(Z_i)$	$\bigoplus H^{BM}_1(Z_i)$	$\bigoplus H^{BM}_2(Z_i)$	$\bigoplus H^{BM}_3(Z_i)$	...
0	0	$H^{BM}_0(X)$	$H^{BM}_1(X)$	$H^{BM}_2(X)$	...

Now, the condition that the square of the boundary operator is zero for the first term of our spectral sequence is dual to the relation given by the Theorem 1.1.5.

This provides another proof to the Theorem 1.1.5 modulo torsion.

1.2 Application to Parshin's Residues.

1.2.1 Parshin Residues and the Reciprocity Law.

In this section we review the definition of the Parshin residue and the Reciprocity Law.

Let V_n be an algebraic variety of dimension n. Let $V_n \supset \cdots \supset V_0$ be a flag of subvarieties of dimensions $\dim V_k = k$.

Consider the following diagram:

$$
\begin{array}{ccccccc}
V_n \supset & V_{n-1} & \supset \ldots \supset & V_1 \supset V_0 & & & \\
\uparrow{\scriptstyle p_n} & \uparrow{\scriptstyle p_n} & & & & & \\
\widetilde{V}_n \supset & W_{n-1} & & & & & \\
 & \uparrow{\scriptstyle p_{n-1}} & & & & & \\
 & \widetilde{W}_{n-1} \supset \ldots & & & & &
\end{array}
\tag{1.1}
$$

$$
\begin{array}{c}
\ldots \supset W_1 \\
\uparrow{\scriptstyle p_1} \\
\widetilde{W}_1 \supset W_0
\end{array}
$$

where

1. $p_n : \widetilde{V}_n \to V_n$ is the normalization;

2. $W_{n-1} \subset \widetilde{V}_n$ is the union of $(n-1)$-dimensional irreducible components of the preimage of V_{n-1};

3. for every $k = 1, 2 \ldots, n-1$

   (a) $p_k : \widetilde{W}_k \to W_k$ is the normalization;

   (b) $W_{k-1} \subset \widetilde{W}_k$ is the union of $(k-1)$-dimensional irreducible components of the preimage of V_{k-1}.

Definition 1.2.1. We call diagram 1.1 the *normalization diagram* of the flag $V_n \supset \cdots \supset V_0$.

Definition 1.2.2. The flag $V_n \supset \cdots \supset V_0$ of irreducible subvarieties together with a choice of a point $a_\alpha \in W_0$ is called a *Parshin point.*

Choosing a point $a_\alpha \in W_0$ is equivalent to choosing irreducible components in every W_i, $i = n-1, \ldots, 0$. Indeed, $\widetilde{W}_i$ is normal and, therefore, locally irreducible at every point. In particular, it is locally irreducible at the image of a_α. Let $\widetilde{W}_i^\alpha$ be the irreducible component of $\widetilde{W}_i$, containing the image of a_α. Let $W_i^\alpha = p_i(\widetilde{W}_i^\alpha)$. Note, that W_i^α is an irreducible component of W_i.

In order to define the Parshin residue, one needs to define the *local parameters* at a Parshin point, which play the role of the normalizing parameter in one-dimensional case. After that, one uses these parameters to define a sequence of residual meromorphic forms $\omega_{n-1}, \dots, \omega_0$ on $W_{n-1}^\alpha, \dots, W_0^\alpha$.

The local parameters are defined as follows:

$W_{i-1}^\alpha \subset \widetilde{W}_i$ is a hypersurface in a normal variety. It follows that there exists a (meromorphic) function u_i on $\widetilde{W}_i$ which has zero of order 1 at a generic point of W_{i-1}^α. Since meromorphic functions are the same on W_i and $\widetilde{W}_i$, one can consider u_i as a function on W_i. Then one can extend (in an arbitrary way) u_i to $\widetilde{W}_{i+1}$ and so on. For simplicity, we denote all these functions by u_i. Now u_i is defined on V_n, and can be consecutively restricted to W_j for $j \geq i$.

Definition 1.2.3. Functions $(u_1, \dots, u_n)$ are called *local parameters* at the Parshin point $P = \{V_n \supset \cdots \supset V_0, a_\alpha\}$.

Let ω be a meromorphic n-form on V_n. One can show that the differentials $du_1, \dots, du_n$ are linearly independent at a generic point of V_n. Therefore, one can write

$$\omega = f du_1 \wedge \cdots \wedge du_n,$$

where f is a meromorphic function on V_n.

Now we define the forms ω_i:

Take a generic point $p \in W_{n-1}^\alpha$. Both $\widetilde{V}_n$ and W_{n-1} are smooth at p. Moreover, parameters $u_1, \dots, u_n$ provide an isomorphism of a neighborhood of p to an open subset in $\mathbb{C}^n$, and W_{n-1}^α is given by the equation $u_n = 0$ in this neighborhood. Restrict the function f to the transversal section to W_{n-1} at p, given by fixing the parameters $u_1, \dots, u_{n-1}$. The restriction can be expanded into a Laurent series in u_n. It is easy to see that the coefficients of this expansion depend analytically on p. Moreover, one can see that the coefficients are meromorphic functions on W^α_{n-1}. Let f_{-1} be the coefficient at u_n^{-1} in this expansion. Then $\omega_{n-1} = f_{-1} du_1 \wedge \cdots \wedge du_{n-1}$ is a meromorphic $(n-1)$-form on W^α_{n-1}.

Repeating this procedure one more time one gets a meromorphic $(n-2)$-form on W^α_{n-2}. Finally, after n steps, one gets a function ω_0 on the one-point set $W^\alpha_0 = \{a_\alpha\}$.

Definition 1.2.4. The *residue of* ω *at the Parshin point* $P = \{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ is $res_P(\omega) = \omega_0(a_\alpha)$.

Parshin proves that the residue is independent on the choice of local parameters.

Definition 1.2.5. The sum of residues over all $a \in W_0$ is called the *residue at the flag* $F = \{V_n \supset \cdots \supset V_0\}$ and denoted $res_F(\omega) = \sum_{a \in W_0} res_{\{F,a\}}(\omega)$.

Theorem 1.2.1. *([P1],[B],[L]) Let* ω *be a meromorphic* n*-form on* V_n*. Fix a partial flag of irreducible subvarieties* $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$*, where* V_k *is omitted* ($0 < k < n$)*. Then*

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} res_{V_n \supset \cdots \supset X \supset \cdots \supset X_0}(\omega) = 0,$$

where the sum is taken over all irreducible k*-dimensional subvarieties* X*, such that* $V_{k-1} \supset X \supset V_{k+1}$*. (In this formula only finitely many summands are non zero.)*

In addition, if V_1 *is compact then one has the same relation for* $k = 0$*.*

1.2.2 Residues via Leray Coboundary Operators and the Reciprocity Law.

We want to apply the stratification theory to study the Parshin points and residues. Therefore, we need to stratify all the spaces in the normalization diagram in such a way that the stratifications respect the normalization maps $p_1, \ldots, p_2$. The following Lemma easily follows from the well known results on existence of Whitney stratifications (see section 1.7 in [GM], for example):

Notation. Let X be an irreducible (complex analytic) variety considered with a fixed Whitney stratification. Then by $\breve{X}$ we denote the stratum of maximal dimension. If X is reducible, then by $\breve{X}$ we denote the union of strata of maximal dimension.

Lemma 1.2.1. *Fix a Parshin point $P = \{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ and local parameters $u_1, \ldots, u_n$. There exist Whitney stratifications $\mathbf{S}, \mathbf{S}_{\widetilde{V}}, \mathbf{S}_{\widetilde{W}_{n-1}}, \ldots, \mathbf{S}_{\widetilde{W}_1}$ of $V_n, \widetilde{V}_n, \widetilde{W}_{n-1}, \ldots, \widetilde{W}_1$ correspondingly, such that:*

1. *$V_{n-1}, \ldots, V_0$ are unions of strata of $\mathbf{S}$;*
2. *$W_{n-1}, W_{n-2}, \ldots, W_0$ are unions of strata of the corresponding stratifications;*
3. *for all $i = 1, \ldots, n$, the local parameter u_i is regular and non-vanishing on $\breve{V}_n, \breve{\widetilde{V}}_n, \breve{\widetilde{W}}_{n-1}, \ldots, \breve{\widetilde{W}}$*
4. *for all $i = 1, \ldots, n$, the restriction of the normalization map p_i to any stratum in the source is a covering over a stratum in the image.*

There is an important corollary about stratifications $\mathbf{S}_{\widetilde{\mathbf{W}}_\mathbf{i}}$:

Lemma 1.2.2. *The stratum $\breve{W}_{i-1} \in \mathbf{S}_{\widetilde{\mathbf{W}}_\mathbf{i}}$ consist of regular points of $\widetilde{W}_i$.*

Proof. Let $x \in \breve{W}_{i-1}$ be a point, such that $\widetilde{W}_i$ is singular at x. Note, that by dimension reasons and condition of the frontier, there are only two strata intersecting a small neighborhood of x: $\breve{W}_{i-1}$ and $\breve{\widetilde{W}}_i$. Note, that function u_i is regular in $\breve{\widetilde{W}}_i$ and at a generic point of $\breve{W}_{i-1}$. Therefore, by extension theorem for normal varieties, u_i is regular at x.

Note also, that u_i is non-vanishing in $\breve{\widetilde{W}}_i$ and has zero of order 1 at a generic point of W_{i-1}. Therefore, $\{u_i = 0\}$ coincide with W_{i-1} near x. Moreover, easy to see, that the germ of u_i at x generates the ideal of the germ of W_{i-1} at x. Indeed, if g is a function, regular at x and vanishing on W_{i-1}, then $\frac{g}{u_i}$ is regular at x by the extension theorem for normal varieties.

Now, let $f_1, \ldots, f_{i-1}$ be any coordinate system on W_{i-1} at x. Easy to see, that the functions $u_i, f_1, \ldots, f_{i-1}$ generate the maximal ideal in the local ring of $\{x\} \subset \widetilde{W}_i$. Therefore, x is a smooth point of $\widetilde{W}_i$. $\square$

Our goal is to show, that

$$res_F(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\Delta_F} \omega,$$

where $F := \{V_n \supset \cdots \supset V_0\}$ and $\Delta_F = \phi_{\check{V}_{n-1},\check{V}_n} \circ \cdots \circ \phi_{\check{V}_0,\check{V}_1}([V_0]) \in H_n(\check{V}_n)$.

Moreover, we'll show that Δ_F naturally splits into the sum $\Delta_F = \sum\limits_{a_i \in W_0} \Delta_{\{F,a_i\}}$, such that

$$res_{\{F,a_i\}}(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\Delta_{\{F,a_i\}}} \omega.$$

Note, that according to the construction of the Leray coboundary operator, Δ_F is represented by a smooth compact real n-dimensional submanifold $\tau_F \subset \check{V}_n$. Moreover, τ_F is obtained from a point by the following procedure: there are n steps, and on each step we take the total space of an oriented fibration with 1-dimensional compact fiber over the result of the previous step. Therefore, τ_F is a union of n-dimensional tori. We'll show, that the connected components of τ_F are in natural one-to-one correspondence with the points of W_0 and the connected component τ_{F,a_i} corresponding to $a_i \in W_0$ represent Δ_{F,a_i}.

Fix control data on the stratification $\mathbf{S}$ of V_n. Let us use these control data to construct the representative $\tau_F \subset \check{V}_n$ of Δ_F. Let us also denote by $\tau_k \subset \check{V}_k$ the representative of $\Delta_k = \phi_{\check{V}_{k-1},\check{V}_k} \circ \cdots \circ \phi_{\check{V}_0,\check{V}_1}([V_0]) \in H_k(\check{V}_k)$, constructed in the same way.

Let us introduce the following notations:

1. $\widehat{U}_0 := \check{V}_0$;

2. $\widehat{U}_k := \pi^{-1}_{\check{V}_{k-1},\check{V}_k}(\widehat{U}_{k-1})$, for $k = 1, \ldots, n$.

Note, that for $k > 0$, $\widehat{U}_k$ is the preimage of $\widehat{U}_{k-1} \times \mathbb{R}_+$ under the map $(\pi_{\check{V}_{k-1},\check{V}_k}, \rho_{\check{V}_{k-1},\check{V}_k}) : U_{\check{V}_{k-1},\check{V}_k} \to \check{V}_{k-1} \times \mathbb{R}_+$. Since $\check{V}_{k-1}$ and $\check{V}_k$ are consecutive strata, it follows from the property (A12), that the restriction $(\pi_{\check{V}_{k-1},\check{V}_k}, \rho_{\check{V}_{k-1},\check{V}_k})|_{\widehat{U}_k}$ is a proper submersion to the $\widehat{U}_{k-1} \times \mathbb{R}_+$.

After composing these maps n times, one gets the following Lemma:

Lemma 1.2.3. *$(\rho_{\breve{V}_0}, \dots, \rho_{\breve{V}_{k-1}}) : \widehat{U}_k \to (\mathbb{R}_+)^k$ is a proper submersion. Therefore, $\widehat{U}_k$ is diffeomorphic to $\tau_k \times (\mathbb{R}^k_+)$.*

Consider the preimages $U_k = (p_n \circ \dots \circ p_k)^{-1}(\widehat{U}_k) \subset \widetilde{W}_k$. According to the Lemma 1.2.1, $U_k \subset \breve{\widetilde{W}}_k$ and $(p_n \circ \dots \circ p_k)|_{U_k} : U_k \to \widehat{U}_k$ is a covering.

Denote $\overline{U}_k := U_k \cup p_{k-1}(U_{k-1})$ for $k = n, n-1, \dots, 1$.

Lemma 1.2.4. *$\overline{U}_k \subset \widetilde{W}_k$ is an open subset consisting of regular points of $\widetilde{W}_k$.*

Proof. $\widehat{U}_k \cup \widehat{U}_{k-1}$ is an open subset in $\breve{V}_k \cup \breve{V}_{k-1}$, and $\overline{U}_k$ is its preimage under $p_n \circ \dots \circ p_k)|_{\breve{\widetilde{W}}_k \cup \breve{W}_{k-1}}$. Also, by Lemma 1.2.2, $\breve{W}_{k-1}$ consist of regular points of $\widetilde{W}_k$. □

We need the following Lemma about lifting the control data:

Lemma 1.2.5. *Let V and V' be two stratified spaces, consisting of two strata each: $V = X \sqcup Y$, $X < Y$, and $V' = X' \sqcup Y'$, $X' < Y'$. Let and $p : V' \to V$ be a map, such that $p|_{X'}$ is a covering over X and $p|_{Y'}$ is a covering over Y. Let $U_X \subset Y$, $\pi_X : U_X \to X$, and $\rho_X : U_X \to \mathbb{R}_{\geq 0}$ be the control data on V. Then there exist control data $U_{X'}, \pi_{X'}, \rho_{X'}$ on V', such that*

1. *$\rho_X \circ p = \rho_{X'}$;*
2. *$\pi_X \circ p = p \circ \pi_{X'}$.*

Proof. We set the tubular neighborhood $U_{X'} := p^{-1}(U_X)$. The tubular function $\rho_{X'}$ is defined by the property (1). The retraction $\rho_{X'}$ is defined uniquely by the property (2) and continuity. □

We apply the Lemma 1.2.5 to the $V = \widehat{U}_k \sqcup \widehat{U}_{k-1}$ and $V' = \overline{U}_k = p_{k-1}(U_{k-1}) \sqcup U_k$. Let $\pi_{p_{k-1}(U_{k-1})} : \overline{U}_k \to p_{k-1}(U_{k-1})$ and $\rho_{p_{k-1}(U_{k-1})} : U_k \to \mathbb{R}_{\geq 0}$ be the corresponding retraction and tubular function. We have the following corollary:

Corollary 3. *For any $k = n, n-1, \dots, 1$ the connected components of U_k are in natural one-to-one correspondence with the connected components of U_{k-1}.*

Proof. Indeed, the map from the connected components of U_k to the connected components of $p_{k-1}(U_{k-1})$ is given by the retraction $\rho_{p_{k-1}(U_{k-1})}$, and the existence of the inverse to this map follows from the fact that $p_{k-1}(U_{k-1}) \subset \overline{U}_k$ is a complex hypersurface in the manifold $\overline{U}_k$. Finally, $p_{k-1}|_{U_{k-1}}$ is an isomorphism to the image. □

Pick a point $a_\alpha \in W_0$. Denote $U_1^\alpha, \ldots, U_n^\alpha$ the corresponding connected components of $U_1, \ldots, U_n$ correspondingly. Denote also $\overline{U}_k^\alpha := U_k^\alpha \cup p_{k-1}(U_{k-1}^\alpha)$ the corresponding connected components of $\overline{U}_k$.

Denote $\widetilde{\tau}_k = (p_n \circ \cdots \circ p_k)^{-1}(\tau_k)$. Note, that $\widetilde{\tau}_k \subset U_k$ is a union of connected components, one in each U_k^α. We denote those connected components $\widetilde{\tau}_k^\alpha \subset U_k^\alpha$.

Lemma 1.2.6. $\phi_{p_{k-1}(U_{k-1}),U_k} \circ (p_{k-1}|_{U_{k-1}})_*([\widetilde{\tau}_{k-1}^\alpha]) = [\widetilde{\tau}_k^\alpha]$.

Proof. Obvious from the construction. □

Now we use the local parameters $(u_1, \ldots, u_n)$ to construct cycles $\gamma_k^\alpha \subset U_k^\alpha$ such that, on the one side, it is obvious that

$$res_{\{F,a_\alpha\}}(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma_n^\alpha} \omega,$$

and, on the other, γ_k^α is homologically equivalent to $\widetilde{\tau}_k^\alpha$ in U_k^α.

Function u_k is regular and non-vanishing in $U_k^\alpha \subset \widetilde{\breve{W}}_k$ and has zero of order one at a generic point of $p_{k-1}(U_{k-1}) \subset \overline{U}_k$. It follows immediately, that u_k is regular on $\overline{U}_k$ and the equation $u_k = 0$ defines $p_{k-1}(U_{k-1})$ in $\overline{U}_k$.

The following Lemma easily follows from the above observation:

Lemma 1.2.7. *There exist smooth positive real functions* $\epsilon_1, \ldots, \epsilon_n, \;\; \epsilon_k : \mathbb{C}^{k-1} \to \mathbb{R}_+$, *and open subsets* $B_k \subset U_k, \;\; k = 1, \ldots, n$, *such that*

$$(u_1, \ldots, u_k) : B_k \to A_k := \{(z_1, \ldots, z_k) : |z_i| < \epsilon_i(z_1, \ldots, z_{i-1}),\; i = 1, \ldots, k\} \subset \mathbb{C}^k$$

is a biholomorphism. (Note, that ϵ_1 *is a constant.)*

Let $\delta_1, \ldots, \delta_n \in \mathbb{R}_+$ be small enough, so that $\{(z_1, \ldots, z_n) || z_i| = \delta_i,\ i = 1, \ldots, n\} \subset A_n$.

Definition 1.2.6. Denote $\gamma_k^\alpha = \{x \in B_k : |u_i(x)| = \delta_i,\ i = 1, \ldots, k\}$, and $\gamma_0^\alpha = a_\alpha$.

It follows immediately from the definition of the Parshin residue, that

$$res_{\{F, a_\alpha\}}(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma_n^\alpha} \omega.$$

Lemma 1.2.8. *γ_k^α and $\widetilde{\tau}_k^\alpha$ defines the same homology class in $H_k(U_k)$.*

Proof. We prove this Lemma by induction. For $k = 0$ one has $\gamma_0^\alpha = \widetilde{\tau}_0^\alpha = a_\alpha$.

For the induction step, one uses the Lemma 1.2.6 and the similar observation for cycles γ_k^α. □

So, we proved the following Theorem:

Theorem 1.2.2.

$$res_F(\omega) = \frac{1}{(2\pi i)^n} \int\limits_{\Delta_F} \omega,$$

where $F := \{V_n \supset \cdots \supset V_0\}$ *and* $\Delta_F = \phi_{\check{V}_{n-1}, \check{V}_n} \circ \cdots \circ \phi_{\check{V}_0, \check{V}_1}([V_0]) \in H_n(\check{V}_n)$.

In order to get the Parshin's Reciprocity Law from the Theorem 1.1.5 and the above consideration, one needs a fixed stratification of V, such that all non-zero residues of the given form ω are in the flags consisting of closures of strata of the stratification. It turns out, that any Whitney stratification, such that ω is regular on the top dimensional stratum is good enough. More precisely, we have the following Theorem:

Theorem 1.2.3. *Let V be an n-dimensional variety and ω a meromorphic n-form on V. Let $\mathbf{S}_\omega$ be a Whitney stratification V, such that ω is regular on V^0.*

Let $F = \{V_n \supset \cdots \supset V_0\}$ be a flag of irreducible subvarieties of V, $\dim V_i = i$. Suppose that at least one of V_i's is not the closure of a stratum of $\mathbf{S}_\omega$. Then $res_{F, a_\alpha}\omega = 0$ for all $a_\alpha \in W_0$.

Proof. Consider the normalization diagram for the flag F. Let $a_\alpha \in W_0$ and $(u_1, \ldots, u_n)$ be local parameters. Let $\mathbf{S}$ be a stratification of V satisfying conditions of the Lemma 1.2.1, and such that all strata of the stratification $\mathbf{S}_\omega$ are unions of strata of $\mathbf{S}$. As usual, we denote by $\breve{V}_k$ the stratum of $\mathbf{S}$ which is open and dense in V_k.

The proof of the theorem is based on two observations:

1. Let $X' < Y'$ be consecutive strata of the $\mathbf{S}_\omega$ and let $X < Y$ be the consecutive strata of $\mathbf{S}$, such that X is an open dense subset in X', and Y is an open dense subset in Y'. Let $i_X : X \hookrightarrow X'$ and $i_Y : Y \hookrightarrow Y'$ be the embeddings. Then it easily follows from the construction of coboundary operators $\phi_{X,Y}$ and $\phi_{X',Y'}$ and the independence of these operator from the choice of the control data, that $\phi_{X',Y'} \circ i_{X*} = i_{Y*} \circ \phi_{X,Y}$.

2. Let Y' be a stratum of $\mathbf{S}_\omega$ and let $X < Y$ be consecutive strata of $\mathbf{S}$, such that $(X \cup Y) \subset Y'$ and Y is open and dense in Y'. Then $i_* \circ \phi_{X,Y} = 0$, where $i : Y \hookrightarrow Y'$ is the embedding. Moreover, if A is a representative of a homology class in $H_*(X)$ and B is the representative of the $\phi_{X,Y}([A]) \in H_{*+\dim Y - \dim X - 1}(Y)$, then every connected component of B is homologically equivalent to 0 in Y. Indeed, one can use the control data to embed the mapping cone of $\pi_X|_B : B \to A$ into Y'.

Let k be the largest number, such that $\breve{V}_k$ is a subset of a stratum of $\mathbf{S}_\omega$ of dimension bigger then k. For $m = k+1, \ldots, n$ Let $\breve{V}'_m$ be the strata of $\mathbf{S}_\omega$, such that $\breve{V}_m \subset \breve{V}'_m$. Note, that $\dim \breve{V}'_m = m$ and $\breve{V}_m$ is open and dense in $\breve{V}'_m$. Moreover, by dimension reasons and the condition of the frontier, $\breve{V}_k \subset \breve{V}'_{k+1}$.

Let $i_m : \breve{V}_m \hookrightarrow \breve{V}'_m$ be the embedding. Then, according to the first observation, one has

$$i_{n*} \circ \phi_{\breve{V}_{n-1},\breve{V}_n} \circ \cdots \circ \phi_{\breve{V}_1 \breve{V}_0} = \phi_{\breve{V}'_{n-1},\breve{V}'_n} \circ \cdots \circ \phi_{\breve{V}'_{k+1},\breve{V}'_{k+2}} \circ i_{k+1*} \circ \phi_{\breve{V}_k,\breve{V}_{k+1}} \circ \cdots \circ \phi_{\breve{V}_1 \breve{V}_0}.$$

On the other side, according to the second observation, $i_{k+1*} \circ \phi_{\check{V}_k, \check{V}_{k+1}} = 0$. Therefore,

$$i_{n*} \circ \phi_{\check{V}_{n-1}, \check{V}_n} \circ \cdots \circ \phi_{\check{V}_1 \check{V}_0} = 0,$$

and, since ω is regular in $\check{V}_n'$, $res_F \omega = 0$. Moreover, easy to see, that every connected component of the standard representative of the $\phi_{\check{V}_{n-1}, \check{V}_n} \circ \cdots \circ \phi_{\check{V}_1 \check{V}_0}([V_0])$ is homologically equivalent to 0. Therefore, $res_{F,a} \omega = 0$ for any $a \in W_0$. □

Corollary. There is only finitely many non-zero Parshin's residues for a given meromorphic form.

Note, that Parshin Reciprocity Law follows from the Theorems 1.2.2, 1.2.3, and 1.1.5.

Chapter 2

Toric Neighborhoods of Parshin's Points.

2.1 Flags of Lattices and the Associated Systems of Toric Varieties.

2.1.1 Ordered Abelian Groups.

Definition 2.1.1. An Abelian group G subdivided into subsets $G = G_- \sqcup \{0\} \sqcup G_+$ is called an *ordered abelian group* if

(1) $a \in G_- \Rightarrow -a \in G_+$;

(2) G_+ is a semigroup.

We write $a > b$ if $a - b \in G_+$ and $a < b$ if $a - b \in G_-$. Easy to see that this gives a total ordering of G. The elements of G_+ are called positive and the elements of G_- are called negative.

Definition 2.1.2. A subgroup $H \subset G$ is called *isolated* if for any two positive elements $g_1 > g_2 > 0$ such that $g_1 \in H$ it follows that $g_2 \in H$.

It follows immediately that if $h > 0$, $g > 0$, $h \in H$, and $g \notin H$ than $g > h$.

Theorem 2.1.1. *Let $H^1 \subset G$ and $H^2 \subset G$ be two isolated subgroups. Then either $H_1 \subset H_2$ or $H_2 \subset H_1$.*

Definition 2.1.3. Let $G = H^k \supset H^{k-1} \supset \cdots \supset H^0 = \{0\}$ be the tower of all isolated subgroups of G. Then k is the *rank of* G.

Theorem 2.1.2. *Suppose that G is isomorphic to $\mathbb{Z}^n$ and has rank n. Let $G = H^n \supset H^{n-1} \supset \cdots \supset H^0 = \{0\}$ be the isolated subgroups of G. Then for all k H^k is isomorphic to $\mathbb{Z}^k$ and the order on G is isomorphic to the lexicographic order with respect to any basis $(e_n, \ldots, e_1)$ of G such that for all k $(e_k, \ldots, e_1)$ is a basis in H^k. (The lexicographic order range the elements first with respect to the coefficient at e_n, than e_{n-1}, etc.)*

Definition 2.1.4. Let G be an ordered abelian group of rank greater than 1. Let $G' \subsetneq G$ be its maximal proper isolated subgroup. Than $H_+ := G_+ \backslash G'$ is called the *upper half-space* in G.

For groups of rank 1 we set $H_+ = G_+$ and for $G = \{0\}$ we set $H_+ = \{0\}$.

2.1.2 Flags of Lattices. Injective Systems of Cones and Projective Systems of Toric Varieties.

Let L^n be an ordered abelian group of rank n isomorphic to $\mathbb{Z}^n$. Let $L^n \supset \hat{L}^{n-1} \supset \cdots \supset \hat{L}^1 \supset L^0$ be the isolated subgroups of L^n. Let $L^i \subset \hat{L}^i$ for $1 \leq i \leq n-1$ be subgroups of full rank such that $L^n \supset L^{n-1} \supset \cdots \supset L^0$.

Definition 2.1.5. $L^n \supset L^{n-1} \supset \cdots \supset L^0$ is called a *flag of lattices.*

We use the multiplicative notations for the operation in L^n.

Let H^k_+ be the upper half-space in L^k for $k = 0, \ldots, n$. We are interested in the semigroup $L = H^0_+ \cup \cdots \cup H^n_+$ (i.e. we take 0, then all the positive elements of L^1, then all the positive elements in L^2 which are not in $\hat{L}^1$, etc.). Note, that L is not finitely generated. In particular, it doesn't correspond to any algebraic variety. However, one can consider L as a union of countably many cones, which are finitely generated.

Definition 2.1.6. Let $\hat{\mathcal{C}}$ be the set of all simple cones $\hat{C}$ in L^n such that for all $k = 1, \dots n$ exactly k generators of $\hat{C}$ belong to $\hat{L}^k_+$ (i.e. one of the generators is exactly the generator of $\hat{L}^1_+$, another one belong to $\hat{L}^2_+$, etc.).

Definition 2.1.7. Let $\hat{C} \in \hat{\mathcal{C}}$ and let $(x_1, \dots, x_n)$ be the generators of $\hat{C}$ such that $x_i \in \hat{L}_k$ for $i = 1, \dots, n$. Then we call $(x_1, \dots, x_n)$ the *standard generators* of $\hat{\mathcal{C}}$.

The cones from $\hat{\mathcal{C}}$ are not subsets of L. So we need to intersect them with L.

Definition 2.1.8. Let $\mathcal{C} = \{C = \hat{C} \cap L : \hat{C} \in \hat{\mathcal{C}}\}$.

Lemma 2.1.1. *Elements of $\mathcal{C}$ are finitely generated.*

Proof. Let $C \in \mathcal{C}$ and let $(x_1, \dots, x_n)$ be the standard generators of $\hat{C}$. Let $k_1, \dots, k_n$ be the smallest positive integers such that $x_i^{k_i} \in C$. Denote $y_i := x_i^{k_i}$. Let $\hat{P} \subset \hat{C}$ be the subset in $\hat{C}$ consisting of all the monomials $x_1^{d_1} \dots x_n^{d_n}$ such that $0 \leq d_i \leq k_i$ for $i = 1, \dots, n$. Let $P = \hat{P} \cap C$. P is obviously finite. Let us prove that it generates C.

Any element $x_1^{m_1} \dots x_n^{m_n} \in \hat{C}$ can be factored in the following way: $x_1^{m_1} \dots x_n^{m_n} = y_1^{l_1} \dots y_n^{l_n} x_1^{d_1} \dots x_n^{d_n}$, where $x_1^{d_1} \dots x_n^{d_n} \in \hat{P}$, $l_i \geq 0$, and $d_i \neq 0$ if and only if $m_i \neq 0$ for $i = 1, \dots, n$. Therefore, we only need to prove that if $x_1^{m_1} \dots x_n^{m_n} \in C$ then $x_1^{d_1} \dots x_n^{d_n} \in C$ as well.

Let $x_1^{m_1} \dots x_n^{m_n} \in C$. Let j be the biggest number such that $m_j \neq 0$, i.e. $x_1^{m_1} \dots x_n^{m_n} = x_1^{m_1} \dots x_n^{m_j}$ and $m_j \neq 0$. Then $d_n = \dots = d_{j+1} = 0$ and $d_j \neq 0$ as well. Since $x_1^{m_1} \dots x_n^{m_j} \in C$, it follows that $x_1^{m_1} \dots x_n^{m_j} \in L^j$ and $x_1^{m_1} \dots x_n^{m_j} \notin \hat{L}^{j-1}$. Since $y_1, \dots, y_j \in L^j$, $x_1^{d_1} \dots x_n^{d_n} \in L^j$ as well. Also, since $d_j \neq 0$, $x_1^{d_1} \dots x_n^{d_n} \notin \hat{L}_{j-1}$. So, $x_1^{d_1} \dots x_n^{d_n} \in L$ and, therefore, $x_1^{d_1} \dots x_n^{d_n} \in C$. □

Lemma 2.1.2. *Let $K \subset L^n_+$ be a finite set of positive elements in L^n. Then there exist $\hat{C} \in \hat{\mathcal{C}}$ such that $K \subset \hat{C}$.*

Proof. It is enough to prove that if $\hat{C} \subset \hat{\mathcal{C}}$ and $a \in L_+^n$ then there exist another cone $\hat{C}' \in \hat{\mathcal{C}}$ such that $\hat{C} \subset \hat{C}'$ and $a \in \hat{C}'$. Indeed, using this fact one can start from any cone in $\hat{\mathcal{C}}$ and add elements of K one by one.

Let $(x_1, \dots, x_n)$ be the standard generators of $\hat{C}$. Let $a = x_1^{k_1} \dots x_n^{k_n}$. Let j be the biggest number such that $k_j \neq 0$. Note that $k_j > 0$, otherwise a is not positive. Take the cone $\hat{C}'$ generated by $(x_1, \dots, x_{j-1}, x'_j, x_{j+1}, \dots, x_n)$, where $x'_j = x_1^{min(k_1,0)} \dots x_{j-1}^{min(k_{j-1},0)} x_j$ (i.e. x'_j is obtained from $a = x_1^{k_1} \dots x_j^{k_j}$ by removing all the factors with positive powers and replacing $x_j^{k_j}$ by x_j).

$\hat{C}' \in \hat{\mathcal{C}}$ because $x'_j \in L_j$ and the transition matrix from $(x_1, \dots, x_n)$ to $(x_1, \dots x_{j-1}, x'_j, x_{j+1}, \dots, x_n$ is integrally invertible. Indeed, $x_j = x_1^{-min(k_1,0)} \dots x_{j-1}^{-min(k_{j-1},0)} x'_j$.

We need to check that $\hat{C} \subset \hat{C}'$ and $a \in \hat{C}'$. Indeed, all the generators of $\hat{C}$ except for x_j are also generators of $\hat{C}$ and $x_j \in \hat{C}'$ because $x_j = x_1^{-min(k_1,0)} \dots x_{j-1}^{-min(k_{j-1},0)} x'_j$ and $-min(k_i, 0) \geq 0$ for $i = 1, \dots, j-1$. Also, $a = x_1^{k_1} \dots x_n^{k_n} = x_1^{k_1} \dots x_j^{k_j} = x_1^{k_1} \dots x_{j-1}^{k_{j-1}} (x_1^{-min(k_1,0)} \dots x_{j-}$ $x_1^{k_1 - k_j min(k_1,0)} \dots x_{j-1}^{k_{j-1} - k_j min(k_{j-1},0)} (x')_j^{k_j}$. Since $k_j > 0$, it follows, that $k_i - k_j min(k_i, 0) \geq 0$. Therefore, $a \in \hat{C}'$. □

Corollary 4. *Let $G \subset L_+^n$ be any finitely generated semigroup in L_+^n. Then there exist $\hat{C} \in \hat{\mathcal{C}}$ such that $G \subset \hat{C}$.*

Corollary 5. *Let $G \subset L$ be any finitely generated semigroup in L. Then there exist $C \in \mathcal{C}$ such that $G \subset C$.*

Corollary 6. $L = \bigcup_{C \in \mathcal{C}} C$.

Corollary 7. *Both $\hat{\mathcal{C}}$ and $\mathcal{C}$ are injective systems under inclusion.*

Although the elements of $\mathcal{C}$ are not cones in the usual sense, we still call them cones. If $C \in \mathcal{C}$ then the corresponding element $\hat{C} \in \hat{\mathcal{C}}$ is called the normalization of C.

There is the correspondence between the elements of $\mathcal{C}$ and non-normal affine toric varieties T_C. (One can construct the variety T_C as follows: semigroup algebra of C is finitely generated and therefore correspond to the affine variety which we denote by T_C.) We denote the set of toric varieties T_C for all $C \in \mathcal{C}$ by $\mathcal{T}$.

Any $C \in \mathcal{C}$ has exactly one k-dimensional face which span L_k, for any $k = 0, \ldots, n$. Therefore, one can identify one of the k-dimensional orbits in all $T \in \mathcal{T}$ for each $k = 0, \ldots, n$. We denote these orbits $T^0, T^1, \ldots, T^n$. So, $T^i \subset T$ for all $T \in \mathcal{T}$ and $i = 0, \ldots, n$. Note also, that $T^k \cup T^{k-1}$ is an affine subvariety in every $T \in \mathcal{T}$ for $i = 1, \ldots, n$. However, any union of more than two consequent orbits is not a subvariety.

Let $C, C' \in \mathcal{C}$ and $C' \subset C$. Then there is the natural map $\phi_{C,C'} : T_C \to T_{C'}$. Note, that $\phi_{C,C'}$ is identity on $T^0, T^1, \ldots, T^n$. Since the maps $\phi_{C,C'}$ goes in the back direction, $\mathcal{T}$ is a projective system with respect to these maps.

Take $T_C \in \mathcal{T}$. Although it is not normal, its normalization $\widetilde{T}_C$ is isomorphic to $\mathbb{C}^n$. The standard generators $(x_1, \ldots, x_n)$ of the cone $\hat{C}$ give coordinates on $\widetilde{T}$. We'll use $(x_1, \ldots, x_n)$ as coordinates for T as well (although some points are going to be glued together). We call these coordinates the *standard coordinates* for the varieties $T \in \mathcal{T}$.

Switching from T_C to $T_{C'}$ corresponds to the change of coordinates

$$
\begin{aligned}
x'_1 &= x_1 \\
x'_2 &= x_1^{d_{12}} x_2 \\
&\vdots \\
x'_n &= x_1^{d_{1n}} \ldots x_{n-1}^{d_{(n-1)n}} x_n,
\end{aligned}
$$

where

$$
\begin{pmatrix}
1 & d_{12} & d_{13} & \ldots & d_{1n} \\
0 & 1 & d_{23} & \ldots & d_{2n} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
$$

is the transition matrix from the standard generators of the cone $\hat{C}$ to the standard generators of the cone $\hat{C}'$.

The following lemma is important in the study of the Laurent series of L :

Lemma 2.1.3. *Let $\hat{C}' \in \hat{\mathcal{C}}$ and $S \subset \hat{C}'$, $S \neq \emptyset$. Than S contains its minimal element s_{min} and there exist another cone $\hat{C} \in \hat{\mathcal{C}}$ such that $S \subset \hat{C} + s_{min}$.*

Proof. Let $(x_1, \ldots, x_n)$ be the standard generators of $\hat{C}'$. Note, that according to the Theorem 2.1.2 the order in L_n coincides with the lexicographic order with respect to the basis $(x_1, \ldots, x_n)$ (first, with respect to the power of x_n, then the power of x_{n-1}, etc.). Let m_n be the smallest integer such that $x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} x_n^{m_n} \in S$ for some integers $k_1, \ldots, k_{n-1}$. Let m_{n-1} be the smallest integer such that $x_1^{k_1} \ldots x_{n-2}^{k_{n-2}} x_{n-1}^{m_{n-1}} x_n^{m_n} \in S$ for some integers $k_1, \ldots, k_{n-2}$. Continuing in the same way we get the integers $m_1, \ldots, m_n$ such that $s_{min} = x_1^{m_1} \ldots x_n^{m_n} \in S$ is the minimal element in S.

Let $S' = \{a - s_{min} : a \in \hat{C}' \text{ and } a > s_{min}\}$. Clearly, $S \subset S' + s_{min}$. Therefore, it is enough to find $\hat{C} \in \hat{\mathcal{C}}$ such that $S' \subset \hat{C}$.

Let $y_1 = x_1$, $y_2 = x_1^{-m_1} x_2$, $\ldots$, $y_n = x_1^{-m_1} \ldots x_{n-1}^{-m_{n-1}} x_n$. Easy to see that $y_k \in \hat{L}_+^k$ for $k = 1, \ldots, n$. Moreover, the transition matrix from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ is upper triangular with units on diagonal. Therefore, the cone generated by $(y_1, \ldots, y_n)$ is simple and belong to $\hat{\mathcal{C}}$. Denote this cone by $\hat{C}$. Let's prove that $S' \subset \hat{C}$. Indeed, let $s = x_1^{l_1} \ldots x_n^{l_n} \in S'$. It follows from the definition of S', that $l_k \geq -m_k$ for $k = 1, \ldots, n$ and that the last non-zero power, say l_p, is positive (otherwise $s < 0$). Then $s = y_p(x_1^{l_1+m_1} \ldots x_{p-1}^{l_{p-1}+m_{p-1}} x_p^{l_p-1})$. Note, that all the powers in the last formula are non-negative and that $x_k \in \hat{C}$ for all $k = 1, \ldots n$. Therefore, $s \in \hat{C}$. □

2.1.3 Laurent Power Series.

Definition 2.1.9. Let $F(L)^*$ be the set of all formal infinite linear combinations $f = \sum_{p \in L^n} f_p p$ of the elements of L^n such that

1. There exist at least one point in the torus T^n such that f is convergent at this point.

2. There exist a cone $\hat{C} \in \hat{\mathcal{C}}$ and an element $p_0 \in L^n$ such that the Newton polyhedron of f is a subset of $\hat{C} + p_0$.

Let $F(L) = F(L)^* \sqcup \{0\}$ (i.e. we add the series with zero coefficients to $F(L)^*$). We call $F(L)$ the set of Laurent series of L.

Lemma 2.1.4. *Let $f \in F(L)^*$. Then one can choose the cone $\hat{C}$ and an element $p_0 \in L^n$ in such a way that the Newton polyhedron of f is a subset of $\hat{C} + p_0$ and $f_{p_0} \neq 0$.*

Proof. Let $Sup_f = \{p \in L^n : f_p \neq 0\}$. Since $f \in F(L)$, $Sup_f \subset \hat{C}' \in \hat{\mathcal{C}}$. Now we just apply the Lemma 2.1.2 to Sup_f to get p_0 and $\hat{C}$. □

Note, that the $p_0 \in L^n$ from Lemma 2.1.4 is unique for each $f \in F(L)^*$. Basically, it is the smallest element in L^n such that the corresponding coefficient of f is not zero. Therefore, one gets a map $\nu : F(L)^* \to L^n$.

Lemma 2.1.5. *Let $f_1, \ldots, f_m \in F(L)$. Then there exist $T \in \mathcal{T}$ such that $\frac{f_i}{\nu(f_i)}$ converge to holomorphic non-zero functions in a neighborhood of the origin in the normalization $\widetilde{T}$ of T for $i = 1, \ldots, m$ (here $(x_1, \ldots, x_n)$ are the standard coordinates on T).*

Proof. Let $\hat{C}_i \in \hat{\mathcal{C}}$ be such that the Newton polyhedron of f_i is a subset of $\hat{C}_i + \nu(f_i)$. Let $\hat{C} \in \hat{\mathcal{C}}$ be such that $\hat{C}_i \subset \hat{C}$ for all $i = 1, \ldots, m$ (the existence of $\hat{C}$ follows from the Lemma 2.1.2). Let $C = \hat{C} \cap L$, $T = T_C$, and $(x_1, \ldots, x_n)$ be the standard coordinates in T. Then, for any $i = 1, \ldots, m$, $\frac{f_i}{\nu(f_i)}$ is a Taylor series in $(x_1, \ldots, x_n)$ converging at least at one point in $T^n \subset T$ and, therefore, converging in a neighborhood of zero in $\widetilde{T}$. Moreover, since the coefficient of f_i at $\nu(f_i)$ is not zero, f_i is not zero at the origin. □

Remark. Note, that $\nu(f_i)$'s are monomials in $(x_1, \ldots, x_n)$. So, $f_1, \ldots, f_m$ are *almost monomial* in the neighborhood of the origin in $\widetilde{T}$, i.e. each of them is equal to a monomial multiplied by a non-zero holomorphic function.

Theorem 2.1.3. *$F(L)$ is a field and $\nu : F(L)^* \to L^n$ is a homomorphism of the multiplicative group $F(L)^*$ of $F(L)$ to the ordered abelian group L^n.*

Proof. The theorem follows immediately from the previous lemma. Indeed, if $f, g \in F^*(L)$ then there exist $T \in \mathcal{T}$ with coordinates $(x_1, \ldots, x_n)$ such that both $\frac{f}{\nu(f)}$ and $\frac{g}{\nu(g)}$ converge to holomorphic non-zero functions in a neighborhood of the origin in $\widetilde{T}$. Let $\nu(f) = x_1^{k_1} \ldots x_n^{k_n}$ and $\nu(g) = x_1^{m_1} \ldots x_n^{m_n}$. Then $\frac{fg}{\nu(f)\nu(g)}$, $\frac{f+g}{x_1^{\min(k_1,m_1)} \ldots x_n^{\min(k_n,m_n)}}$, and $\frac{f^{-1}}{\nu(f)^{-1}}$ also converge to holomorphic functions in a neighborhood of the origin. Moreover, $\frac{fg}{\nu(f)\nu(g)}$ and $\frac{f^{-1}}{\nu(f)^{-1}}$ don't vanish at the origin. Therefore, $fg, f^{-1}, f+g \in F(L)$, $\nu(fg) = \nu(f)\nu(g)$, and $\nu(f^{-1}) = \nu(f)^{-1}$. □

Definition 2.1.10. A homomorphism of the multiplicative group of a field F to an ordered abelian group G is called a valuation on F.

According to the Theorem 2.1.3, the field $F(L)$ is endowed with a valuation.

Lemma 2.1.6.

1. *Let r be a meromorphic function on T^n (which is the same as a meromorphic function on any $T \in \mathcal{T}$). Then there exist a Laurent series $f \in F(L)$ normally converging to r in $\{0 < |x_1| < \epsilon_1, 0 < |x_2| < \epsilon_2, \ldots, 0 < |x_n| < \epsilon_n\} \subset T^n$, where $(x_1, \ldots, x_n)$ are the standard coordinates on one of the varieties $T \in \mathcal{T}$.*

2. *Let $T \in \mathcal{T}$. Let r be a meromorphic function on a neighborhood of the origin in T. Then there exist a Laurent series $f \in F(L)$ normally converging to r in $\{0 < |x_1| < \epsilon_1, 0 < |x_2| < \epsilon_2, \ldots, 0 < |x_n| < \epsilon_n\} \subset T^n$, where $(x_1, \ldots, x_n)$ are the standard coordinates on one of the varieties $T' \in \mathcal{T}$.*

Proof. Both statements immediately follows from the Theorem 2.1.3. Indeed, the Taylor expansions of functions holomorphic at the origin in the normalization of any $T \in \mathcal{T}$ belong to $F(L)$ and $F(L)$ is closed under division by non-zero elements. □

$F(L)$ is basically the field of functions, which are meromorphic in a "good" neighborhood of the flag $T^0, T^1, \ldots, T^n$. Note, that $F(L)$ and the valuation on it depend only on L^n, and don't depend on other elements of the flag $L^0 \subset L^1 \subset \cdots \subset L^n$. However, if one wants to consider the subalgebra in $F(L)$ of functions which are holomorphic in a "good" neighborhood of the flag $T^0, T^1, \ldots, T^n$, it will depend on the whole flag $L^0 \subset L^1 \subset \cdots \subset L^n$ (or, equivalently, on the semigroup L).

Definition 2.1.11. Let $O(L) \subset F(L)$ be the subset of all series $f = \sum_{p \in L^n} f_p p$ in $F(L)$ such that $f_p \neq 0 \Rightarrow p \in L$.

Remark. Let $f \in F(L)$ and $f = \nu(f)\phi(x_1, \ldots, x_n)$ where $(x_1, \ldots, x_n)$ are the coordinates on the appropriate $T \in \mathcal{T}$ and ϕ is a holomorphic function in the neighborhood of the origin in the normalization $\widetilde{T} \simeq \mathbb{C}^n$. Then the series of f belong to $O(L)$ if and only if $\nu(f) \geq 0$ and ϕ can be pushed forward to T.

Lemma 2.1.7. $O(L) \subset F(L)$ *is a subalgebra.*

Proof. Follows immediately from the definition. □

Remark. Let $C \in \mathcal{C}$ be a cone. Denote by $O(C)$ the ring of series of elements of C, converging in a neighborhood of the origin in T_C. In other words, $O(C)$ is the ring of germs of analytic functions on T_C at the origin. Then $O(L) = \bigcup_{C \in \mathcal{C}} O(C)$. Therefore, $O(L)$, in a sense, is the ring of germs of functions on the inverse limit of $\mathcal{T}$ at the origin.

Lemma 2.1.8. *Any* $T \in \mathcal{T}$ *satisfy the following continuation property. Let* $U \subset T$ *be any open subset and* $f : U \to \mathbb{C}$ *be any continuous function, holomorphic in the complement to an analytic subset* $\Sigma \subset U$ *of codimension* 1. *Then* f *is holomorphic in* U.

Proof. It is enough to check that f is holomorphic in a neighborhood of every point in U. Moreover, every point in T has a neighborhood isomorphic to a neighborhood of the origin in a variety constructed in the same way as T, but for a different flag of lattices. Therefore, it is enough consider the case when U contains the origin and to check that f is holomorphic in a neighborhood of the origin.

Let $(x_1, \dots, x_n)$ be the standard coordinates on T and $C \in \mathcal{C}$ be the corresponding cone. Let $\pi : \widetilde{T} \to T$ be the normalization map. Then $\tilde{f} = f \circ \pi$ is continuous function in a neighborhood of the origin in $\widetilde{T} \simeq \mathbb{C}^n$ and holomorphic in the compliment to the analytic subset $\pi^{-1}(\Sigma)$ of codimension 1. Therefore, by Riemann Extension Theorem, it is holomorphic in this neighborhood and can be expanded into the Taylor series $\tilde{f} = \sum a_{\overline{k}} x^{\overline{k}}$ in it.

Since f is continuous in a neighborhood of the origin in T, it follows that if $a_{\overline{k}} \neq 0$ then $x^{\overline{k}} \in C$. Therefore, $\sum a_{\overline{k}} x^{\overline{k}} \in O(C)$ and f is regular at the origin in T. □

2.1.4 Changes of Variables.

Let $(f_1, \dots, f_n)$ be an n-tuple of functions from $F(L)$ and $x_1 = \nu(f_1), \dots, x_n = \nu(f_n)$. Suppose that:

1. $(x_1, \dots, x_n)$ are the standard coordinates on a variety in $\mathcal{T}$;
2. for any integers $k_1, \dots, k_n$ such that $x_1^{k_1} \dots x_n^{k_n} \in L$ we have $f_1^{k_1} \dots f_n^{k_n} \in O(L)$.

Let L_f^n be the lattice of monomials in $f_1, \dots, f_n$. Easy to see that the restriction $\phi := \nu|_{L_f^n} : L_f^n \to L^n$ is an isomorphism. Consider the flag of lattices $L_f^0 \subset L_f^1 \subset \dots \subset L_f^n$, where $L_f^k = \phi^{-1}(L^k)$ for $k = 0, \dots, n$ and the order on L_f^n is induced by ϕ. Denote by L_f the corresponding semigroup, $\mathcal{C}_f$ the corresponding system of cones and $\mathcal{T}_f$ the corresponding system of toric varieties.

Theorem 2.1.4. *There exist a toric variety $T_f \in \mathcal{T}_f$ with the standard coordinates $(g_1, \dots, g_n)$, and a toric variety $T \in \mathcal{T}$ such that the n-tuple $(g_1, \dots, g_n)$ provides an analytic isomorphism of a neighborhood of the origin in T to a neighborhood of the origin in T_f (note, that $g_1, \dots, g_n$ are monomials in $f_1, \dots, f_n$).*

Proof. According to the Theorem 2.1.5 there exist a toric variety $T \in \mathcal{T}$ such that $\frac{f_k}{x_k}$ converge to a non-zero holomorphic function in a neighborhood of zero of the normalization $\widetilde{T}$ for $1 \leq k \leq n$. Let $(y_1, \dots, y_n)$ be the standard coordinates in T and $C \in \mathcal{C}$ be the corresponding cone. The transition matrix from $(x_1, \dots, x_n)$ to $(y_1, \dots, y_n)$ is integer and upper-triangular with units on diagonal. For instance, let $y_k = x_1^{d_{1k}} \dots x_{k-1}^{d_{(k-1)k}} x_k$ for $k = 1, \dots, n$. Let $g_k = f_1^{d_{1k}} \dots f_{k-1}^{d_{(k-1)k}} f_k$ for $k = 1, \dots, n$. Let $\hat{C}_f \in \hat{\mathcal{C}}_f$ be the cone generated by $(g_1, \dots, g_n)$ and let $T_f \in \mathcal{T}_f$ be the corresponding toric variety. Then $\nu(g_k) = y_k$ for $k = 1, \dots, n$ and $\frac{g_k}{y_k}$ converge to holomorphic functions in a neighborhood of the origin in the normalization $\widetilde{T}$ of T for $k = 1, \dots, n$. Let $h_k = \frac{g_k}{y_k}$. Then $\frac{\partial g_i}{\partial y_j}(0) = \frac{\partial h_i y_i}{\partial y_j}(0) = h_i(0)\frac{\partial y_i}{\partial y_j}(0) + \frac{\partial h_i}{\partial y_j}(0) y_i(0) = h_i(0)\delta_{i,j}$. Since $h_i(0) \neq 0$ for $i = 1, \dots, n$, the Inverse Function Theorem is applicable. Therefore, $(g_1, \dots, g_n)$ provides an isomorphism $\hat{G} : \widetilde{U} \to \widetilde{U_f}$ of a polydisk $\widetilde{U}$ with center at the origin of $\widetilde{T}$ to a neighborhood of the origin $\widetilde{U_f}$ in $\widetilde{T}_f$.

Let $\pi(\widetilde{U}) = U \subset T$ and $\pi_f(\widetilde{U_f}) = U_f \subset T_f$. We need to show, that the isomorphism $\hat{G} : \widetilde{U} \to \widetilde{U_f}$ can be pushed down to an isomorphism $G : U \to U_f$. According to the Lemma 2.1.8, it is enough to show that G is a homeomorphism. Moreover, since the topology of U (U_f respectively) is the factor topology of $\pi : \widetilde{U} \to U$ ($\pi_f : \widetilde{U_f} \to U_f$ respectively), it is enough to show, that G is a bijection.

First, we construct the map G and then prove that it is surjective and injective. We get that it is regular for free, but the above arguments allow us to avoid considering the inverse map. The condition 2 provides that the generators of C_f are regular (and, therefore, well defined) on U. Indeed, they are regular on $\widetilde{U}$ and they belong to $O(L)$ by condition 2. Surjectivity follows immediately from the diagram. All we need to prove is the injectivity.

Easy to see, that $\hat{G}$ maps the coordinate cross to the coordinate cross and, in particular, maps $\widetilde{T}^k \cap \widetilde{U}$ to $\widetilde{T}_f^k \cap \widetilde{U}_f$, where $\widetilde{T}^k$ and $\widetilde{T}_f^k$ are respectively the coordinate subspaces $\widetilde{T}^k = \{y_n = \dots = y_{k+1} = 0\} \subset \widetilde{T}$ and $\widetilde{T}_f^k = \{g_n = \dots = g_{k+1} = 0\} \subset \widetilde{T}_f$. Therefore, for any point $x \in U$ the number of preimages of x in $\widetilde{U}$ is bigger or equal to the number of preimages of $G(x)$ in $\widetilde{U}_f$. But the preimages of x are mapped injectively by $\hat{G}$ to the preimages of $G(x)$. Therefore, $\hat{G}|_{\pi^{-1}(x)}$ is a bijection to $\pi_f^{-1}(G(x))$, which implies the injectivity of G. □

Switching from $\mathcal{T}$ to $\mathcal{T}'$ is called a *change of variables.* Easy to see, that a change of variables gives an isomorphism from $F(L)$ to $F(L_f)$.

The same arguments works for any bigger cone $C' \supset C$. More precisely, we have the following

Corollary 1. *Let $C \in \mathcal{C}$ and $C_f \in \mathcal{C}_f$ be as in the Theorem 2.1.4. Let $C' \in \mathcal{C}$ be such that $C \subset C'$. Let $C'_f \in \mathcal{C}_f$ be such that the transition matrix from the standard generators of the $\hat{C}'_f$ to the standard generators of the $\hat{C}_f$ is the same as the transition matrix for the generators of $\hat{C}'$ and $\hat{C}$ respectively. Then there exist neighborhoods of the origins in $T_{C'}$ and $T_{C'_f}$ U' and U'_f and the unique isomorphism $G' : U' \to U'_f$ such that $G \circ \phi_{C',C} = \phi_{C'_f,C_f} \circ G'$ on U'.*

So, the change of variables provides isomorphisms between the neighborhoods of the origins in the elements of $\mathcal{T}$ and $\mathcal{T}_f$ respectively, at least starting from some $T \in \mathcal{T}$ and $T_f \in \mathcal{T}_f$. These isomorphisms commute with the maps $\phi_{C',C} : T_{C'} \to T_{C'_f}$.

2.1.5 Residue.

One can consider the free one-dimensional module $\Omega(L)$ over $F(L)$ with the generator $\omega_{T^n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. Note, that ω_{T^n} doesn't depend on the choice of coordinates $(x_1, \ldots, x_n)$. Then for every element of $\omega \in \Omega(L)$ there exist a toric variety $T \in \mathcal{T}$ with coordinates $(x_1, \ldots, x_n)$ such that $\omega = x_1^{d_1} \ldots x_n^{d_n} \phi dx_1 \wedge \cdots \wedge dx_n$ where ϕ is a holomorphic non-zero function in a neighborhood of the origin in $\widetilde{T}$. We call the elements of $\Omega(L)$ the germs of meromorphic n-forms at the flag of orbits $T^n, T^{n-1} \ldots, T^0$. According to the Lemma 2.1.6, any meromorphic form on a neighborhood of the origin in any $T \in \mathcal{T}$ can be expanded into a power series from $\Omega(L)$.

Definition 2.1.12. Let $\omega \in \Omega(L)$, $\omega = (\sum_{p \in L^n} a_p p)\omega_{T^n}$. Then the residue of ω is given by $res(\omega) = a_1$.

Lemma 2.1.9.

$$res(\omega) = \frac{1}{(2\pi i)^n} \int_{\tau^n} \omega,$$

where $\tau^n = \{(x_1, \ldots, x_n) \in T^n : |x_1| = \epsilon_1, \ldots, |x_n| = \epsilon_n\}$ *for* $(x_1, \ldots, x_n)$ *– coordinates on a toric variety* $T \in \mathcal{T}$*, and* ϵ_i *are small enough, so that* ω *converges on* τ^n*. The orientation on* τ^n *is provided by the form* $\frac{1}{(2\pi i)^n}\omega_{T^n}|_{\tau^n}$ *(note, that this form is real).*

Proof. Follows immediately from the Fubini's Theorem and the formula for the one-dimensional residues. □

Suppose that the n-tuple of functions $(f_1, \ldots, f_n)$ defines a change of variables. Let $\omega \in \Omega(L)$ be a germ of a meromorphic form. Let $\hat{G} : \widetilde{U} \to \widetilde{U}_f$ be an isomorphism of neighborhoods of the origins in the normalizations of the appropriate toric varieties $T \in \mathcal{T}$ and $T_f \in \mathcal{T}_f$ provided by the change of variables (see Theorem 2.1.4). According to the Corollary 1 from the Theorem 2.1.4, one can assume that ω converges to a meromorphic form in $U \subset T$. Then one can push forward ω using the isomorphism $\hat{G}$. The result can be expanded as a power series from $\Omega(L_f)$. Therefore, we get a map from $F_* : \Omega(L) \to \Omega(L_f)$. Easy to see, that this map is an isomorphism of $F(L)$-modules ($\Omega(L_f)$ is an $F(L)$-module via the isomorphism of $F(L)$ and $F(L_f)$ provided by the change of variables).

Lemma 2.1.10. *Changes of variables doesn't change the residue, i.e.* $res(F_*(\omega)) = res(\omega)$.

Proof. Follows from the Lemma 2.1.9 and the observation that $\hat{G}$ restricts to an isomorphism of $\widetilde{U} \cap T^n$ and $\widetilde{U}_f \cap T_f^n$ and this isomorphism is homotopic to the identity map (here we identify T^n and T_f^n with the standard tori via the standard coordinates on T and T_f). □

2.1.6 Algebraic functions.

Let X be an analytic (algebraic) variety and let $g_0, \ldots, g_k$ be regular functions on X. Consider the equation $g_0 + g_1 t \cdots + g_k t^k = 0$. Let $U = X \backslash (sing(X) \cup \{g_k = 0\} \cup \Sigma)$, where $\Sigma = \{Dis = 0\}$, where Dis is the discriminant of the polynomial $g_0 + g_1 t \cdots + g_k t^k$. Then there exist the k-sheeted covering $p : W \to U$ and a regular function f on W, such that for every point $x \in U$ the values of f on the preimage $p^{-1}(x)$ are exactly the roots of the equation $g_0 + g_1 t \cdots + g_k t^k$. In such a situation, we say that f is an algebraic function on X. We say that Σ is the divisor of branching of f.

Lemma 2.1.11. *Let X be an analytic (algebraic) variety and let $g_0, \ldots, g_k$ be regular functions on X. Suppose that there is an open subset $U \subset X$, such that $X \backslash U$ is a finite union of subvarieties of codimension 1, and a holomorphic function f on U such that f satisfy the equation $g_0 + g_1 t \cdots + g_k t^k = 0$ on U. Then f can be continued to a meromorphic function on X.*

Proof. Consider the function $\tilde{f} = g_k f$. Easy to see that it is holomorphic on U and satisfy the integral equation $g_0 g_k^{k-1} + g_1 g_k^{k-2} t + \cdots + g_{k-1} t^{k-1} + t^k = 0$ on U. Let's prove that $\tilde{f}$ is locally bounded on X, i.e. for any point $x \in X$ there exist a neighborhood V of x, such that $\tilde{f}$ is bounded in $V \cap U$. Indeed, assume that it is not true. The coefficients $g_i g_k^{k-i-1}$ are regular on X, and, therefore, locally bounded. So, there exist a neighborhood V of x and a constant $M > 1$, such that $|g_i g_k^{k-i-1}| < M$ for $0 \le i \le k-1$. Since $\tilde{f}$ is not bounded in $V \cap U$, there exist a point $y \in V \cap U$, such that $|\tilde{f}(y)| > kM$. Then $|\tilde{f}^k(y)| > kM|\tilde{f}^{k-1}(y)|$. But $|g_i(y) g_k^{k-i-1}(y) \tilde{f}^i(y)| < M|\tilde{f}^i(y)| \le M|\tilde{f}^{k-1}(y)|$ for $0 \le i \le k-1$. So, $|g_0 g_k^{k-1} + g_1 g_k^{k-2} \tilde{f}(y) + \cdots + g_{k-1} \tilde{f}^{k-1}(y)| < |\tilde{f}^k(y)|$. Therefore, $\tilde{f}$ doesn't satisfy the equation $g_0 g_k^{k-1} + g_1 g_k^{k-2} t + \cdots + g_{k-1} t^{k-1} + t^k = 0$ at y, which is a contradiction.

Let $p : \widetilde{X} \to X$ be the normalization. Then $\tilde{f} \circ p$ is holomorphic in the complement of a finite union of subvarieties of codimension 1 in the normal variety $\widetilde{X}$ and is locally bounded in $\widetilde{X}$. Therefore, it is regular on $\widetilde{X}$. The normalization map p is a birational isomorphism, so $\tilde{f}$ is meromorphic on X. Finally, $f = \frac{\tilde{f}}{g_k}$, therefore f is also meromorphic on X. □

Let now $g_0, \dots, g_k \in F(L)$. One can choose $T \in \mathcal{T}$ such that g_k and the discriminant of the equation $g_k t^k + \cdots + g_0 = 0$ are almost monomial in a polydisk D with center at the origin of the normalization $\widetilde{T} \simeq \mathbb{C}^n$. Let $X = D \cap T^n$. Let $p : W \to X$ be the corresponding k-sheeted covering. Suppose that W is connected (which is equivalent to say that the equation $g_k t^k + \cdots + g_0 = 0$ is irreducible). We need the following Lemma:

Lemma 2.1.12. *Let $N = k!$. Consider the map $P : \mathbb{C}^n \to \mathbb{C}^n$ given by $P(x_1, \dots, x_n) = (x_1^N, \dots, x_n^N)$. Let $W' = P^{-1}(X)$. Then there exist a map $\phi : W' \to W$ such that $p \circ \phi = P$.*

Proof. According to the classical theory, the connected coverings of X are classified up to isomorphism by the subgroups of the fundamental group $\pi_1(X)$ and the number of sheets corresponds to the index of the subgroup. In more details, the induced homomorphism $p_* : \pi_1(W) \to \pi_1(X)$ is injective and its isomorphism type defines the isomorphism type of the covering $p : W \to X$.

Since $\pi_1(X) \simeq \mathbb{Z}^n$, it follows, that $p_*(\pi_1(W)) \subset \pi_1(X)$ is a sublattice of full rank and index k. Let $(a_1, \dots, a_n)$ be the basis of $\pi_1(X)$ corresponding to the loops going around coordinate hyperplanes in positive direction. Then $a_i^N \in p_*(\pi_1(W))$ for all i. Consider now the covering $P : W' \to X$. Easy to see, that $P_*(\pi_1(W')) \subset \pi_1(X)$ is generated by $(a_1^N, \dots, a_n^N)$. Therefore, $\pi_1(W') \subset \pi_1(W)$ (here we identified $\pi_1(W)$ and $\pi_1(W')$ with their images in $\pi_1(X)$). Therefore, there exist a covering $\phi : W'' \to W$ such that $\phi_*(\pi_1(W'')) = \pi_1(W') \subset \pi_1(W) \subset \pi_1(X)$. But then the composition $p \circ \phi : W'' \to X$ and $P : W' \to X$ correspond to the same subgroup in $\pi_1(X)$ and, therefore, isomorphic. □

According to the Lemmas 2.1.12 and 2.1.11, the equation $g_k t^k + \cdots + g_0 = 0$ has a meromorphic solution f in $P^{-1}(D)$. Moreover, f is holomorphic in $W' = P^{-1}(D) \cap (\mathbb{C}^*)^n$. Therefore, we have the following theorem:

Theorem 2.1.5. *Let $g_0, \dots, g_k \in F(L)$ be such that the equation $g_k t^k + \cdots + g_0 = 0$ is irreducible. Let $N = k!$. Then there exist a toric variety $T \in \mathcal{T}$ with the coordinates $(x_1, \dots, x_n)$, and the integers $m_1, \dots, m_n$ such that the root f of the equation $g_k t^k + \cdots +$*

$g_0 = 0$ *can be written in the form of the Piezo series* $f = \sqrt[N]{x_1}^{m_1} \dots \sqrt[N]{x_n}^{m_n} \sum\limits_{i_1,\dots,i_n \geq 0} f_{i_1,\dots,i_n} \sqrt[N]{x_1}^{i_1} \dots$ *converging in a neighborhood of the origin in the normalization* $\widetilde{T}$.

2.2 Toric Neighborhoods of Parshin's points.

2.2.1 Branched coverings and Generic components of the preimage of a hypersurface.

Definition 2.2.1. Let X and Y be non-empty pure-dimensional algebraic (analytic) varieties of the same dimension. An algebraic (analytic) map $f : Y \to X$ is called a *branched covering* if it is proper, surjective, and is a local isomorphism at a generic point (i.e. there is an open dense subset $U \subset Y$ consisting of smooth points of Y such that for every point $y \in U$ $f(y)$ is a smooth point of X and the differential of f has full rang at y).

Note, that a composition of two branched coverings is again a branched covering.

Definition 2.2.2. Let $f : Y \to X$ be a branched covering. Let $H \in X$ be a hypersurface in X. Then the *generic component of the preimage* (shortly, the *generic preimage*) $H_f \subset Y$ is the union of those irreducible components $\hat{H}_i$ of the full preimage $f^{-1}(H)$ for which the restriction $f|_{\hat{H}_i} : \hat{H}_i \to H$ is a local isomorphism at a generic point.

Note, that if $f : Y \to X$ is a blow-up with a smooth center $C \subset X$, then the generic preimage H_f coincide with the strict transform of H unless C and H has common irreducible components.

Let $f : Y \to X$ and $g : Z \to Y$ be branched coverings. Let $H \subset X$ be a hypersurface. Then, clearly, $(H_f)_g = H_{f \circ g}$. Indeed, the restriction of $f \circ g$ to an irreducible component of $g^{-1}(f^{-1}(H))$ is a local isomorphism at a generic point if and only if g is a local isomorphism at a generic point of this component and f is a local isomorphism at a generic point of the image of this component.

Lemma 2.2.1. *$f|_{H_f} : H_f \to H$ is a branched covering.*

Proof. By Sard's Lemma, the image of those components of $f^{-1}(H)$ which are not components of H_f has measure 0 in H. Therefore, H_f is non-empty and its image is dense in H. Then, since H_f is a closed subset in Y, $f|_{H_f}$ is proper. So, we only need to prove that $f|_{H_f}$ is surjective to H. Indeed, let $x \in H$. Let $K \subset H$ be a compact neighborhood of x in H. Since $f|_{H_f}$ is proper, $N = f^{-1}(K) \cap H_f$ is compact. So, $f(N)$ is compact and, therefore, closed. Since $f(H_f)$ is dense in H, there exist a sequence $\{y_m\} \subset H_f$, such that $f(y_m) \to x$ and $\{f(y_m)\} \in K$. Then $\{y_m\} \subset N$ and $\{f(y_k)\} \subset f(N)$. Therefore, $x \in f(N)$. □

Note, that the normalization map is always a branched covering.

Theorem 2.2.1. *Let X be a pure-dimensional analytic (algebraic) variety. Let $\pi : Y \to X$ be a degree one branched covering and let Y be normal. Let $p : \widetilde{X} \to X$ be the normalization. Let $H \subset X$ be a hypersurface. Then there is a natural map $\widetilde{\pi} : H_\pi \to H_p$ which is a degree one branched covering.*

Proof. According to the universal properties of the normalization, the map $\pi : Y \to X$ factors through the normalization, i.e. there exist a map $\tilde{\pi} : Y \to \widetilde{X}$ such that $\pi = p \circ \tilde{\pi}$. Moreover, $(H_p)_{\tilde{\pi}} = H_\pi$. Therefore, it is enough to consider the case when X is normal. So, $\widetilde{X} = X$, $H_p = H$, $\tilde{\pi} = \pi$, and we need to prove that $\pi|_{H_\pi} : H_\pi \to H$ is a degree one branched covering.

Let $sing(X) \subset X$ and $sing(Y) \subset Y$ be the singular loci of X and Y correspondingly. Since π is a proper map and $sing(Y)$ is a closed subvariety, it follows that $\pi(sing(Y)) \subset X$ is a closed subvariety in X. Since X and Y are normal, the codimensions of $sing(X)$, $sing(Y)$, and, therefore, $\pi(sing(Y))$ are at least 2. Let $X' = X \backslash (sing(X) \cup \pi(sing(Y)))$. Let $H' = H \cap X'$. Note, that H' is a complement to a closed subvariety of codimension at least 1 in H. Let $Y' = \pi^{-1}(X')$. Now X' and Y' are smooth, $\pi' := \pi|_{Y'} : Y' \to X'$ is still a degree one branched covering, and $H' \subset X'$ is a hypersurface. Note, that $H'_{\pi'} = H_\pi \cap Y'$, and it is a complement to a subvariety of codimension at least 1 in H_π (although $Y\backslash Y'$ can have codimension 1).

Let $crit(\pi') \subset Y$ be the critical locus of π. Let $\widetilde{X} = X'\backslash\pi'(crit(\pi'))$. Let $\widetilde{Y} = \pi'^{-1}(\widetilde{X})$. Then $\widetilde{\pi} = \pi|_{\widetilde{Y}} : \widetilde{Y} \to \widetilde{X}$ is an isomorphism. Indeed, it is non-degenerate and, therefore, unbranched covering of degree 1.

We need the following

Lemma 2.2.2. *Let M and N be analytic manifolds and $f : M \to N$ be a degree one branched covering. Then $codim(f(crit(f))) \geq 2$.*

Proof. Suppose that $codim(f(crit(f))) = 1$. Then there exist $p \in crit(f)$ such that

1. p is a smooth point of the hypersurface $crit(f)$;
2. $f^{-1}(f(crit(f)))$ coincide with $crit(f)$ in a neighborhood of p;
3. $f(p)$ is a smooth point of the codimension 1 irreducible component of $f(crit(f))$;
4. $f|_{crit(f)}$ is a local isomorphism at p.

Therefore, there exist coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively in a neighborhood U of p in M and in a neighborhood V of $f(p)$ in N, such that

1. $f(U) \subset V$.
2. $U \cap crit(f) = \{x_n = 0\}$;
3. $V \cap crit(f) = \{y_n = 0\}$;
4. $f(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1}, 0)$.

So, the map $f|_U$ is given by

$$y_1 = x_1 + x_n\phi_1(x);$$
$$\vdots$$
$$y_{n-1} = x_{n-1} + x_n\phi_{n-1}(x);$$
$$y_n = x_n^k\phi_n(x),$$

where $k \geq 1$ and ϕ_n is not divisible by x_n. Note, that $\phi_n(p) \neq 0$. Indeed, otherwise $\{\phi_n = 0\} \subset f^{-1}(f(crit(f))) = \{x_n = 0\}$ and ϕ_n is not divisible by x_n.

One can get rid of all the ϕ_i's simply by changing the coordinates in a neighborhood of $p \in M$. Indeed, let

$$t_1 = x_1 + x_n\phi_1(x);$$
$$\vdots$$
$$t_{n-1} = x_{n-1} + x_n\phi_{n-1}(x);$$
$$t_n = x_n \sqrt[k]{\phi_n(x)}.$$

Easy to check that the Jacobian is not zero, so $(t_1, \ldots, t_n)$ are indeed coordinates in a neighborhood of p. In t's coordinates the map f is given by

$$y_1 = t_1;$$
$$\vdots$$
$$y_{n-1} = t_{n-1};$$
$$y_n = t_n^k.$$

Since a general point in N should have only one primage, k should be equal to 1. But then f is non-degenerate at p. □

So, the codimension of $\pi'(crit(\pi'))$ is at least 2. Therefore, $\widetilde{H} := H' \cap \widetilde{X}$ is a complement to a subvariety of degree at least 1 in H'. Its' preimage $\widetilde{\pi}^{-1}(\widetilde{H}) = \widetilde{H}_{\widetilde{\pi}}$ is also a complement to a subvariety of codimension at least 1 in $H'_{\pi'}$. And $\pi|_{\widetilde{H}} : \widetilde{H}_{\widetilde{\pi}} \to \widetilde{H}$ is an isomorphism. So, $\pi|_{H_\pi} : H_\pi \to H$ is a degree one branched covering. □

2.2.2 Resolution of Singularities for Flags.

To avoid difficulties with the resolution of singularities, we need to assume some compactness condition on the analytic varieties. For simplicity, let us assume that all the analytic varieties are restrictions of bigger analytic varieties to relatively compact open subsets.

We need the following Theorem which is a direct corollary of the famous Hironaka Theorem (...) on resolution of singularities:

Theorem 2.2.2. *Let X be a variety. Let $Y_1, \ldots, Y_K$ be closed subvarieties in X. Then there exist a branched covering of degree one $\pi : \tilde{X} \to X$ such that:*

1. *$\tilde{X}$ is smooth;*

2. *$\pi|_{\tilde{X}\setminus D}$ is an isomorphism to $reg(X)\setminus(Y_1 \cup \cdots \cup Y_k)$, where $D = H_1 \cup \cdots \cup H_N$ is a union of smooth exceptional hypersurfaces H_i, which simultaneously have only normal crossings. We denote $\mathcal{D} = \{H_1, \ldots, H_N\}$ the set of exceptional hypersurfaces (let us always assume the exceptional hypersurfaces irreducible).*

3. *For any $k = 1, \ldots, K$ $\pi^{-1}(Y_k)$ is a union of hypersurfaces from $\mathcal{D}$;*

In order to improve the resolution, we'll need to do additional blow-ups with centers in intersections of exceptional hypersurfaces. We'll need some simple properties of this type of blow-ups:

Lemma 2.2.3. *Let $\tilde{X}$, $D = H_1 \cup \cdots \cup H_N$, and $\mathcal{D} = \{H_1, \ldots, H_N\}$ be as in Theorem 2.2.2. Let $C = H_{i_1} \cap \cdots \cap H_{i_k}$ and $\pi_C : \tilde{X}_C \to \tilde{X}$ be the blow-up with center in C. Let $H_C = \pi_C^{-1}(C)$ and $\tilde{H}_i \subset \tilde{X}^C$ be the strict transform of $H_i \in \mathcal{D}$. Denote $D_C := \tilde{H}_1 \cup \cdots \cup \tilde{H}_N \cup H_C$ and $\mathcal{D}_C = \{\tilde{H}_1, \ldots, \tilde{H}_N, H_C\}$. Then*

1. *D_C again has only normal crossing;*

2. *$\tilde{H}_{i_1} \cap \cdots \cap \tilde{H}_{i_k} = \emptyset$;*

3. *$\pi_C|_{\tilde{H}_{i_1} \cap \cdots \cap \tilde{H}_{i_{k-1}}}$ is an isomorphism to $H_{i_1} \cap \cdots \cap H_{i_{k-1}}$;*

4. *if $C \not\subset H_j \in \mathcal{D}$ (i.e. $j \neq i_1, \ldots, i_k$) then $\pi_C^{-1}(H_j) = \tilde{H}_j$.*

Proof.

1. is a standard property of blow-ups;

2. is trivial;

3. follows immediately from the standard fact that if $W \supset U \supset V$ are smooth manifolds, $\pi_V : W_V \to W$ is the blow-up with center in V and U_V is the strict transform of U, then $\pi_V|_{U_V} : U_V \to U$ is the blow-up with center in V. Indeed, let $W = \tilde{X}$, $U = H_{i_1} \cap \cdots \cap H_{i_{k-1}}$, and $V = C$. Then $U_V = H_{i_1} \cap \cdots \cap H_{i_{k-1}}$ and V is a hypersurface in U, so $\pi_V|_{U_V}$ is an isomorphism to U;

4. it follows that C is transversal to H_j because H_j is a hypersurface. In particular, the normal bundle of $H_j \cap C$ in H_j is canonically isomorphic to the normal bundle to C in $\tilde{X}$ restricted to $H_j \cap C$. Therefore, the restriction $\pi_C|_{\pi_C^{-1}(H_j)}$ is the blow-up of H_j with center in $H_j \cup C$.

□

We'll do the additional blow-ups in $n-1$ steps. We call the consequent strict transforms of the original hypersurfaces $H_1, \ldots, H_N$ the 0-type hypersurfaces and the consequent strict transforms of the new hypersurfaces appearing after kth step — the k-type hypersurfaces.

On the kth step we blow-up the intersections of $n-k+1$ 0-type hypersurfaces. Note, that after the $(k-1)$th step the 0-type hypersurfaces cannot meet by more then $n-k+1$ at one point. Therefore, the centers for the blow-ups on the kth step are disjoint and one can blow them up simultaneously. Also, the k-type hypersurfaces are always disjoint for $k > 0$ and the 0-type hypersurfaces are disjoint after $(n-1)$th step. It is convenient to label the k-type hypersurfaces by $(n-k+1)$-tuples of 0-type hypersurfaces.

Applying the above procedure, one get the following Lemma:

Lemma 2.2.4. *In the Theorem 2.2.2, one can assume also that $\mathcal{D}$ satisfy the following conditions:*

1. *Let $H_i, H_j \in \mathcal{D}$ and $H_i \cap H_j \neq \emptyset$. Then either $\pi(H_i) \subset \pi(H_j)$ or $\pi(H_i) \supset \pi(H_j)$;*

2. *Let* $H_{i_1}, \ldots, H_{i_k} \in \mathcal{D}$, $C := H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset$, *and* $\pi(H_{i_1}) \subset \cdots \subset \pi(H_{i_k})$. *Then for any irreducible component* C^0 *of* C $\pi(C^0) = \pi(H_1)$.

Proof. Follows immediately from the Lemma 2.2.3. □

As an immediate corollary of the Lemma 2.2.4, we get the following result on resolutions for flags of subvarieties.

Definition 2.2.3. Let $(x_1, \ldots, x_n)$ be a coordinate system in U (i.e. $(x_1, \ldots, x_n)$ maps U isomorphically to an open neighborhood of the origin in $\mathbb{C}^n$). A meromorphic function f is called *almost monomial* in U with respect to $(x_1, \ldots, x_n)$ if $f = x_1^{d_1} \ldots x_n^{d_n} \phi$, where $d_1, \ldots, d_n$ are integers and ϕ is a holomorphic non-zero function on U.

Theorem 2.2.3. *Let* $V_n \supset V_{n-1} \supset \cdots \supset V_0$ *be a flag of algebraic (closed analytic) subvarieties with* $\dim V_i = i$. *Let* $f_1, \ldots, f_k$ *be meromorphic functions on* V_n. *Then there exist a flag of smooth algebraic (closed analytic) subvarieties* $\overline{V}_n \supset \overline{V}_{n-1} \supset \cdots \supset \overline{V}_0$ *and a map* $\pi : \overline{V}_n \to V_n$, *such that*

1. π *is a degree one branched covering (in fact, a composition of blow-ups);*

2. $\overline{V}_k$ *is the generic component of the preimage of* V_k *with respect to* $\pi|_{\overline{V}_{k+1}}$ *for* $k = n-1, n-2, \ldots, 0$;

3. *for any point* $a \in \overline{V}_i$ *there exist a system of coordinates (called good coordinates)* $(x_1, \ldots, x_n)$ *in a neighborhood* U *of* a *in* $\overline{V}_n$ *such that*

   (a) $\overline{V}_j \cap U = \{b \in U : x_n(a) = \cdots = x_{j+1}(a) = 0\}$ *for* $j = i, i+1, \ldots, n$;

   (b) $\pi^* f_1, \ldots, \pi^* f_k$ *are almost monomial in* U *with respect to* $(x_1, \ldots, x_n)$.

Proof. We apply the Theorem 2.2.2 and Lemma 2.2.4 to V_n with subvarieties $V_{n-1}, \ldots, V_0$ and all the divisors of $f_1, \ldots, f_k$. Let $\pi : \overline{V}_n \to V_n$ be the resulting resolution map and $\mathcal{D} = \{H_1, \ldots, H_N\}$ be the exceptional hypersurfaces. Denote $\mathcal{D}_k = \{H_i \in \mathcal{D} : \pi(H_i) =$

$V_k\}$ and $D_k = \bigcup_{H_i \in \mathcal{D}_k} H_i$. Let $\overline{V}_n \supset \overline{V}_{n-1} \supset \cdots \supset \overline{V}_0$ be the flag of consequent generic preimages.

Lemma 2.2.5. *$\overline{V}_k = D_{n-1} \cap \cdots \cap D_k$ and $\overline{V}_k$ is smooth for all $k = 0, \ldots, n$.*

Proof. We first prove that $\overline{V}_k = D_{n-1} \cap \cdots \cap D_k$ by induction starting from $\overline{V}_{n-1} = D_{n-1}$, which is trivial by dimension.

Suppose that we already proved $\overline{V}_{k+1} = D_{n-1} \cap \cdots \cap D_{k+1}$. The inclusion $D_{n-1} \cap \cdots \cap D_k \subset \overline{V}_k$ follows from the condition 2 of the Lemma 2.2.4 and the dimension counting. Indeed, $dim(D_{n-1} \cap \cdots \cap D_k) = n - ((n-1) - (k-1)) = k = dim(V_k)$ and $\pi(D_{n-1} \cap \cdots \cap D_k) = \pi(D_k)$ (unless $D_{n-1} \cap \cdots \cap D_k = \emptyset$ in which case the inclusion is trivial). Therefore, every irreducible component of $D_{n-1} \cap \cdots \cap D_k$ is mapped surjectively to V_k, and, by dimension, $\pi|_{D_{n-1} \cap \cdots \cap D_k}$ is a local isomorphism at a general point.

Suppose now that $X \subset D_{n-1} \cap \cdots \cap D_{k+1}$ is an irreducible subvariety of codimension 1 such that $\pi(X) = V_k$. Since $\pi^{-1}(V_k)$ is a union of hypersurfaces from $\mathcal{D}$, it follows that $X \subset H \in \mathcal{D}$, such that $\pi(H) \supset V_k$. Moreover, there exist $H_{n-1} \in \mathcal{D}_{n-1}, \ldots, H_{k+1} \in \mathcal{D}_{k+1}$ such that $X \in H_{n-1} \cap \cdots \cap H_{k+1} \cap H$. Consider $\pi(H)$. It is an irreducible subvariety in V_n such that it either contains or is contained in V_i for every $i = n-1, \ldots, k+1$ and it contains V_k. Therefore, it coincide with one of the V_i's for $i = n-1, \ldots, k$. We need to prove that $\pi(H) = V_k$. Suppose it is wrong and $\pi(H) = V_i$ for $i > k$. Then $\pi(H_{n-1} \cap \cdots \cap H_{k+1} \cap H) = V_{k+1}$ which is impossible by dimension.

The only way $D_{n-1} \cap \cdots \cap D_k$ can be singular is if two of its irreducible components intersect each other. That means that there are two different sets of hypersurfaces $H_{n-1}, \ldots, H_k$ and $H'_{n-1}, \ldots, H'_k$ with $H_i, H'_i \in \mathcal{D}_i$, such that $C := H_{n-1} \cap \cdots \cap H_k \cap H'_{n-1} \cap \cdots \cap H'_k \neq \emptyset$. Since there is at least $n-k+1$ different hypersurfaces, $dim(C) < k$. However, by the condition 2 of the Lemma 2.2.4, $\pi(C) = V_k$ which is impossible by dimension. □

The Theorem 2.2.3 is proved.

□

Definition 2.2.4. The flag $\overline{V}_n \supset \overline{V}_{n-1} \supset \cdots \supset \overline{V}_0$ together with the map $\pi : \overline{V}_n \to V_n$ is called the *resolution of singularities of the flag* $V_n \supset V_{n-1} \supset \cdots \supset V_0$, *respecting the functions* $f_1, \ldots, f_k$.

2.2.3 Toric Parameters and Toric Neighborhoods.

Consider the following diagram:

$$
\begin{array}{ccccc}
\overline{V}_n \supset & \overline{V}_{n-1} \supset & \ldots \supset & \overline{V}_1 \supset & \overline{V}_0 \\
\downarrow \pi & \downarrow \pi & & \downarrow \pi & \downarrow \pi \\
V_n \supset & V_{n-1} \supset & \ldots \supset & V_1 \supset & V_0 = \{a\} \\
\uparrow p_n & \uparrow p_n & & & \\
\widetilde{V}_n \supset & W_{n-1} & & & \\
& \uparrow p_{n-1} & & & \\
& \widetilde{W}_{n-1} \supset & \ldots & & \\
& & \ldots \supset & W_1 & \\
& & & \uparrow p_1 & \\
& & & \widetilde{W}_1 \supset & W_0
\end{array}
$$

where the first line is the resolution of singularities of the flag $V_n \supset \cdots \supset V_0$ and the rest is the normalization diagram.

The following Lemma follows immediately from the Theorem 2.2.1:

Lemma 2.2.6. *For every* $k = 0, \ldots, n-1$ *there exist a natural degree one branched covering* $\varphi_k : \overline{V}_k \to \widetilde{W}_k$*, such that all the diagram commutes. In particular,* $\varphi_0 : \overline{V}_0 \to \widetilde{W}_0 = W_0$ *is a bijection of finite sets.*

Let $(u_1, \ldots, u_n)$ be local parameters. Let $b_\alpha \in \overline{V}_0$. According to the Theorem 2.2.3 there exist good coordinates $(x_1, \ldots, x_n)$ near b_α, such that $(v_1, \ldots, v_n) := (u_1 \circ$

$\pi, \ldots, u_n \circ \pi)$ are almost monomial near b_α. Let $v_k = x_1^{d_{k1}} \ldots x_n^{d_{kn}} \phi_k$, for $k = 1, \ldots, n$, where $\phi_1, \ldots, \phi_n$ are non-zero holomorphic functions in a neighborhood of b_α.

Lemma 2.2.7. *$d_{ij} = 0$ for $i < j$ and $d_{ii} = 1$ for $i = 1, \ldots, n$.*

Proof. According to the definition of the local parameters, u_m can be restricted consequently from $\widetilde{V}_n$ to W_{n-1}, from $\widetilde{W}_{n-1}$ to W_{n-2} and so on down to W_m and all the restrictions are not identical zeros. Finally, on $\widetilde{W}_m$, u_m has zero of order 1 at a general point of W_{m-1}.

$\varphi_k : \overline{V}_k \to \widetilde{W}_k$ is a local isomorphism at a generic point of $\overline{V}_{k-1}$ (follows from Lemma 2.2.2 applied to $\varphi_k|_{\varphi_k^{-1}(reg(\widetilde{W}_k))}$). Therefore, v_k restricts to a meromorphic function on $\overline{V_k}$ with zero of order 1 at a generic point of $\overline{V}_{k-1}$. □

Definition 2.2.5. The matrix $D := \{d_{ij}\}$ is called the *valuation matrix* of the almost monomial functions $(v_1, \ldots, v_n)$ with respect to $(x_1, \ldots, x_n)$.

Definition 2.2.6. A set $(y_1, \ldots, y_n)$ of almost monomial functions in a neighborhood of $b_\alpha \in \overline{V}_n$ with respect to the coordinates $(x_1, \ldots, x_n)$, such that the corresponding valuation matrix D is lower-diagonal with units on diagonal is called *generalized local parameters.*

Lemma 2.2.7 basically says that if $(u_1, \ldots, u_n)$ are local parameters and the resolution $\pi : \overline{V}_n \to V_n$ is such that $(v_1, \ldots, v_n) := (u_1 \circ \pi, \ldots, u_n \circ \pi)$ are almost monomial near b_α then $(v_1, \ldots, v_n)$ are generalized local parameters. However, later we'll have to use generalized local parameters of more general type.

We have the following simple lemma about the almost monomial functions:

Lemma 2.2.8. *Let $(y_1, \ldots, y_n)$ be almost monomial functions with respect to the good coordinates $(x_1, \ldots, x_n)$ in a neighborhood of b_α. Let $D = \{d_{ij}\}$ be the valuation matrix.*

Let $C = \{c_{ij}\} = D^{-1}$, *and let*

$$z_1 = y_1^{c_{11}} \dots y_1^{c_{1n}};$$
$$z_2 = y_1^{c_{21}} \dots y_1^{c_{2n}};$$
$$\vdots$$
$$z_n = y_1^{c_{n1}} \dots y_n^{c_{nn}}.$$

Then $(z_1, \dots, z_n)$ *are good coordinates in a neighborhood of* b_α *as well. (If* $|detD| \neq 1$ *then* C *has rational entries. However,* $z_1, \dots, z_n$ *don't have any branching near* b_α*, so one should just choose one branch for each* z_i*.)*

Proof. Indeed, the valuation matrix of $(z_1, \dots, z_n)$ is the identity matrix, so

$$z_1 = x_1 h_1;$$
$$z_2 = x_2 h_2;$$
$$\vdots$$
$$z_n = x_n h_n,$$

where $h_1, \dots, h_n$ are holomorphic non-zero functions in a neighborhood of b_α. Then

$$\frac{\partial z_i}{\partial x_j}|_{b_\alpha} = \delta_{ij} h_1(b_\alpha),$$

so,

$$|J(b_\alpha)| = |\{\frac{\partial z_i}{\partial x_j}\}| = h_1(b_\alpha) \cdot \dots \cdot h_n(b_\alpha) \neq 0.$$

Therefore, by the Inverse Function Theory, $(z_1, \dots, z_n)$ are coordinates in a neighborhood of b_α. Also, $\{z_i = 0\} = \{x_i = 0\}$ in a neighborhood of b_α for any $i = 1, \dots, n$. So, $(z_1, \dots, z_n)$ are good coordinates. □

Let $(y_1, \dots, y_n)$ be generalized local parameters and let $D = \{d_{ij}\}$ be the corresponding valuation matrix. Let $\hat{L}^n$ be the lattice of monomials in $y_1, \dots, y_n$ endowed with the lexicographic order with respect to the basis $(y_1, \dots, y_n)$ (first with respect to y_n, then y_{n-1} and so on). Consider the flag $\hat{L}^n \supset \hat{L}^{n-1} \supset \dots \supset \hat{L}^0$ of isolated subgroups in $\hat{L}^n$. Let $\hat{\mathcal{C}}$ be the corresponding system of cones, $\hat{L}$ be the corresponding semigroup, and $\hat{\mathcal{T}}$

be the corresponding system of toric varieties. Note, that all the toric varieties in $\hat{\mathcal{T}}$ are normal and, moreover, isomorphic to $\mathbb{C}^n$.

Note that the valuation matrix D and its inverse C are lower-triangular integer matrix with units on the diagonal in this case. Therefore,

$$z_1 = y_1^{c_{11}} \dots y_1^{c_{1n}};$$
$$z_2 = y_1^{c_{21}} \dots y_1^{c_{2n}};$$
$$\vdots$$
$$z_n = y_1^{c_{n1}} \dots y_n^{c_{nn}},$$

are standard coordinates on a variety $\hat{T}_z \in \hat{\mathcal{T}}$.

Therefore, $(z_1, \dots, z_n)$ provides an isomorphism $\psi^{-1} : W \to \hat{U}$ between a neighborhood W of $b_\alpha \in \overline{V}_n$ and a neighborhood $\hat{U}$ of the origin in $\hat{T}_z$. Moreover, $\psi(\hat{U}^k) \subset \overline{V}_k$, where $\hat{U}^k = \hat{T}^k \cap \hat{U}$, and $\psi(\hat{U}^n) = W \backslash \{z_1 \cdot \dots \cdot z_n = 0\}$.

Let $\hat{\phi} := \pi \circ \psi : \hat{U} \to V_n$. We proved the following

Theorem 2.2.4. *The map $\hat{\phi}$ has the following properties:*

1. *$\hat{\phi}(\hat{U}^k) \subset V_k$ for $k = 0, \dots, n$;*

2. *$\hat{\phi}|_{\hat{U}^n}$ is an isomorphism to the image;*

3. *for any hypersurface $H \subset V^n$ one can choose the resolution $\pi : \overline{V}_n \to V_n$ in such a way, that $H \cap \hat{\phi}(\hat{U}^n) = \emptyset$.*

Corollary 1. *Let f be a meromorphic function in a neighborhood of $a \in V^n$. One can choose the variety $\hat{T} \in \hat{\mathcal{T}}$, neighborhood of the origin $\hat{U} \subset \hat{T}$, and the map $\hat{\phi} : \hat{U} \to V^n$ in such a way that there exist a power series $f_v \in F(\hat{L})$ converging to $f \circ \hat{\phi}$ in $\hat{U}^n$. In other words, f expands into a power series in $(v_1, \dots, v_n)$ normally converging to f in $\hat{\phi}(\hat{U}^n)$, and the Newton's polyhedron of this power series belong to a cone from $\hat{\mathcal{C}}$, shifted by an integer vector.*

Corollary 2. *Parameters $(v_1, \dots, v_n)$ induce the homomorphism $v^* : F(V^n) \to F(\hat{L})$ from the field of meromorphic functions on V^n to the field $F(\hat{L})$ of Laurent power series of $\hat{L}$. In particular, the field of meromorphic functions $F(V^n)$ is endowed with the valuation $\nu_v = \nu_{F(\hat{L})} \circ v^*$.*

Corollary 3. *Let ω be a meromorphic n-form on V_n. Let $\omega = f dv_1 \wedge \dots \wedge v_n$. Then $res_{(V^n \supset \dots \supset V_0, a_\alpha = \phi_0(b_\alpha))}(\omega) = res(v^*(f) dv_1 \wedge \dots \wedge dv_n)$.*

Corollary 4. *Let $(v'_1, \dots, v'_n)$ be another set of local parameters. Then the set of Laurent series $(v^*(v'_1), \dots, v^*(v'_n))$ define the change of variables from $\hat{\mathcal{T}}$ to $\hat{\mathcal{T}}'$. Moreover, $v'^* = \psi \circ v^*$, where $\psi : F(\hat{L}) \to F(\hat{L}')$ is the isomorphism given by the change of variables.*

Corollary 5. *Parshin's residue doesn't depend on the choice of local parameters.*

Note, that $\hat{\phi}|_{\hat{U}^k}$ for $k < n$ is not an isomorphism to the image. It is rather a covering, at least locally (one should be careful defining what this locality actually means). Indeed, $\hat{\phi} = \pi \circ \psi$, where $\psi|_{\hat{U}^k}$ is an isomorphism to the image for all $k = 0, \dots, n$, while $\pi|_{\overline{V}_k}$ can be a branched covering of a higher degree with some branching near our flag. We want to improve it in such a way that $\phi|_{U^k}$ is an isomorphism to the image for all $k = 0, \dots, n$. In particular, it will help us to understand the local geometry of V^n near the flag $V^n \supset \dots \supset V^0$. In order to do so, one needs to consider special generalized local parameters, which we call *toric parameters* and more general systems of (non-normal) toric varieties, associated to flags of lattices, different from the flag of isolated subgroups.

Let $\rho_k : \widetilde{V}_k \to V_k$ be the normalization maps for $k = 0, \dots, n$ ($\rho_n = p_n$). Let $\hat{V}_{k-1} \subset \widetilde{V}_k$ be general primages of hypersurfaces $V_{k-1} \subset V_k$ under ρ_k. For every $k = 1, \dots, n$, let t_k be a meromorphic function on $\widetilde{V}_k$ which has zero of order 1 at a general point of $\hat{V}_{k-1}$. One can think of t_k as a meromorphic function on V_k as well and, moreover, for every k let us continue t_k to a meromorphic function on the whole V_n. For simplicity, we'll denote this continuation by t_k as well.

By applying the Theorem 2.2.3 several times one can get the following diagram:

$$\begin{array}{ccccccc}
V_n^n & \supset V_{n-1}^n & \supset \dots \supset & V_1^n & \supset V_0^n \\
\downarrow \pi_n & \downarrow \pi_n & & \downarrow \pi_n & \downarrow \pi_n \\
\vdots & \vdots & & \vdots & \vdots \\
\downarrow \pi_3 & \downarrow \pi_3 & & \downarrow \pi_3 & \downarrow \pi_3 \\
V_n^2 & \supset V_{n-1}^2 & \supset \dots \supset & V_1^2 & \supset V_0^2 \\
\downarrow \pi_2 & \downarrow \pi_2 & & \downarrow \pi_2 & \downarrow \pi_2 \\
V_n^1 & \supset V_{n-1}^1 & \supset \dots \supset & V_1^1 & \supset V_0^1 \\
\downarrow \pi_1 & \downarrow \pi_1 & & \downarrow \pi_1 & \downarrow \pi_1 \\
V_n & \supset V_{n-1} & \supset \dots \supset & V_1 & \supset V_0
\end{array}$$

where $(\pi_1 \circ \pi_2 \circ \dots \circ \pi_k)|_{V_k^k} : V_k^k \to V_k$ is a resolution of singularities of the flag $V_k \supset \dots \supset V_0$ respecting the functions $t_1, \dots, t_k$.

Let $b_\alpha \in V_0^n$ as before. Let $(x_1, \dots, x_k)$ be good coordinates in a neighborhood of the $b_\alpha^k := \pi_{k+1} \circ \dots \circ \pi_n(b_\alpha)$. Let $\tilde{t}_i := (\pi_1 \circ \dots \circ \pi_k)^*(t_i)$ for $i = 1, \dots, k$. Let $D^k = \{d_{ij}^k\}$ be the valuation matrix of $(\tilde{t}_1, \dots, \tilde{t}_k)$ with respect to $(x_1, \dots, x_k)$ and let $C^k = \{c_{ij}^k\} = (D^k)^{-1}$ be the inverse matrix.

Lemma 2.2.9. *$d_{ij}^k = 0$ for $i < j$, $d_{kk}^k = 1$, and $d_{ii}^k \neq 0$ for $i = 1, \dots, k-1$.*

Proof. Similar to Lemma 2.2.7. □

Let

$$\begin{aligned}
t_1^k &= t_1^{c_{11}^k} \dots t_1^{c_{1k}^k}; \\
t_2^k &= t_1^{c_{21}^k} \dots t_1^{c_{2k}^k}; \\
&\vdots \\
t_k^k &= t_1^{c_{k1}^k} \dots t_n^{c_{kk}^k},
\end{aligned}$$

and let L^k be the lattice of monomials in $(t_1^k, \dots, t_k^k)$. Here L_k is a subgroup in the multiplicative group of monomials in $(t_1, \dots, t_n)$ with rational powers. Note, that although t_i^k are multivalued functions globally, they don't any branching near $b_\alpha \in V_n^n$. We choose the branches simultaneously for all $k = 1, \dots, n$. By abuse of notations we denote these branches by t_i^k as well.

Lemma 2.2.10. *$L^n \supset \cdots \supset L^1 \supset L^0 := \{0\}$ is a flag of lattices, where L^n is endowed with the lexicographic order with respect to the basis $(t_1^n, \ldots, t_n^n)$ (first with respect to t_n^n, then t_{n-1}^n, etc.). Moreover, the cone generated by $(t_1^{k-1}, \ldots, t_{k-1}^{k-1})$ is a subset of the cone generated by $(t_1^k, \ldots, t_k^k)$.*

Proof. It is enough to prove that $(t_1^{k-1}, \ldots, t_{k-1}^{k-1})$ are monomials with positive integer powers in $(t_1^k, \ldots, t_{k-1}^k)$. Indeed, we have the map $\pi_k|_{V_{k-1}^k} : V_{k-1}^k \to V_{k-1}^{k-1}$, which is continuous. $(t_1^{k-1}, \ldots, t_{k-1}^{k-1})$ are coordinates in a neighborhood of $b_\alpha^{k-1} = \pi_k(b_\alpha^k)$ in V_{k-1}^{k-1} and $(t_1^k, \ldots, t_{k-1}^k)$ are coordinates in a neighborhood of b_α^k in V_{k-1}^k. Therefore, the powers have to be positive and integer. □

Let $\mathcal{C}$, L, and $\mathcal{T}$ be, respectively, system of cones, semigroup, and system of toric varieties, associated to the flag of lattices $L^n \supset \cdots \supset L^1 \supset L^0$.

Note, that $t_1^n, \ldots, t_n^n$ are generalized local parameters (here we consider $t_1^n, \ldots, t_n^n$ as functions in a neighborhood of $b_\alpha \in V_n^n$). So, one can apply the Theorem 2.2.4 to them.

Note, that lattice $\hat{L}^n = L^n$. So, the flag of isolated subgroups $\hat{L}^n \supset \hat{L}^{n-1} \supset \cdots \supset \hat{L}^0$ is the normalization of the flag of lattices $L^n \supset \cdots \supset L^1 \supset L^0$. Let $\hat{\phi} : \hat{U} \to V_n$ be the map from the Theorem 2.2.4. Here $\hat{U} \subset \hat{T}_{t^n} \in \hat{\mathcal{T}}$ is a neighborhood of the origin. Let $T_{t^n} \in \mathcal{T}$ be the variety which normalization is $\hat{T}_{t^n}$ and let $\nu : \hat{U} \to U$ be the restriction of the normalization map to $\hat{U}$ ($U = \nu(\hat{U})$). Let $U^k = \nu(\hat{U}^k)$ (i.e. the intersection of U with the corresponding toric orbit in T_{t^n}).

Theorem 2.2.5. *The map $\hat{\phi} : \hat{U} \to V_n$ factors through the normalization map $\nu : \hat{U} \to U$, i.e. $\hat{\phi} = \phi \circ \nu$ where $\phi : U \to V_n$. Moreover, $\phi|_{U^k}$ is an isomorphism to the image inside V_k for all $k = 0, 1, \ldots, n$.*

Proof. According to the Lemma 2.1.8 it is enough to show that $\hat{\phi}$ factors through the normalization on the level of sets and that ϕ is continuous (then it is regular). More over, similarly to the Theorem 2.1.4 the continuity follows immediately. The only thing we need to check is that ϕ is well defined on the level of sets, i.e. that for any $x \in U$ and

$y_1, y_2 \in \nu^{-1}(x)$ we have $\hat{\phi}(y_1) = \hat{\phi}(y_2)$. Indeed, let k be the smallest number such that $x \in \overline{U^k}$. The functions $t_k^1, \dots, t_k^k$ provides coordinates on V_k^k near b_α^k. On the other hand, they belong to L_k and, therefore, they are regular on $U \setminus \overline{U^{k-1}}$, in particular, at x.

Note, that $\hat{\phi} = \pi_1 \circ \cdots \circ \pi_n \circ \psi$. Denote $z_i = \pi_{k+1} \circ \cdots \circ \pi_n \circ \psi(y_i) \in V_k^k$. It follows from the above, that $t_k^1, \dots, t_k^k$ don't distinguish z_1 and z_2. Therefore, $z_1 = z_2$.

Consider now the restriction $\phi|_{U^k}$. It follows from the above that $\phi|_{U^k} = \pi_1 \circ \cdots \circ \pi_k \circ \psi_k$, where ψ_k is a regular map from U^k to the complement of the coordinate cross in a neighborhood of b_α^k in V_k^k. Moreover, ψ_k is an isomorphism to the image (since $dt_k^1 \wedge \cdots \wedge dt_k^k$ is a non-degenerate top form both in the image and the preimage). $\pi_1 \circ \cdots \circ \pi_k$ is an isomorphism to the image on the complement to the exceptional divisor. Therefore, $\phi|_{U^k}$ is an isomorphism to the image as well. $\square$

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