MAT267: 4th HW assignment. Due by 11:59pm on March 26.

1. (10 points) Consider the Lyapunov function $\mathcal{L}(\vec{X}) = x_2^2 + U(x_1)$ where x_1 is the first component of $\vec{X}$ and x_2 is the second. Or, if you prefer, consider the "Energy" $\mathcal{E}(x, x') = (x')^2 + U(x)$.

   (a) Find find a first-order system $\vec{X}' = \vec{F}(\vec{X})$ whose solutions satisfy $\frac{d}{dt}\mathcal{L}(\vec{X}(t)) = 0$. Alternately, find a second-order ODE whose solutions satisfy $\frac{d}{dt}\mathcal{E}(x(t), x'(t)) = 0$ and then write that second-order ODE as a first-order system.

   (b) How are steady states of your second-order ODE (or your first-order system) related to critical points of the "potential" $U(x)$?

   (c) Assume $\vec{X}_{ss}$ is a steady state of your first-order system. Find $D\vec{F}(\vec{X}_{ss})$. Find its eigenvectors and eigenvalues.

2. (15 points) On the next page, you'll find the graphs of three different potentials. Each will produce a different first-order system. $U_1(x)$ and $U_3(x)$ are quartic polynomials; $U_2(x)$ is a cubic polynomial.

   (a) Find all three potentials and then find the three corresponding first-order systems. Before moving on to the next step, if $\vec{X}$ has $x_1 = 0$ and $x_2 > 0$, what direction do you expect $\vec{X}$ to go under the flow? (Compute $\vec{F}(\vec{X})$.) Same question for $\vec{X}$ with $x_1 = 0$ and $x_2 < 0$, for $\vec{X}$ with $x_1 > 0$ and $x_2 = 0$, and $\vec{X}$ with $x_1 < 0$ and $x_2 = 0$.

   (b) Use `http://www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html` to explore the phase portraits of your three systems. Choose the "system" tab and change the "Euler" tab to "Runge-Kutta 3/8 rule" and change the "h" to 0.01. Play around by clicking your mouse near the steady states. And click elsewhere. Once you have have a general understanding of the phase portrait, click "clear all curves" and then make a ready-to-hand-in phase portrait. To save the image, click on the picture next to the word "thumbnail" and save the png file.

   (c) If there are critical level sets of $\mathcal{L}$ that divide the plane into regions where you have one type of behaviour in one region and another type of behaviour in another, what are the critical values of $\mathcal{L}$? Could you have predicted them from the graphs of the potentials? What are the different types of behaviours?

   (d) Modify your first-order systems so that $\frac{d}{dt}\mathcal{L}(\vec{X}(t)) \leq 0$ and demonstrate (pen and paper) that your modifications work. Present three phase portraits to show the effect of this extra term on the dynamics. (Try to choose the term so that the effect is noticeable but not so overwhelming that there's no "memory" of the previous dynamics.)

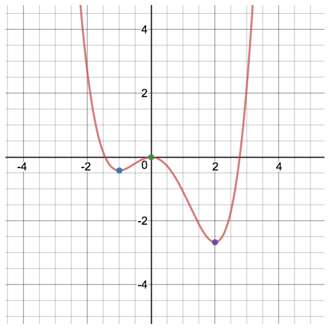

Figure 1: Potential 1. Graph of $U_1(x_1)$. Extrema: $(-1, -5/12)$, $(0, 0)$, and $(2, -8/3)$.

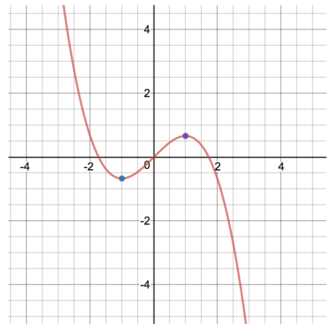

Figure 2: Potential 2. Graph of $U_2(x_1)$. Extrema: $(-1, -2/3)$ and $(1, 2/3)$.

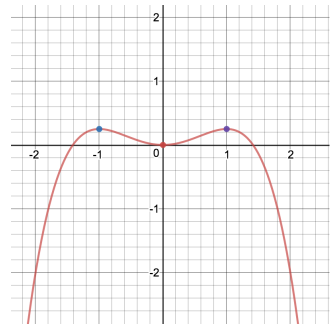

Figure 3: Potential 3. Graph of $U_3(x_1)$. Extrema: $(-1, 1/4)$, $(0, 0)$, and $(1, 1/4)$.

3. (5 points) None of the three potentials have critical points that are also inflection points. Write down a potential whose graph has only one critical point and that critical point is also an inflection point. What is the resulting first-order system? Present a phase plot for the system.

4. (15 points) The three potentials have $U''(x) \neq 0$ at the critical points. If you multiply your potential by $(x - x_c)$ where x_c is one of the critical points, what is the resulting first-order system? Present a phase plot for the system. If you multiply your potential by $(x - x_c)^2$ where x_c is one of the critical points, what is the resulting first-order system? Present a phase plot for the system. *Do this exercise for one of the potentials (you choose). And, for that potential, do it once for a critical point that's a local minimum and then do it for a critical point that's a local maximum. Of course, if you're having a blast, do it for all three potentials and all critical points. And play around with higher powers as well.*

5. (5 points) Now that you've played around with a first-order system that has all of its steady state solutions on the x_1 axis, create a nonlinear first-order system $\vec{X}' = \vec{F}(\vec{X})$ that has one steady state solution in $\mathbb{R}^2$. What is the steady state, what is DF at the steady state, what are the eigenvalues and eigenvectors of DF? Explore its phase portrait numerically.

6. (10 points) Create a nonlinear first-order system $\vec{X}' = \vec{F}(\vec{X})$ that has two steady state solutions in $\mathbb{R}^2$. What are the steady states, what are DF at the steady states, what are the eigenvalues and eigenvectors of the two DF matrices? Explore its phase portrait numerically.

7. Not to hand in (unless you want) and not to do (unless you're curious) Create a nonlinear first-order system $\vec{X}' = \vec{F}(\vec{X})$ that has three non-colinear steady state solutions in $\mathbb{R}^2$. What are the steady states, what are DF at the steady states, what are the eigenvalues and eigenvectors of the two DF matrices? Explore its phase portrait numerically.