# MAT267: 6th HW assignment. Due by 11:59pm on April 9.

Questions are labelled with "hand in" and "don't hand in". Here, "don't hand in" is short for "Make sure you know how to do this and are confident in your answer. But you don't need to write it up nicely to submit it for grading."

1. (10 points)
   (a) Don't hand in. Prove that if $y(t) > 0$ and $y' \leq -\mu\, y$ then $y(t) \leq y(0)\, e^{-\mu\, t}$ and $y(t) \to 0$ as $t \to \infty$.
   (b) Hand in. Prove that if $y(t) < 0$ and $y' \geq -\mu\, y$ then $y(t) \geq y(0)\, e^{-\mu\, t}$ and $y(t) \to 0$ as $t \to \infty$.
2. (10 points) Consider $x' = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is $C^1$. Assume $a$ is an equilibrium point: $f(a) = 0$. Assume $f'(a) \neq 0$.
   (a) Don't hand in. Prove that if $f'(a) < 0$ then there's an $\epsilon > 0$ so that if $x_0 \in (a - \epsilon, a + \epsilon)$ then the initial value problem $x' = f(x)$ with $x(0) = x_0$ has a solution for all $t > 0$ and the solution $x(t) \to a$ as $t \to \infty$.
   (b) Hand in. Prove that if $f'(a) > 0$ then there's an $\epsilon > 0$ so that if $x_0 \in (a - \epsilon, a + \epsilon)$ then the initial value problem $x' = f(x)$ with $x(0) = x_0$ has a solution for all $t < 0$ and the solution $x(t) \to a$ as $t \to -\infty$.
3. (10 points) Consider $x' = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is $C^2$. Assume $a$ is an equilibrium point: $f(a) = 0$. Assume $f'(a) = 0$.
   (a) Don't hand in. Prove that if $f''(a) < 0$ then there's an $\epsilon > 0$ so that if $x_0 \in (a, a + \epsilon)$ then the initial value problem $x' = f(x)$ with $x(0) = x_0$ has a solution for all $t > 0$ and the solution $x(t) \to a$ as $t \to \infty$. And if $x_0 \in (a - \epsilon, a)$ then the initial value problem has a solution for all $t < 0$ and the solution $x(t) \to a$ as $t \to -\infty$.
   (b) Hand in. Prove that if $f''(a) > 0$ there's an $\epsilon > 0$ so that if $x_0 \in (a, a + \epsilon)$ then the initial value problem $x' = f(x)$ with $x(0) = x_0$ has a solution for all $t < 0$ and the solution $x(t) \to a$ as $t \to -\infty$. And if $x_0 \in (a - \epsilon, a)$ then the initial value problem has a solution for all $t > 0$ and the solution $x(t) \to a$ as $t \to \infty$.
4. (5 points) Hand in. Consider $x' = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is $C^k$. Assume $a$ is an equilibrium point: $f(a) = 0$. Assume that $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$ and that $f^{(k)}(a) \neq 0$. What can you say about the behaviour of solutions with initial data "close to" $a$?

5. (15 points) Consider the initial value problems

$$\begin{cases} x' &= f_1(x,y) \\ y' &= f_2(y) \\ x(0) &= x_1 \\ y(0) &= y_1 \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{x}' &= f_1(\tilde{x},\tilde{y}) \\ \tilde{y}' &= f_2(\tilde{y}) \\ \tilde{x}(0) &= x_2 \\ \tilde{y}(0) &= y_1 \end{cases}$$

Note that the dynamics of $y(t)$ are independent of $x(t)$. Also note that the two initial value problems have the same initial value for $y(0)$. Assume that $x_1 < x_2$ and that $f_1$ and $f_2$ are $C^1$ functions. Assume that the solution $(x(t), y(t))$ has interval of existence $(a_1, b_1)$ and the solution $(x(t), y(t))$ has interval of existence $(a_2, b_2)$.

(a) <u>Don't hand in.</u> Prove that if $\frac{\partial f_1}{\partial x} > 0$ then $\tilde{x}(t) - x(t) > x_2 - x_1$ for all $t \in [0, \min\{b_1, b_2\})$.

(b) <u>Hand in.</u> Prove that if $\frac{\partial f_1}{\partial x} < 0$ then $\tilde{x}(t) - x(t) < x_2 - x_1$ for all $t \in [0, \min\{b_1, b_2\})$.

6. <u>Don't hand in.</u> In the the proof of the stable curve theorem, we did everything in cones $C_M^+$ and $C_M^-$. For example, we showed that you can take $\epsilon$ small enough so that

$$(x,y) \in C_M^+ \quad \Longrightarrow \quad y' = -\mu\, y + f_2(x,y) \le -\frac{\mu}{2}\, y.$$

Why is it that we restricted ourselves to cones? What happens if you try to show

$$(x,y) \in B_\epsilon^+ \quad \Longrightarrow \quad y' = -\mu\, y + f_2(x,y) \le -\frac{\mu}{2}\, y$$

where $B_\epsilon^+ = \{(x,y) \,|\, |x| \le \epsilon, 0 < y \le \epsilon\}$. *Hint: consider $(x,y)$ on some curve in $B_\epsilon^+$ that passes through the origin but isn't in $C_M^+$. For example, try $y = M\,x^2$ where $M < 1/\epsilon$.*

7. Exercises from Hirsch, Smale, & Devaney.

(a) <u>Don't hand in.</u> Please do exercise 4 on page 184.

(b) (10 points) <u>Hand in.</u> Please do exercise 5 on page 185.

(c) <u>Don't hand in.</u> Please do exercise 6 on page 185.

(d) (10 points) <u>Hand in.</u> Please do exercise 7 on page 185.

(e) <u>Don't hand in.</u> Please do exercise 8 on page 185.

(f) <u>Don't hand in.</u> Please do exercise 9 on page 185.