MAT267: Please do these problems and submit them by 5pm on Saturday (March 2). If you write them up and hand them in, there's a chance that they'll be marked and that the mark will count towards your course mark. I'll send out email about how to submit the HW via Crowdmark

1. Consider
$$\vec{X}' = A\vec{X} + \vec{B}, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \text{ and } \vec{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}.$$
There are infinitely many steady state solutions. Choose one steady state and use it to define $\vec{X}(t) = \vec{X}_{ss} + \vec{Y}(t)$. Sketch the phase portrait for $\vec{Y}' = A\vec{Y}$. Find the general solution $\vec{X}(t)$ and sketch the phase portrait. Now choose a different steady state and repeat the process. Do you get the same phase portrait? Show that the two general solutions represent the same infinite family of solutions? (Make sure that you can generalize your proof to a general linear system $\vec{X}' = A\vec{X} + \vec{B}$ where A is singular and there are infinitely many steady states.)

2. Now find the general solution from exercise 1 by the direct method: Find P and J so that $A = PJP^{-1}$ and J is in Jordan normal form. Solve $\vec{Z}' = J\vec{Z} + P^{-1}\vec{B}$ and recover the general solution $\vec{X}$ from $\vec{Z}$.

3. Consider
$$\vec{X}' = A\vec{X} + \vec{B}, \qquad \text{where} \qquad A = \begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \text{ and } \vec{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
First, verify that there is no steady state solution and so you cannot take the approach of exercise 1 above. Now, find P and J so that $A = PJP^{-1}$ and J is in Jordan normal form. Solve $\vec{Z}' = J\vec{Z} + P^{-1}\vec{B}$ and recover the general solution $\vec{X}$ from Z. Sketch the phase portrait of the system. *The $\vec{B}$ I gave you happened to be in the kernel of A. But you should check what happens if $\vec{B}$ is not in the range of A and is also not in the kernel of A. $\vec{B} = (1;2)$ for example.*

4. Compute e^{tA} where
$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{1}$$
and α and β are real numbers. Either compute it directly from the power series definition or by first diagonalizing A over the complex numbers.

5. The damped linear oscillator
$$x'' + x' + x = 0$$
has general solution
$$x(t) = e^{-t/2}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

And the forced damped oscillator

$$x'' + x' + x = \sin(2t)$$

has general solution

$$x(t) = e^{-t/2}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{2}{13}\cos(2t) - \frac{3}{13}\sin(2t).$$

On the other hand, we know that we can write $x'' + x' + x = \sin(2t)$ as the first-order system

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(2t) \end{pmatrix} = A\begin{pmatrix} x \\ v \end{pmatrix} + \vec{B}(t)$$

and so we know that

$$\begin{pmatrix} x \\ v \end{pmatrix} = e^{tA}\int e^{-sA}\vec{B}(s)\,ds. \tag{2}$$

The matrix A is similar to a matrix of the form (1); see pages 53-54 of Hirsch, Smale, & Devaney. And so you know how to compute e^{tA} because you've done exercise 4 above and you know how the exponentials of similar matrices are related. Compute (2) and confirm that you get the correct solution.

6. Challenge Problem *I don't expect many/any of you to do this problem. If you do solve it, please email me your solution!*

   Generalize what you observed in exercise 3. Consider $\vec{X}' = A\vec{X} + \vec{B}$. We know that $A = PJP^{-1}$ where J is in Jordan normal form and so if $\vec{Z} = P^{-1}\vec{X}$ then it suffices to solve $\vec{Z}' = J\vec{Z} + P^{-1}\vec{B}$. Show that if $\vec{B}$ is not in the column space of A (i.e. it's not spanned by the columns of A) then at least one component of $\vec{Z}$ will contain a polynomial (of degree at least 1) in t. (The general solution $\vec{Z}(t)$ has n free parameters in it if A is $n \times n$. Show that no matter how you choose those free parameters, the resulting solution will have at least one component that contains a polynomial in t.) What does this mean about the solution $\vec{X}$?